# MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

## Oct 23, 2013: Cohomology III
## Oct 23, 2013: Stability

Fall 2013 course offered through the University of Iowa Division of Continuing Education

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Applied Mathematical and Computational Sciences,
University of Iowa

http://www.math.uiowa.edu/~idarcy/AppliedTopology.html http://ima.umn.edu/2013-2014/W10.28-11.1.13/#Schedule

IMA Annual Program Year Workshop

# Modern Applications of Homology and Cohomology

**October 28 - November 1, 2013**

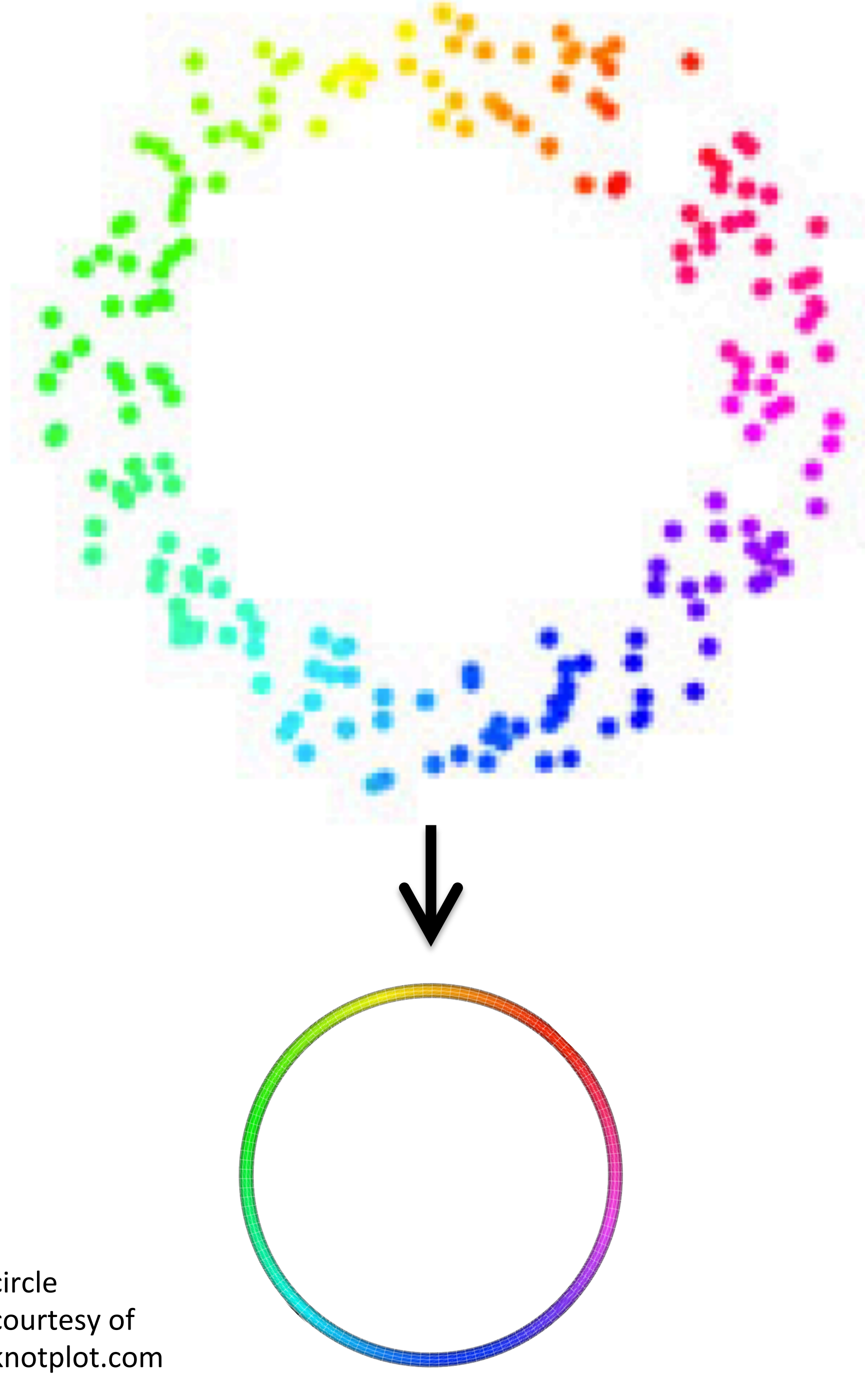
Program Application closed // **Abstracts and Talk Materials** // Live Streaming

**Organizers**

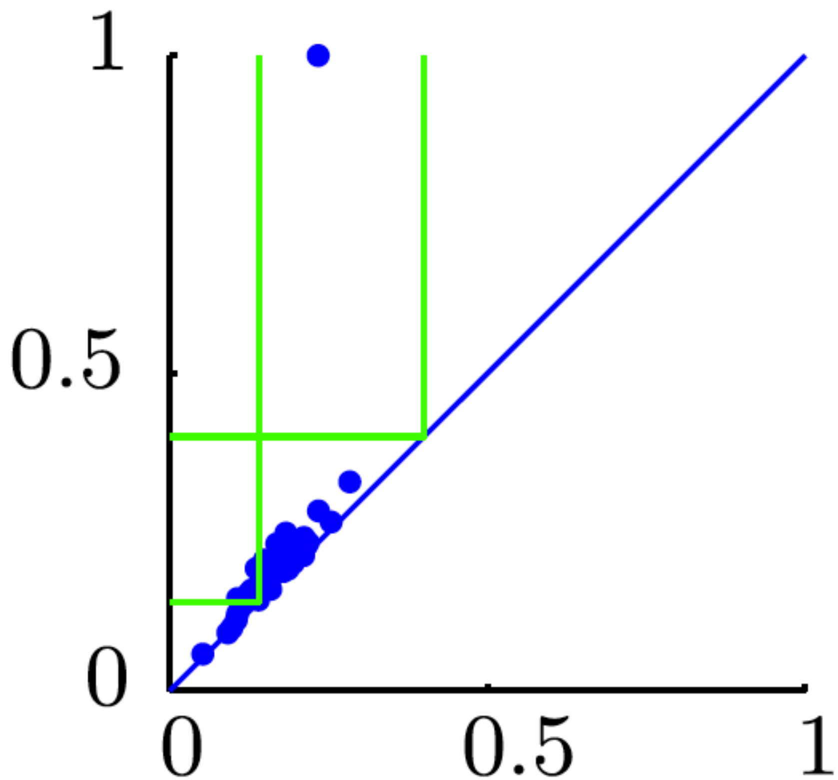

| | |
|---|---|
| Andrew Blumberg | University of Texas, Austin |
| Lek-Heng Lim | University of Chicago |
| Yuan Yao | Peking University |

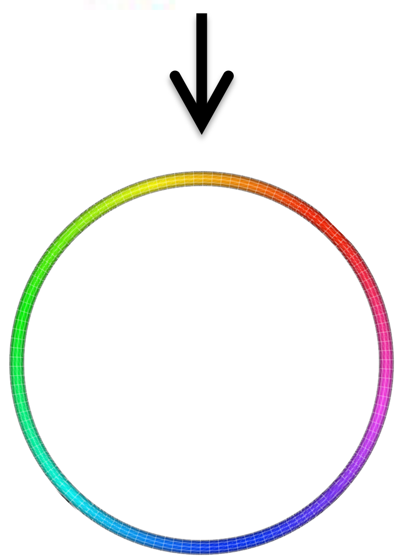
**Goal**

f: D → $S^1$ which "preserves" the structure of the data.



circle courtesy of knotplot.com





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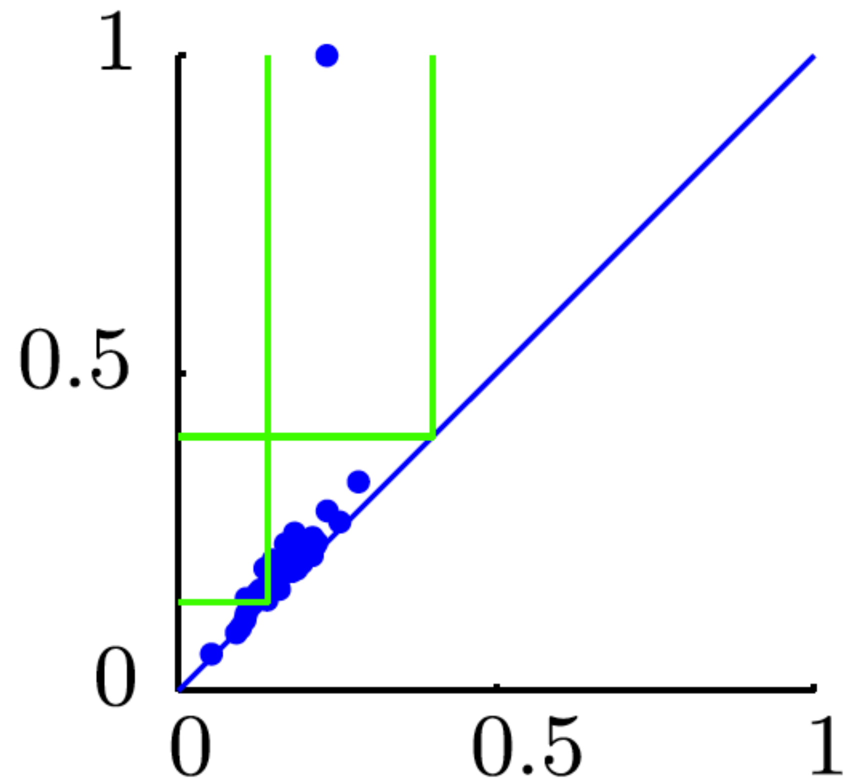


Let $\alpha \in C^i$ ,

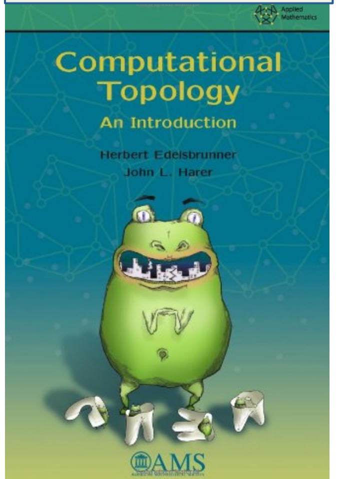
$d_i : C^i = \{ f : X^i \rightarrow R \} \rightarrow C^{i+1} = \{ f : X^{i+1} \rightarrow R \}$

$\alpha$ is a *cocycle* if $d_i\alpha = 0$. I.e., $\alpha$ is in $\mathrm{Ker}(d_1)$

$\alpha$ is a *coboundary* if there exists f in $C^{i-1}$ s.t. $d_{i-1}f = \alpha$
I.e., $\alpha$ is in $\mathrm{Im}(d_{i-1})$

Note $d_i\, d_{i-1}\, f = 0$ for any $f \in C^{i-1}$.

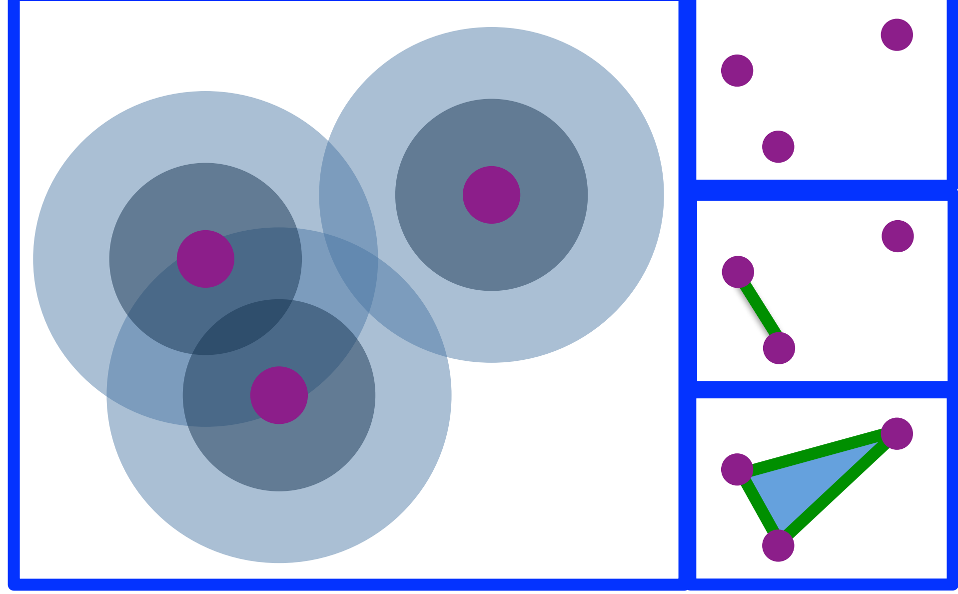
See ch 5




$d_i\, d_{i-1}\, f = 0$ implies $\mathrm{Im}\,(d_{i-1}) \subseteq \mathrm{Ker}\,(d_i)$.

$H^i\,(X;A) = \mathrm{Ker}(d_i)/\,\mathrm{Im}(d_{i-1})$.

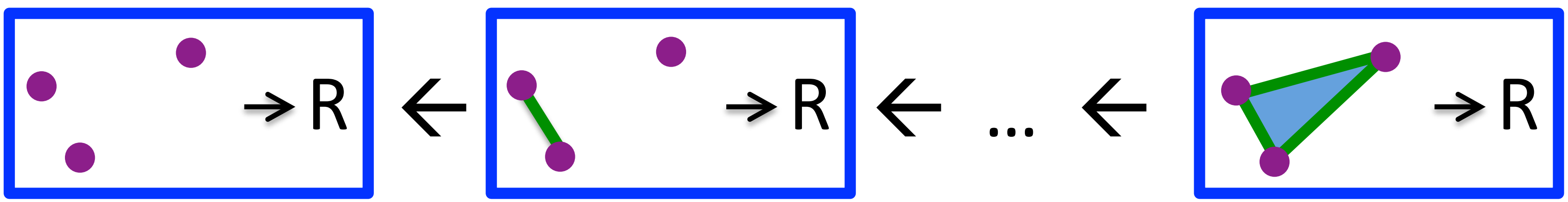
# Persistent Cohomology



$\varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_m$

$X(\varepsilon_1) < X(\varepsilon_2) < \ldots < X(\varepsilon_m)$

$C^i(X(\varepsilon_j), R) = \{ f : X^i(\varepsilon_j) \to R \}$




$C^i(X(\varepsilon_1), R) \leftarrow C^i(X(\varepsilon_2, R)) \leftarrow \ldots \leftarrow C^i(X(\varepsilon_m, R))$

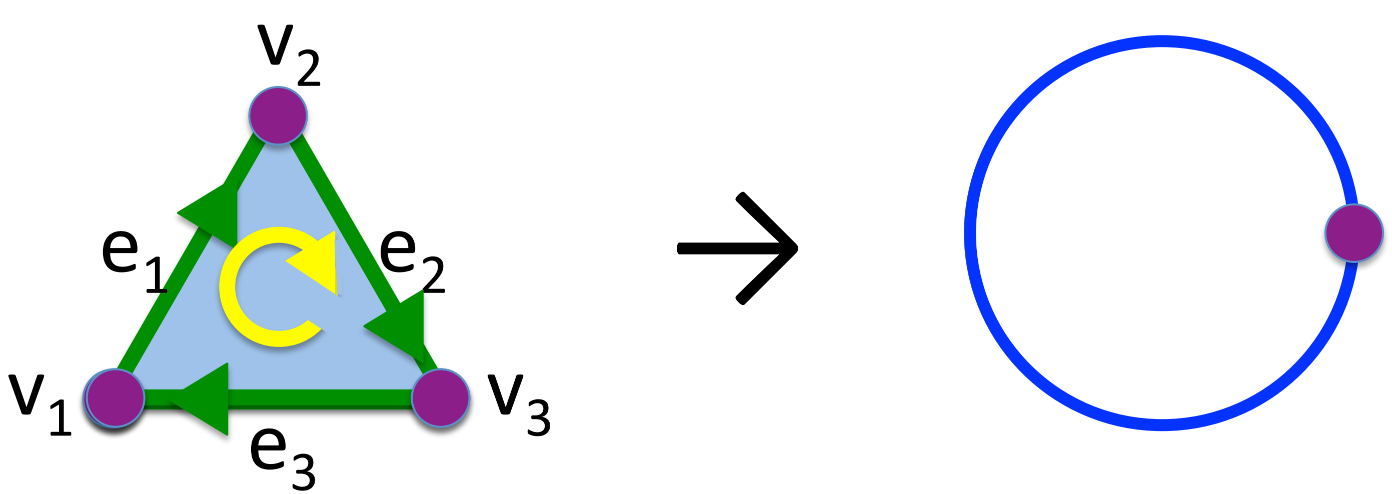
$H^i(X(\varepsilon_1), R) \leftarrow H^i(X(\varepsilon_2, R)) \leftarrow \ldots \leftarrow H^i(X(\varepsilon_m, R))$

Dualities in persistent (co)homology, de Silva, Morozov, Vejdemo-Johansson, Inverse Problems 2011, http://iopscience.iop.org/0266-5611/27/12/124003

Proposition 1: Let $\alpha \in C^1(X;\mathbf{Z})$ be a cocycle. Then there exists a continuous function

$$\theta : X \rightarrow S^1$$

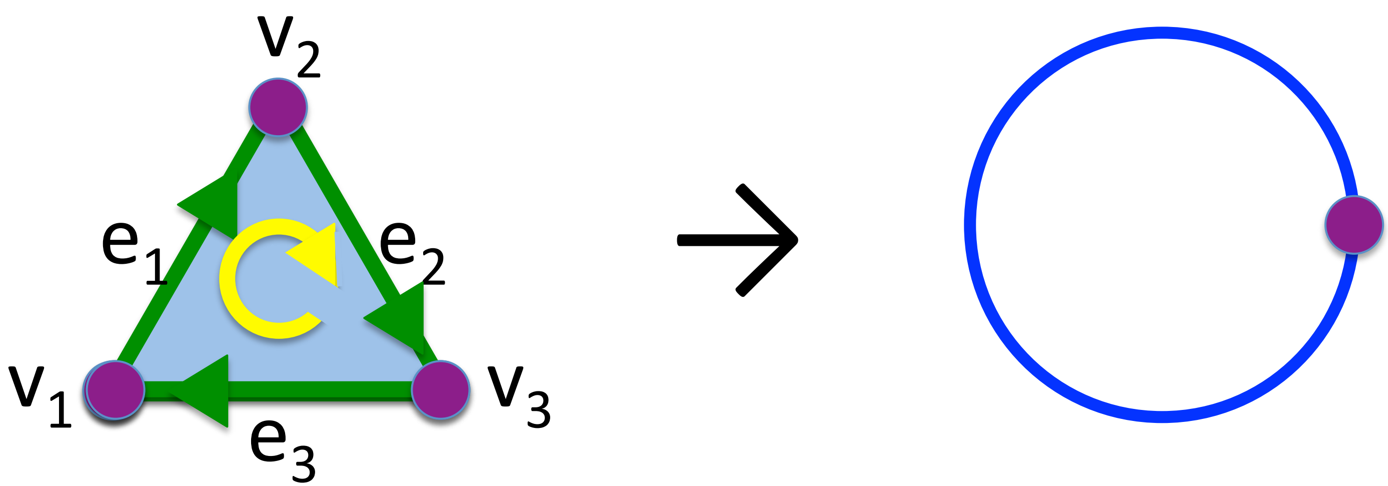
which maps each vertex to 0, and each edge ab around the entire circle with winding number $\alpha(ab)$.



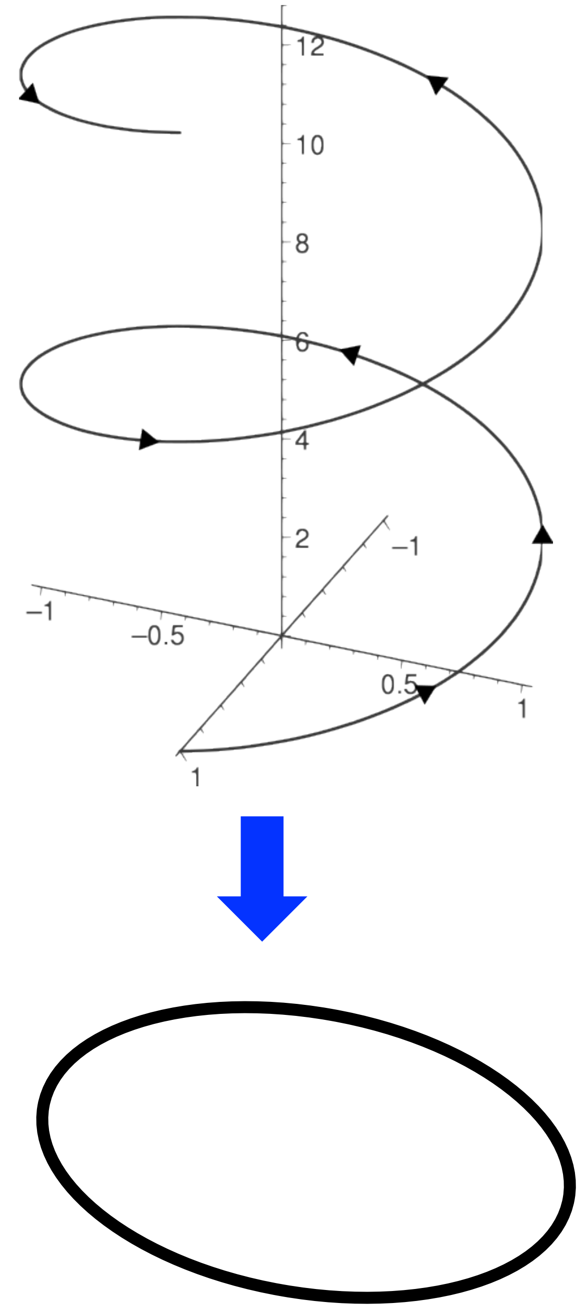
Proposition 2: 2 Let $\overline{\alpha} \in C^1(X;\mathbf{R})$ be a cocycle. Suppose there exists $\alpha \in C^1(X;\mathbf{Z})$ and $f \in C^0(X;\mathbf{R})$ such that $\overline{\alpha} = \alpha + d_0 f$ . Then there exists a continuous function

$$\theta : X \to S^1$$

which maps each edge ab linearly to an interval of length $\overline{\alpha}(ab)$, measured with sign.



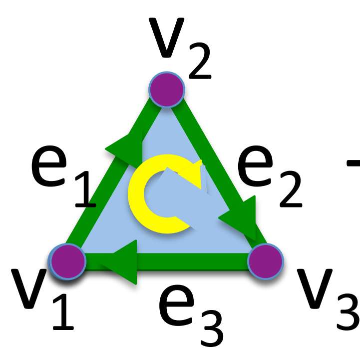
$$\theta : X \to \mathbf{R}/\mathbf{Z} = S^1$$



Given: $\overline{\alpha} \in C^1(X;\mathbf{R})$ is a cocycle, $\alpha \in C^1(X;\mathbf{Z})$

$f \in C^0(X;\mathbf{R})$, $\overline{\alpha} = \alpha + d_0 f$

Create: $\theta : X \to \mathbf{R/Z} = S^1$ s.t. $v_i v_k$ goes to interval of length $\overline{\alpha}(v_i v_k)$




$\to$ $\mathbf{R/Z}$

Given: $\overline{\alpha} \in C^1(X;\mathbf{R})$ be a cocycle, $\alpha \in C^1(X;\mathbf{Z})$

$f \in C^0(X;\mathbf{R})$, $\overline{\alpha} = \alpha + d_0 f$

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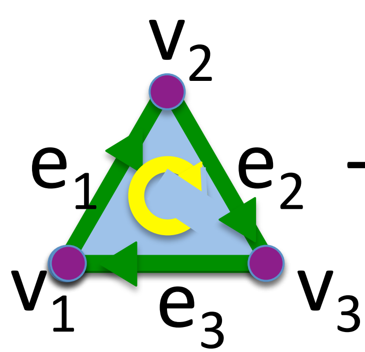
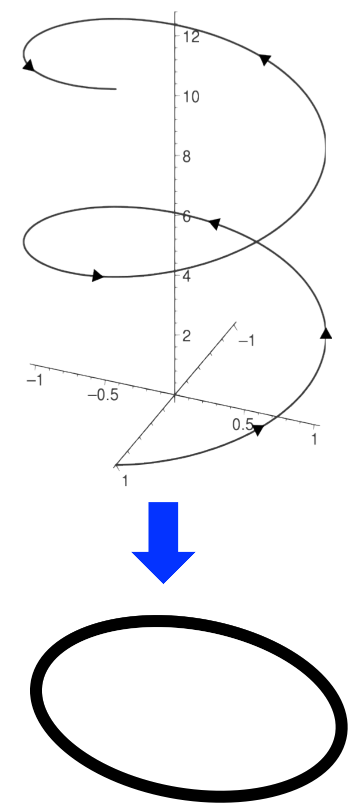



$\to$ $\mathbf{R/Z}$



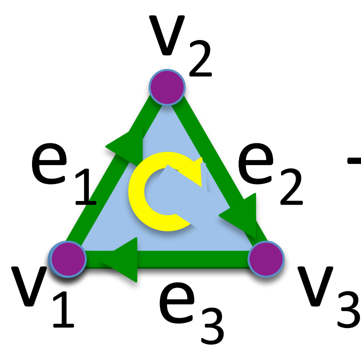
Let $\theta(v_i) = f(v_i) \bmod \mathbf{Z}$

$\theta(v_i) - \theta(v_k) =$

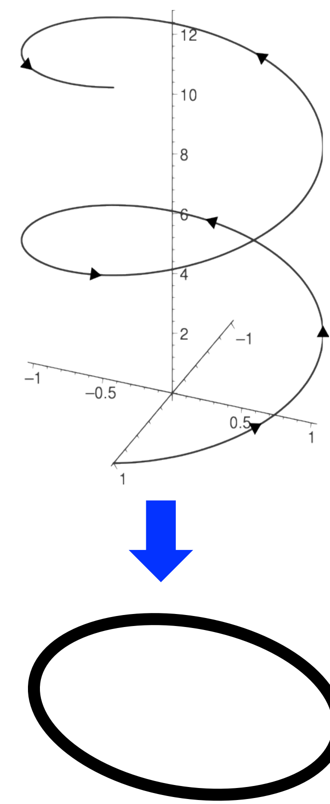
Given: $\overline{\alpha} \in C^1(X;\mathbf{R})$ be a cocycle, $\alpha \in C^1(X;\mathbf{Z})$

$f \in C^0(X;\mathbf{R})$, $\overline{\alpha} = \alpha + d_0 f$

Create: $\theta : X \to \mathbf{R/Z} = S^1$ s.t. $v_i v_k$ goes to interval of length $\overline{\alpha}(v_i v_k)$




$\to$ **R/Z**

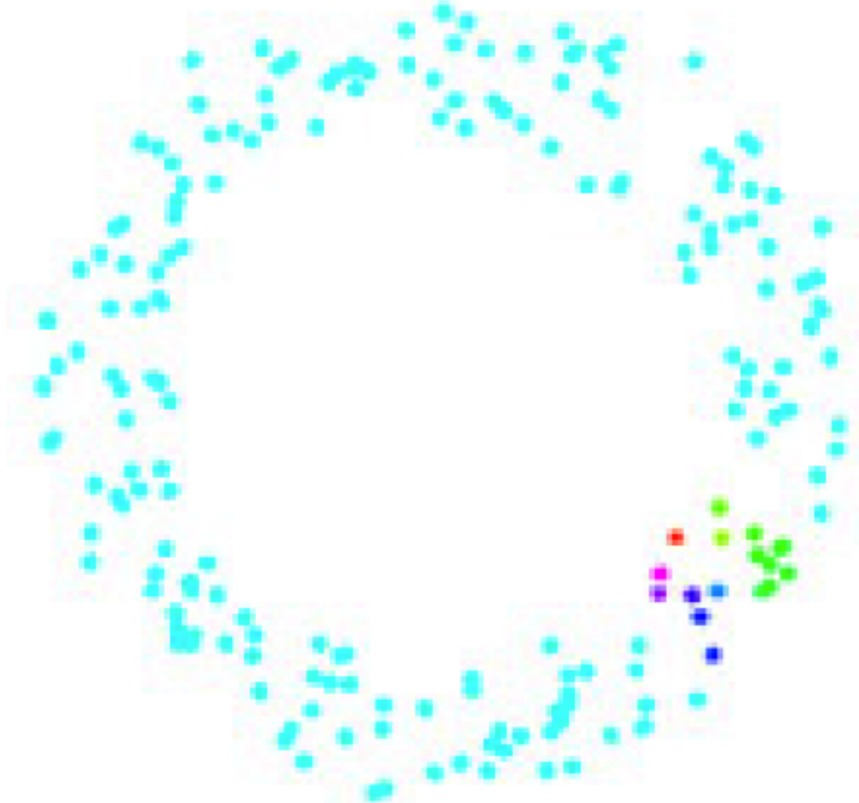


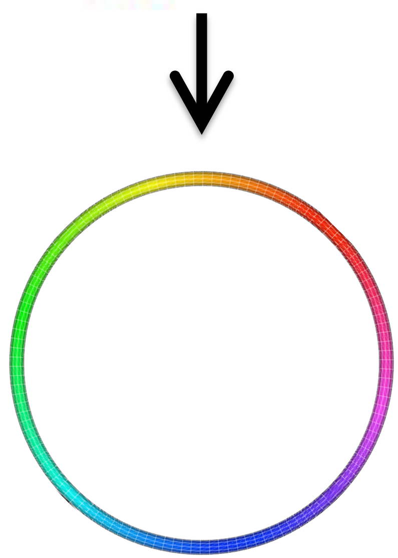
Let $\theta(v_i) = f(v_i) \bmod \mathbf{Z}$

$$\theta(v_i) - \theta(v_k) = f(v_i) - f(v_k)$$

$$= d_0 f(v_i v_k) = \overline{\alpha}(v_i v_k) - \alpha(v_i v_k)$$

$$= \overline{\alpha}(v_i v_k) \mod \mathbf{Z}$$





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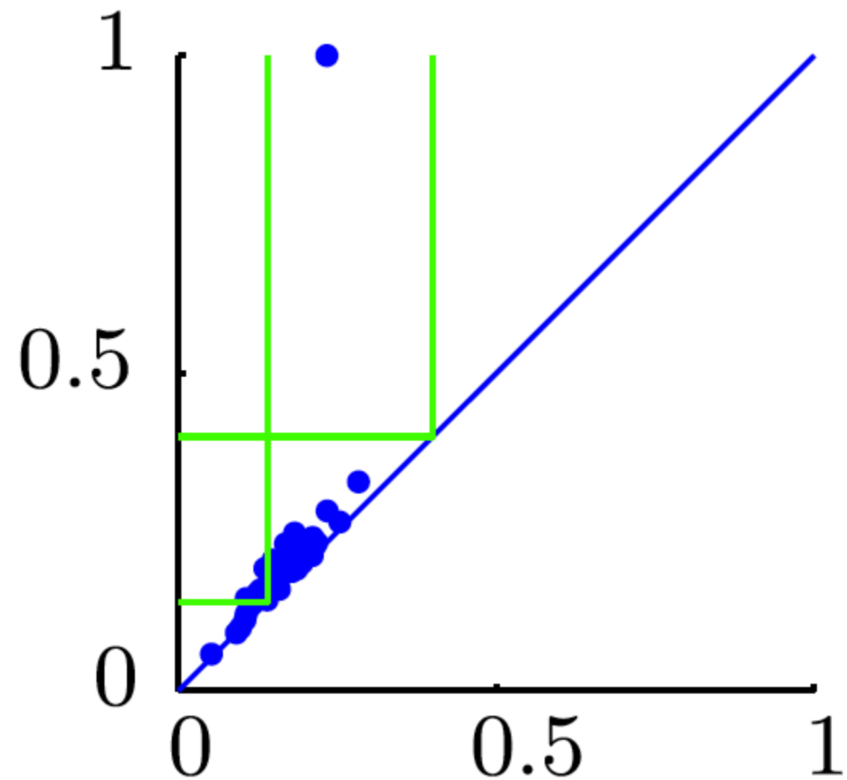

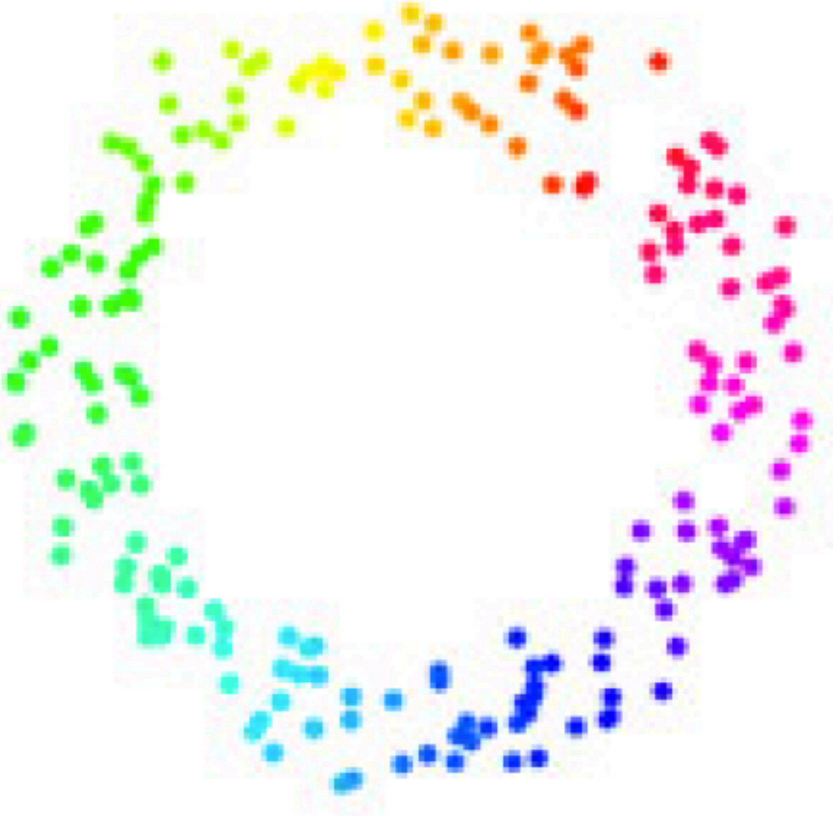

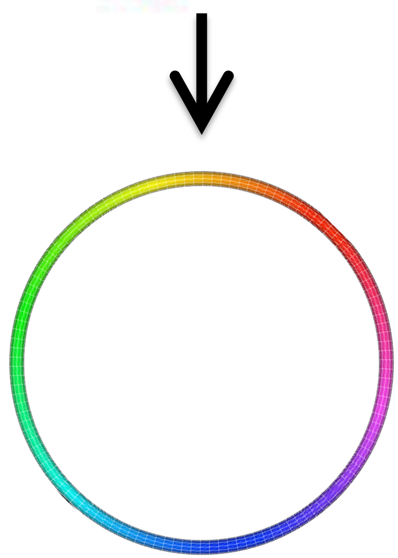

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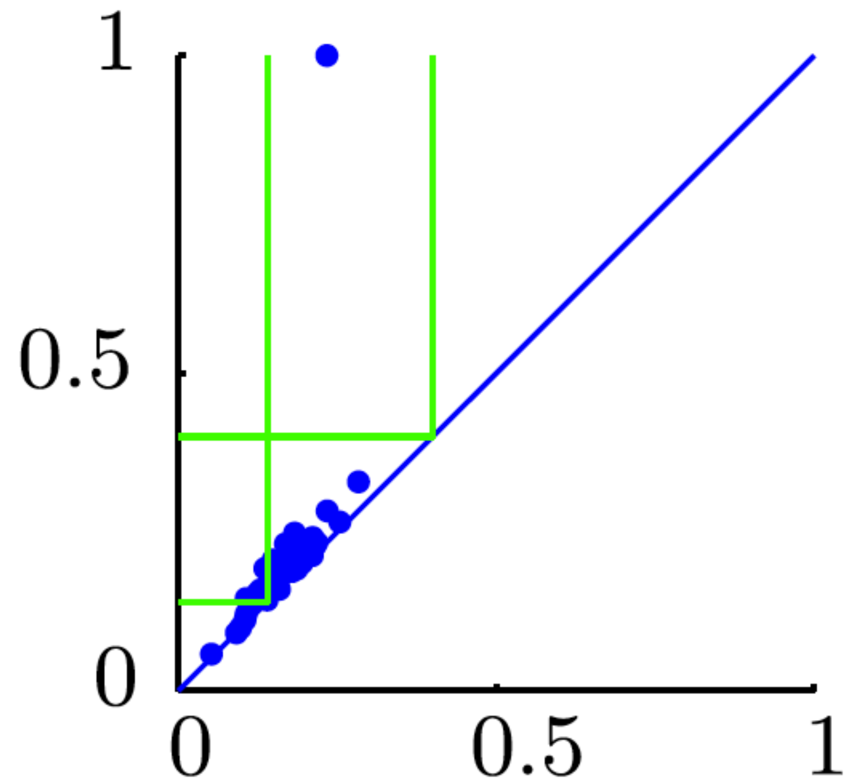

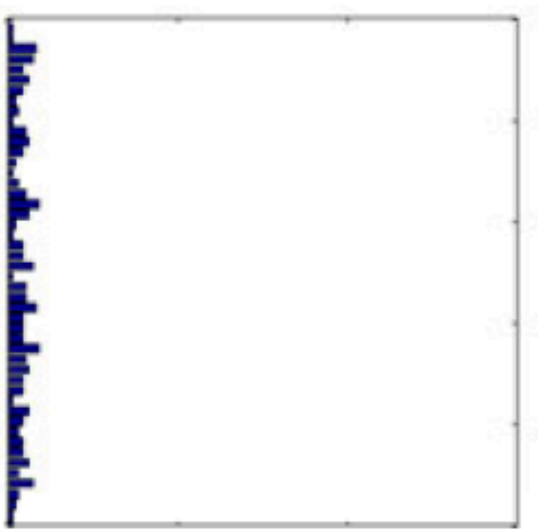

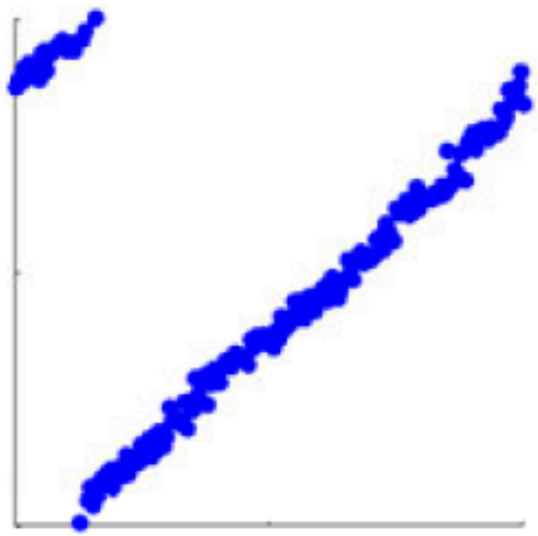

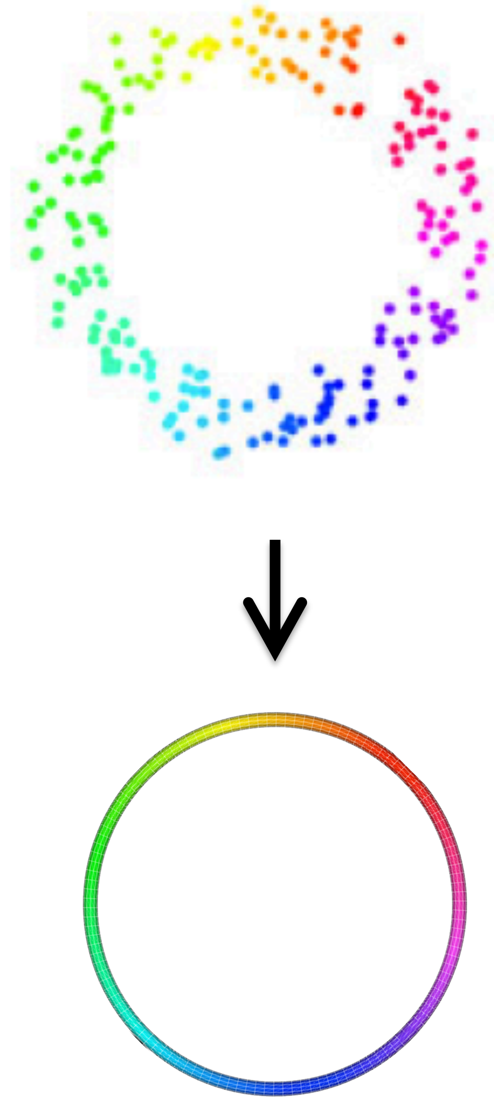

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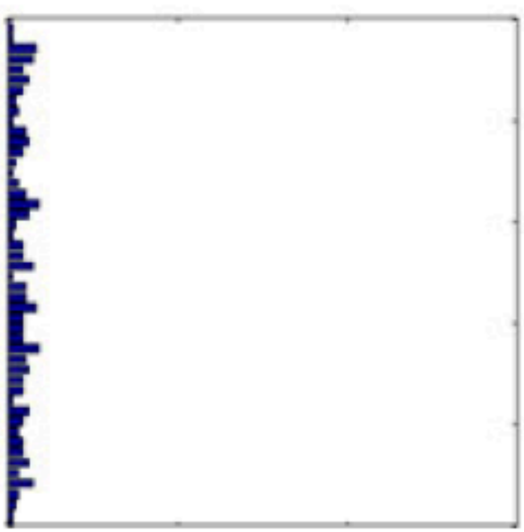

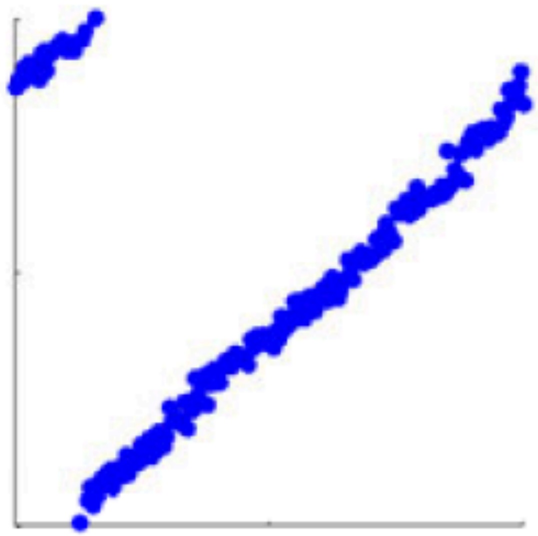

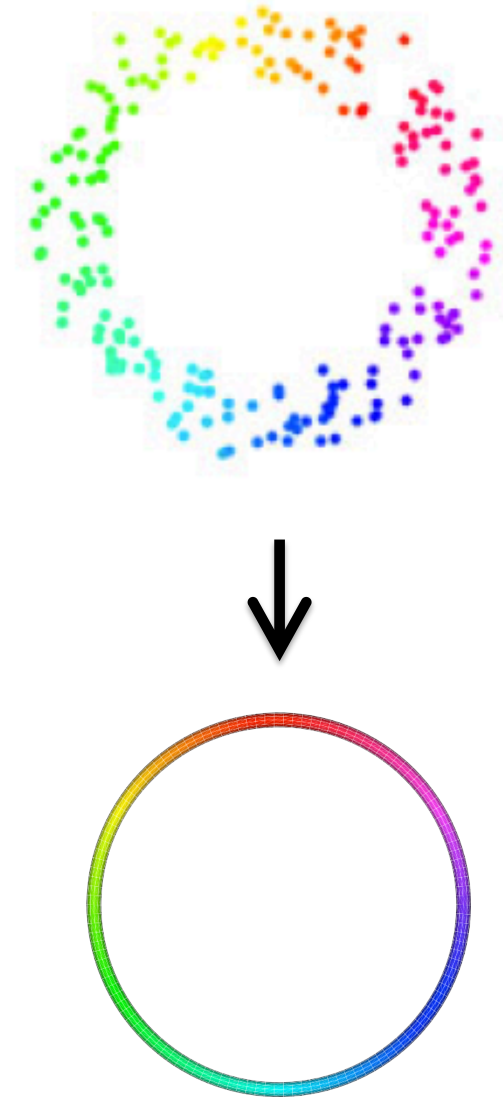

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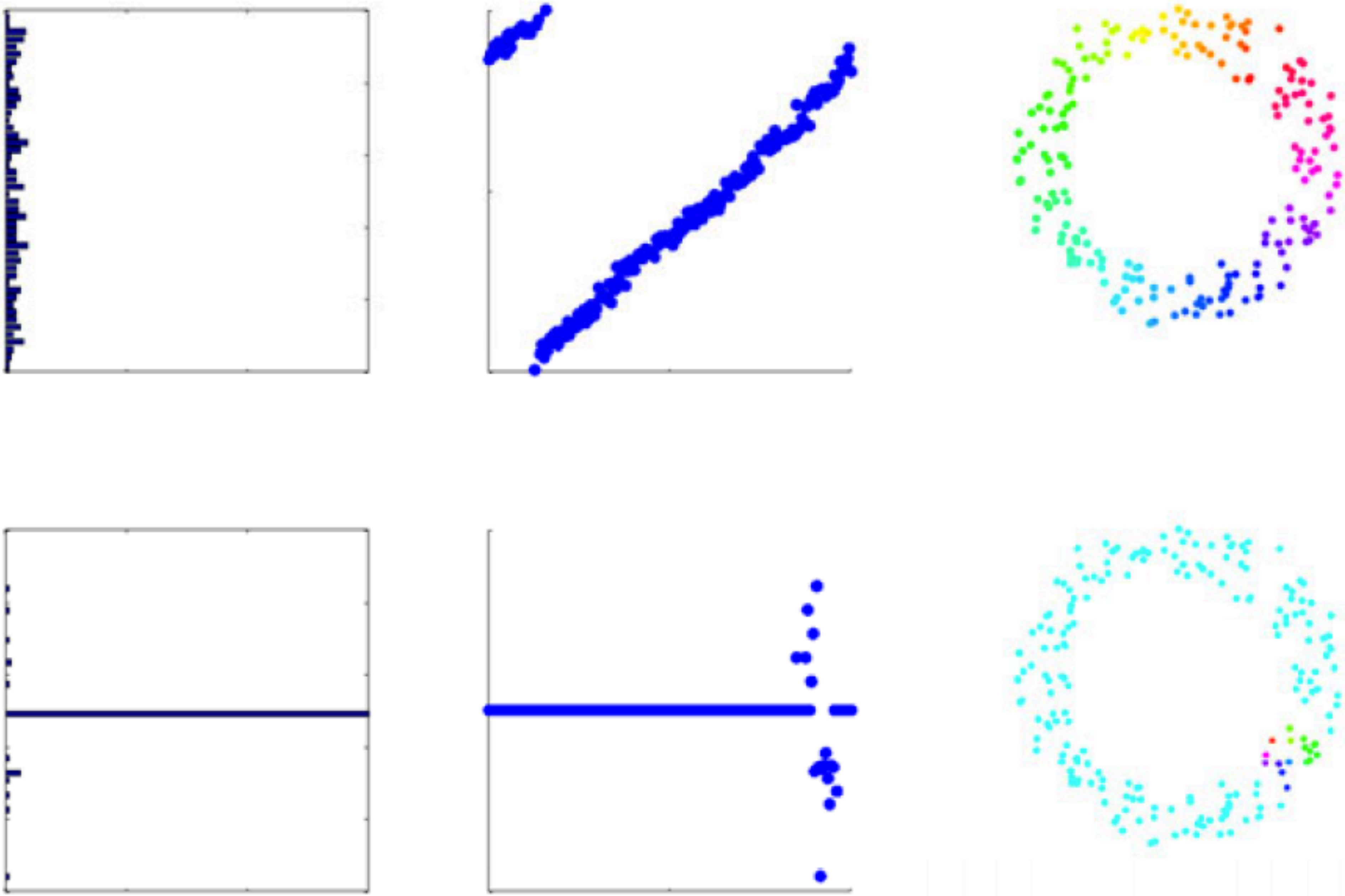

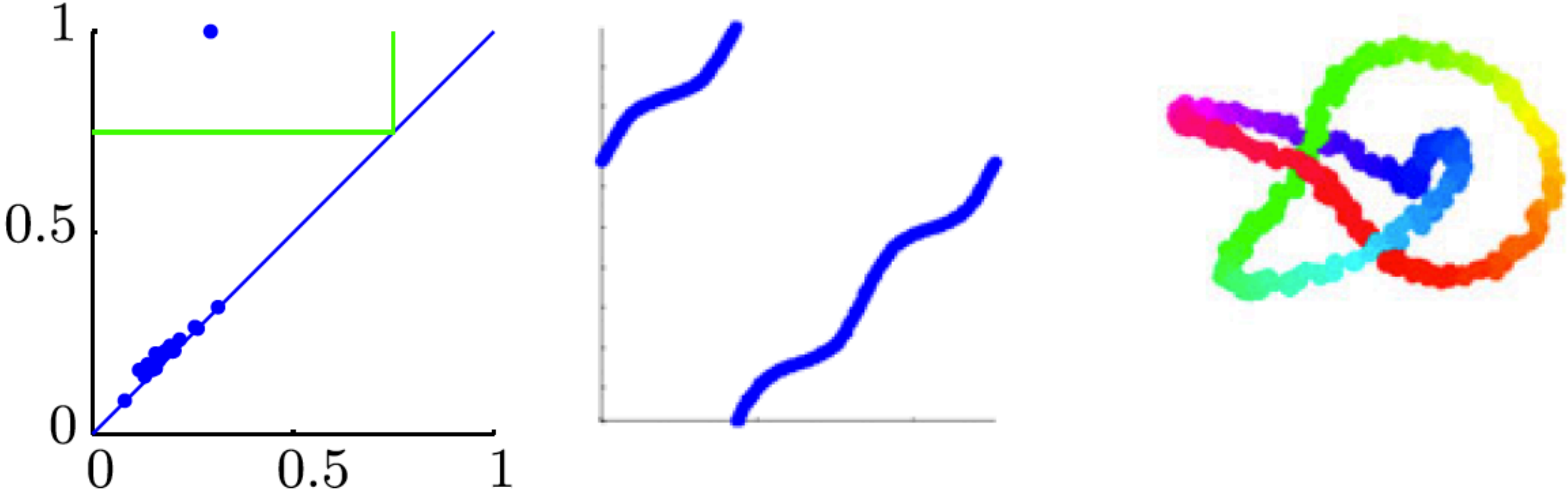
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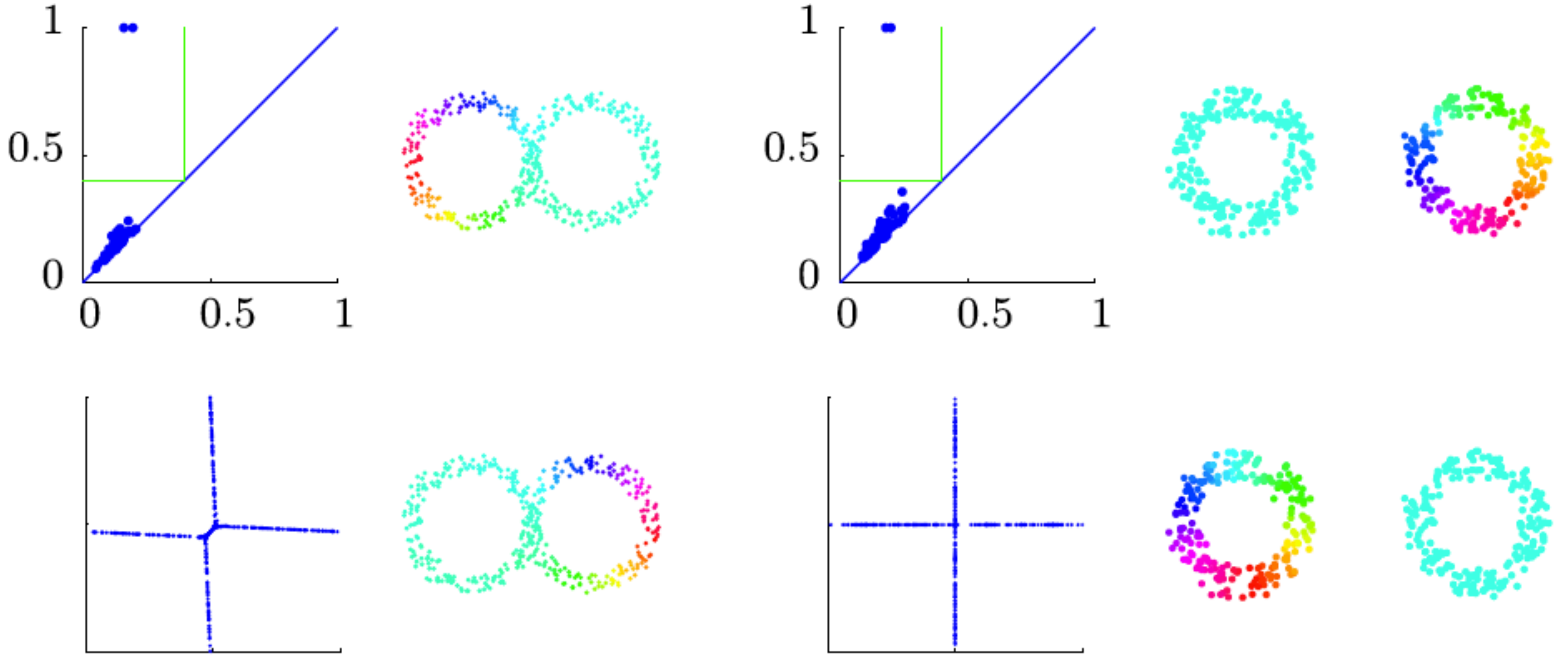
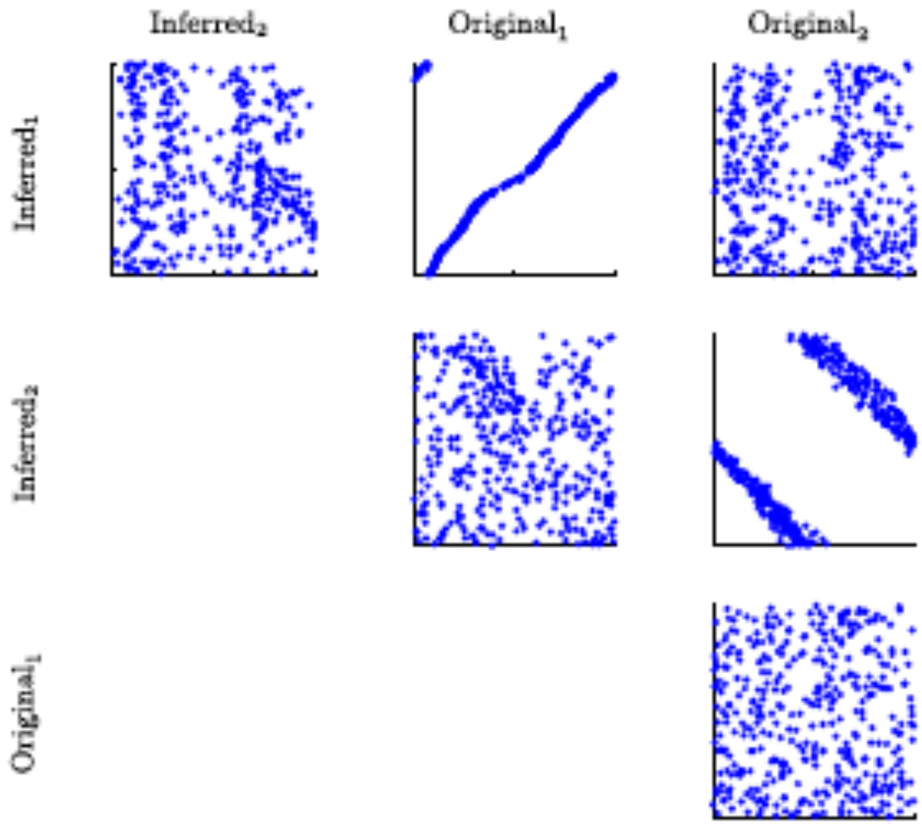
0.5
0
0
0.5
1

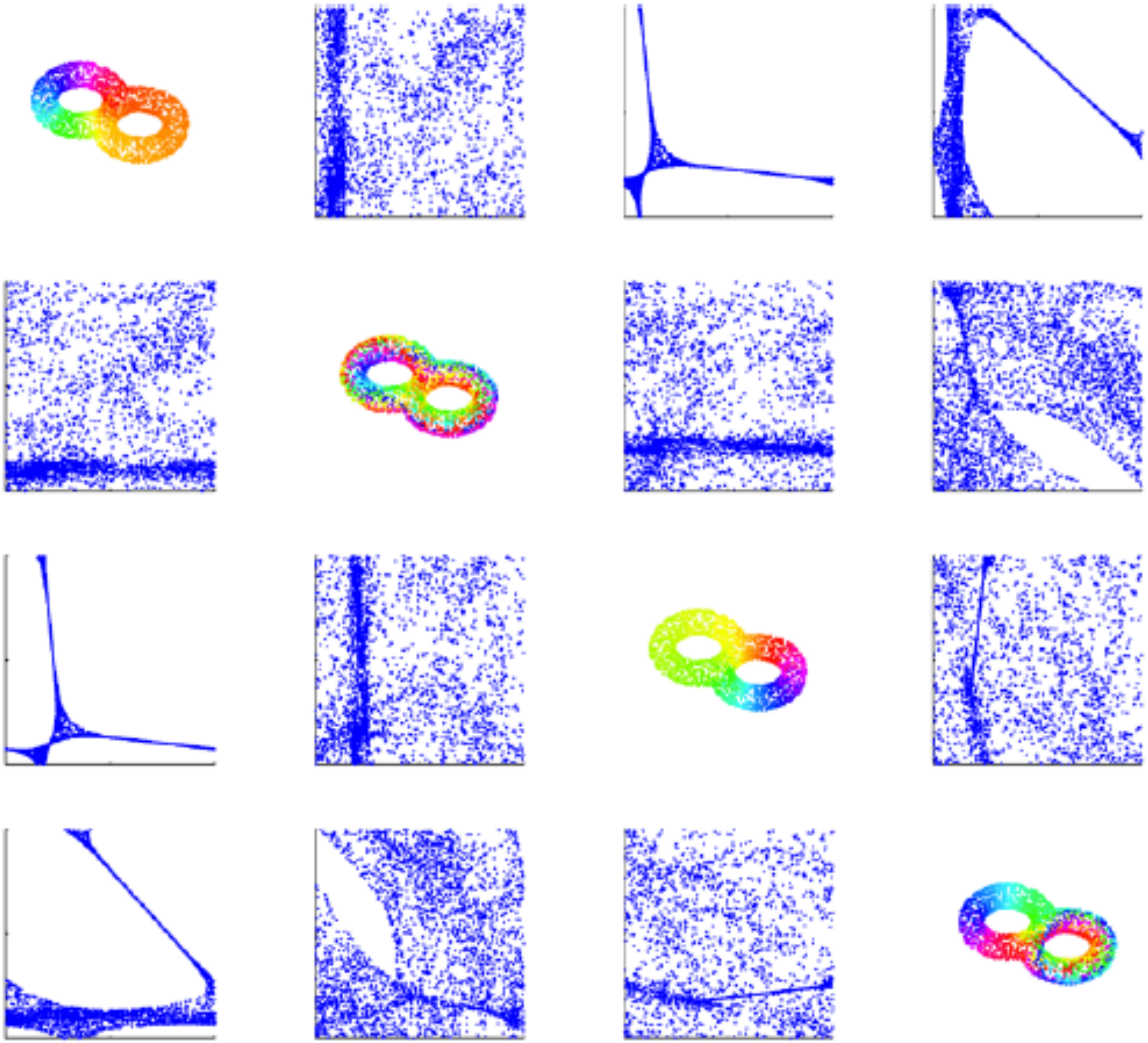


1
0.5
0
0
0.5
1
1
0.5
0
0
0.5
1

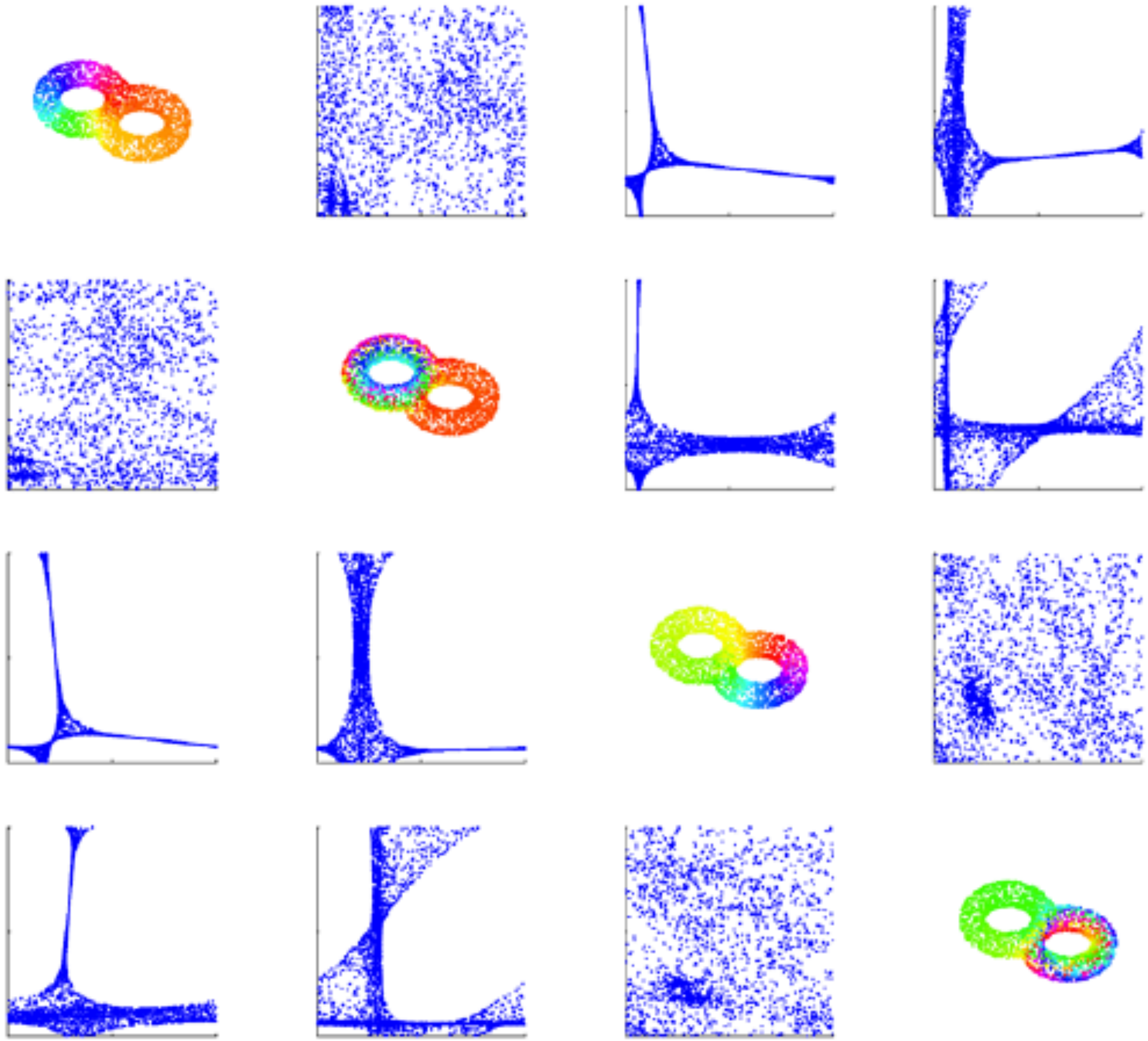

(a) Persistence diagram (left); first inferred coordinate (middle); second inferred coordinate (right).




(b) Correlation scatter plots between the two original and two inferred coordinates.



(a) The four discovered coordinates $\theta_1, \theta_2, \theta_3, \theta_4$ and their matrix of correlation scatter plots.

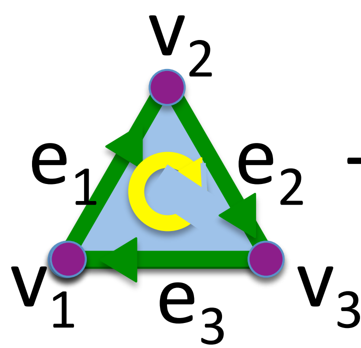


(b) Taking linear combinations for a geometrically more 'natural' basis of circular coordinates: $\phi_1 = \theta_1$, $\phi_2 = \theta_2+\theta_3+\theta_4$, $\phi_3 = \theta_3$, and $\phi_4 = \theta_1 + \theta_4$. The pairs $\phi_1, \phi_2$ and $\phi_3, \phi_4$ respectively parametrize the left and right halves of the double torus.

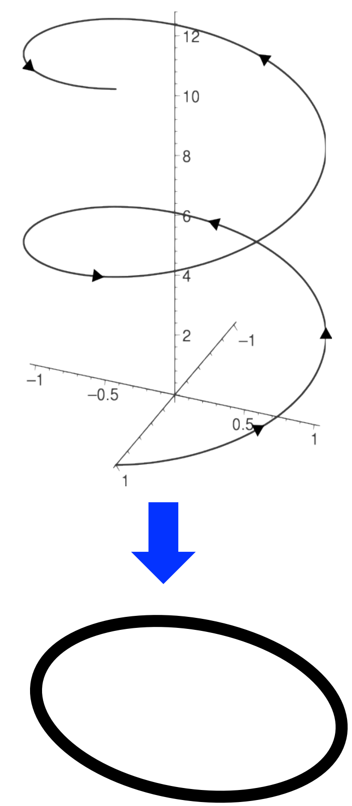
Given: $\overline{\alpha} \in C^1(X;\mathbf{R})$ be a cocycle, $\alpha \in C^1(X;\mathbf{Z})$

$f \in C^0(X;\mathbf{R})$, $\overline{\alpha} = \alpha + d_0 f$

Create: $\theta : X \to \mathbf{R/Z} = S^1$ s.t. $v_i v_k$ goes to interval of length $\overline{\alpha}(v_i v_k)$




$\to$ **R/Z**



Let $\theta(v_i) = f(v_i) \bmod \mathbf{Z}$

$$\theta(v_i) - \theta(v_k) = f(v_i) - f(v_k)$$

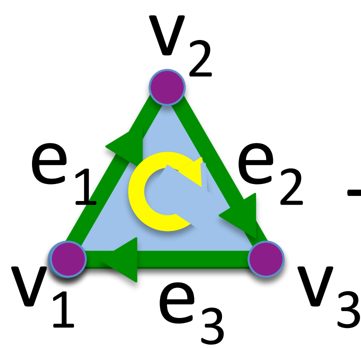
$$= d_0 f(v_i v_k) = \overline{\alpha}(v_i v_k) - \alpha(v_i v_k)$$

$$= \overline{\alpha}(v_i v_k) \mod \mathbf{Z}$$

Given: $\alpha \in C^1(X;\mathbf{Z})$ be a cocycle, $f \in C^0(X;\mathbf{R})$

Let $\overline{\alpha} = \alpha + d_0 f$, a cocycle of $C^1(X;\mathbf{R})$

Create: $\theta : X \to \mathbf{R}/\mathbf{Z} = S^1$ s.t. $v_i v_k$ goes to interval of length $\overline{\alpha}(v_i v_k)$

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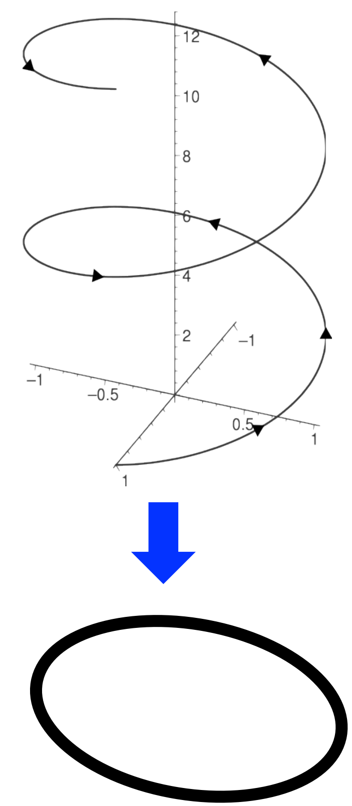


Let $\theta(v_i) = f(v_i) \bmod \mathbf{Z}$

$$\begin{aligned}\theta(v_i) - \theta(v_k) &= f(v_i) - f(v_k)\\ &= d_0 f(v_i v_k) = \overline{\alpha}(v_i v_k) - \alpha(v_i v_k)\\ &= \overline{\alpha}(v_i v_k) \mod \mathbf{Z}\end{aligned}$$

Let $\alpha \in C^1(X;\mathbf{R})$. There is a unique solution $\overline{\alpha}$ to the least-squares minimization problem:

$$\min_{\overline{\alpha}} \{ ||\overline{\alpha}||^2 \mid \text{there exists } f \in C^0(X;\mathbf{R}) \text{ such that } \overline{\alpha} = \alpha + d_0 f \}$$

The (simplified) algorithm:

1a.) Calculate $H^1(X(\varepsilon), \mathbf{Z_p})$, for some prime p.

1b.) Determine persistent cohomology.

1c.) Lift persistent 1-cocycles in $H^1(X(\varepsilon), \mathbf{Z_p})$ to 1-cocycles in $H^1(X(\varepsilon), \mathbf{Z})$.

2.) Simplify the cocyle via least squares minimization.

3.) Use cocyle to create map $\theta : X \to \mathbf{R}/\mathbf{Z} = S^1$ s.t. $v_iv_k$ goes to interval of length $\overline{\alpha}(v_iv_k)$.

# Welcome to Dionysus' documentation!

Dionysus is a C++ library for computing persistent homology. It provides implementations of the following algorithms:

- Persistent homology computation [ELZ02] [ZC05]
- Vineyards [CEM06] (C++ only)
- Persistent cohomology computation (described in [dSVJ09])
- Zigzag persistent homology [CdSM09]
- *Examples* provide useful functionality in and of themselves:
    - *Alpha shape construction* in 2D and 3D
    - *Rips complex construction*
    - Cech complex construction (C++ only)
    - *Circle-valued parametrization*
    - *Piecewise-linear vineyards*

**Contents**


- Get, Build, Install
- Brief Tutorial
- Examples
- Python bindings: module `dionysus`
- Bibliography

# Parametrizing a point set using circle valued functions

The procedure described below is explained in detail in [dSVJ09].

One can use `examples/cohomology/rips-pairwise-cohomology.cpp` to compute persistent pairing of the Rips filtration using the persistent cohomology algorithm. It takes as input a file containing a point set in Euclidean space (one per line) as well as the following command-line flags:

**`-p , --prime`**
: The prime to use in the computation (defaults to 11).