MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Oct 21, 2013: Cohomology

Fall 2013 course offered through the University of Iowa Division of Continuing Education

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http://www.math.uiowa.edu/~idarcy/AppliedTopology.html triangle-zigzag.py

```
from dionysus import Simplex, ZigzagPersistence, \
                    vertex_cmp, data_cmp \
#                    ,enable_log

complex = {Simplex((0,),      0):    None,                # A
          Simplex((1,),      1):    None,                 # B
          Simplex((2,),      2):    None,                 # C
          Simplex((0,1),      2.5):  None,                 # AB
          Simplex((1,2),      2.9):  None,                 # BC
          Simplex((0,2),      3.5):  None,                 # CA
          Simplex((0,1,2),    5):    None}                 # ABC
```

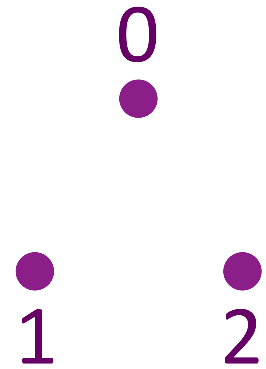
```
print "Complex:"
for s in sorted(complex.keys()): print s
print

triangle idarcy$ python2.7 triangle-zigzag.py
Complex:
<0>
<1>
<2>
<0, 1> 2.500000
<1, 2> 2.900000
<0, 2> 3.500000
<0, 1, 2>
```

```
#enable_log("topology/persistence")
zz = ZigzagPersistence()

# Add all the simplices
b = 1
for s in sorted(complex.keys(), data_cmp):
    print "%d: Adding %s" % (b, s)
```

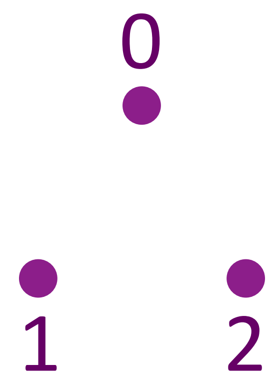
```
1: Adding <0>
2: Adding <1>
3: Adding <2>
```



```
# Add all the simplices
b = 1
for s in sorted(complex.keys(), data_cmp):
    print "%d: Adding %s" % (b, s)
    i,d = zz.add([complex[ss] for ss in s.boundary], b)
    complex[s] = i
    if d:   print "Interval (%d, %d)" % (d, b-1)
    b += 1
```

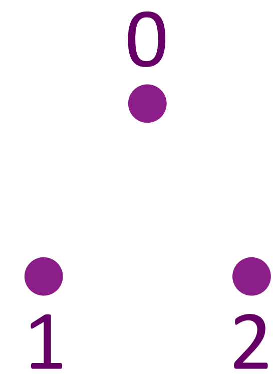
```
# Add all the simplices
b = 1
for s in sorted(complex.keys(), data_cmp):
    print "%d: Adding %s" % (b, s)
    i,d = zz.add([complex[ss] for ss in s.boundary], b)
    complex[s] = i
    if d:   print "Interval (%d, %d)" % (d, b-1)
    b += 1
```

```
4: Adding <0, 1> 2.500000
Interval (2, 3)
5: Adding <1, 2> 2.900000
Interval (3, 4)
6: Adding <0, 2> 3.500000
7: Adding <0, 1, 2>
Interval (6, 6)
```



```
# Remove all the simplices
for s in sorted(complex.keys(), data_cmp, reverse = True):
    print "%d: Removing %s" % (b, s)
    d = zz.remove(complex[s], b)
    del complex[s]
    if d:   print "Interval (%d, %d)" % (d, b-1)
    b += 1
```

```
8: Removing <0, 1, 2>
9: Removing <0, 2> 3.500000
Interval (8, 8)
10: Removing <1, 2> 2.900000
11: Removing <0, 1> 2.500000
```



12: Removing <2>

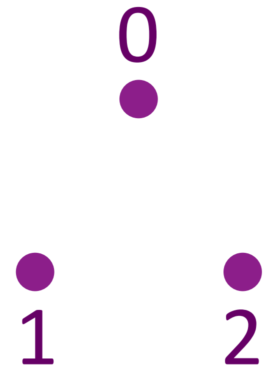
Interval (10, 11)

13: Removing <1>

Interval (11, 12)

14: Removing <0>

Interval (1, 13)



Discrete Comput Geom (2011) 45: 737–759
DOI 10.1007/s00454-011-9344-x

Persistent Cohomology and Circular Coordinates

Vin de Silva · Dmitriy Morozov ·
Mikael Vejdemo-Johansson

http://link.springer.com/article/10.1007%2Fs00454-011-9344-x

NKI (2002) breast cancer data:

gene expression levels of 24,000 from 272 tumors

➔ Data = 272 points living in $\mathbf{R}^{24000}$

NKI (2002) breast cancer data:

gene expression levels of 24,000 from 272 tumors

➔ Data = 272 points living in $\mathbf{R}^{24000}$

In reality one cleans the data first, removing unreliable data, etc.

One may also remove dimensions (genes) that appear irrelevant.

NKI (2002) breast cancer data:
gene expression levels of 24,000 from 272 tumors

➔ Data = 272 points living in $\mathbf{R}^{24000}$

Since our dataset consists of only 272 points, it can't be 24000-dimensional.

For example, suppose our very fake data is

$\{ (9, 8, 7, 6, 5, 4, 3, 2, 3)\, t_i \mid i = 1, 2, 3, \ldots 1000 \} =$

$\{ (9, 8, 7, 6, 5, 4, 3, 2, 3), \{ (18, 16, 14, 12, 10, 8, 6, 4, 6), \ldots \}$

Dimensionality Reduction:

Given dataset $D \subset \mathbf{R}^N$

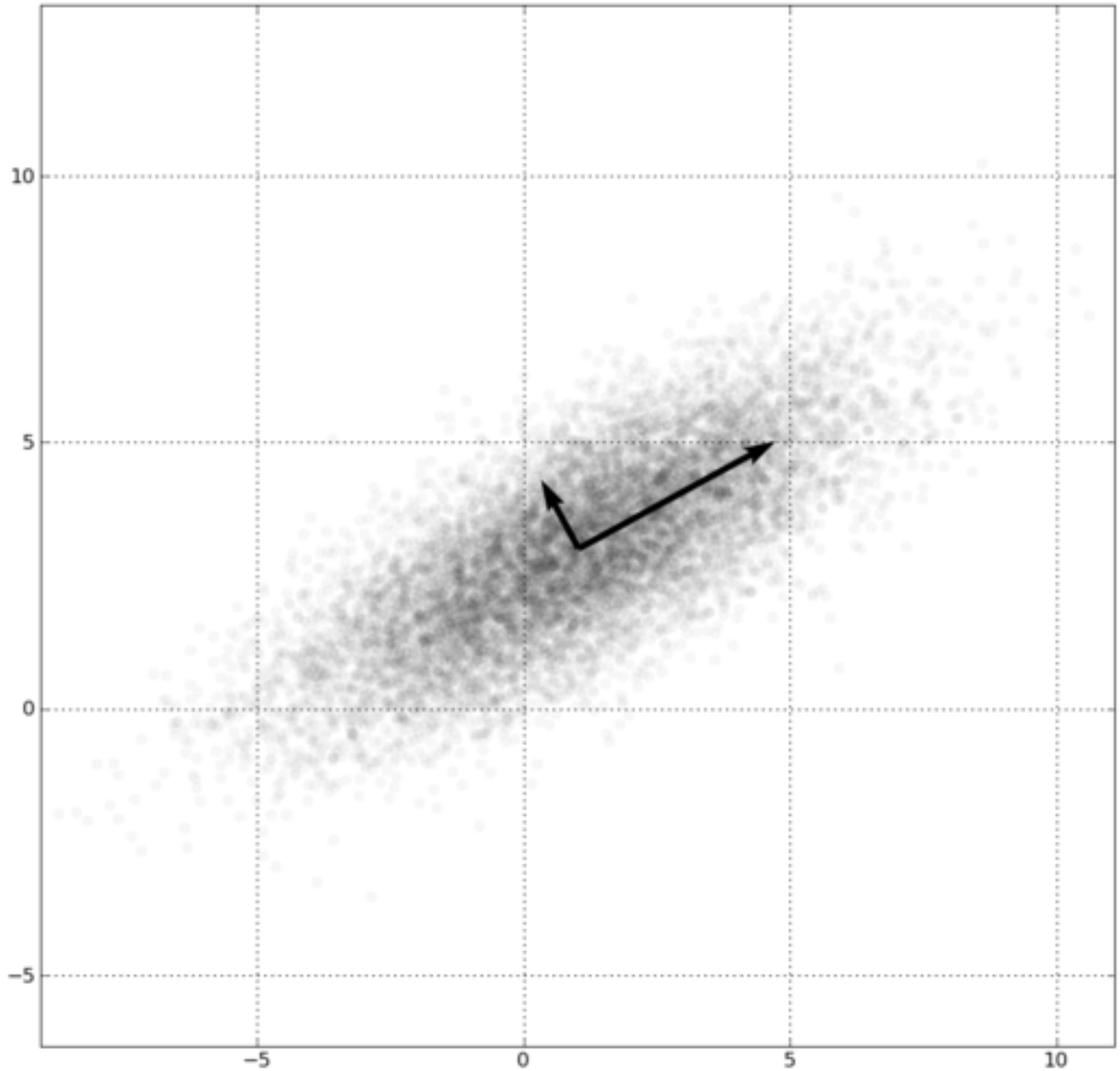
Want: embedding $f: D \rightarrow \mathbf{R}^n$ where $n << N$
which "preserves" the structure of the data.

Example:

$\{ (9, 8, 7, 6, 5, 4, 3, 2, 3)\, t_i \mid i = 1, 2, 3, \ldots 1000 \} \subset \mathbf{R}^9$

$f: D \rightarrow R$, $\quad f((9, 8, 7, 6, 5, 4, 3, 2, 3)\, t_i) = \quad t_i$

Example: Principle component analysis (PCA)



http://en.wikipedia.org/wiki/File:GaussianScatterPCA.png

Many reduction methods:

f_1: D → **R**, f_2: D → **R**, … f_n: D → **R**

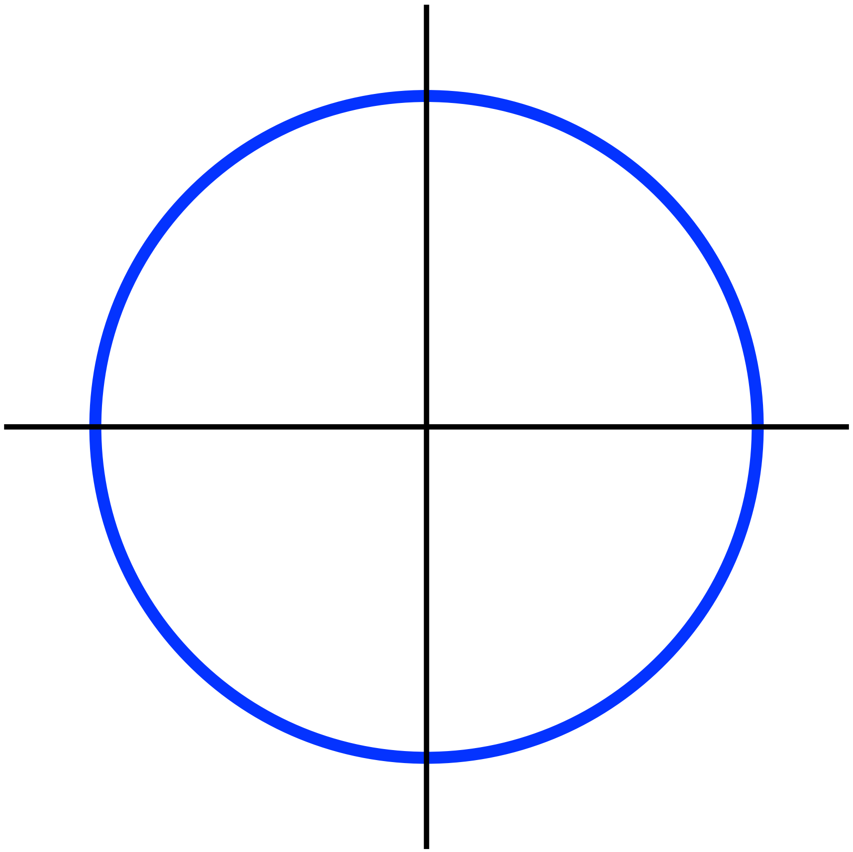
$(f_1, f_2, \ldots f_n)$: D → $\mathbf{R}^n$

Many are linear, M: $\mathbf{R}^N \rightarrow \mathbf{R}^n$, $M\mathbf{x} = \mathbf{y}$

But there are also non-linear dimensionality reduction algorithms.

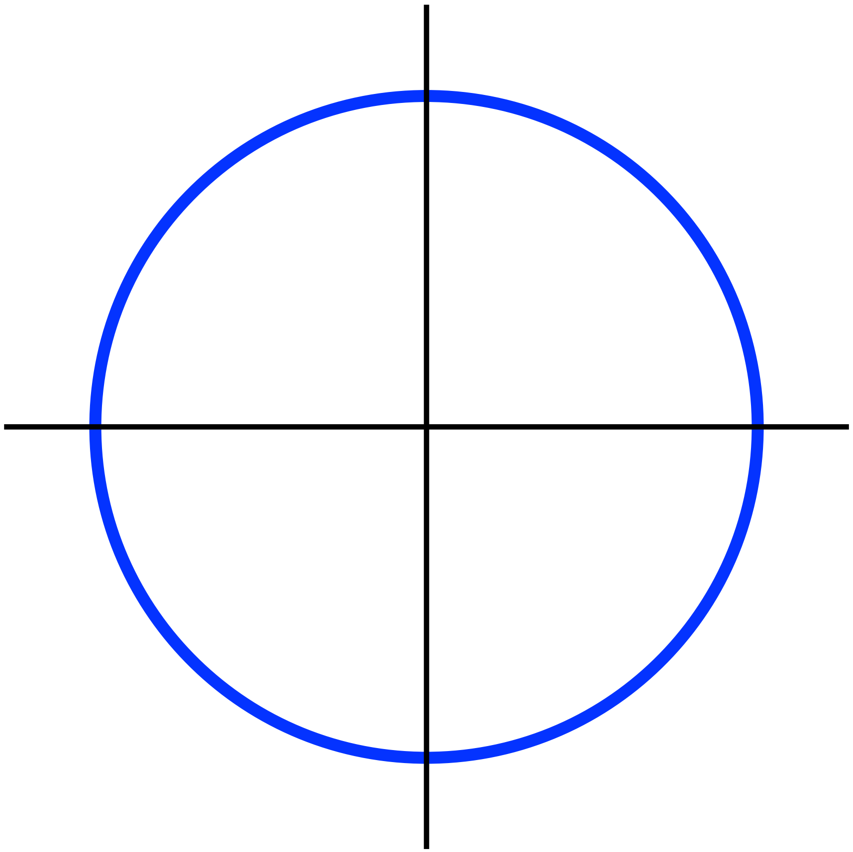
f_1: D → **R**, f_2: D → **R**, … f_n: D → **R**

(f_1, f_2, … f_n): D → $\mathbf{R}^n$

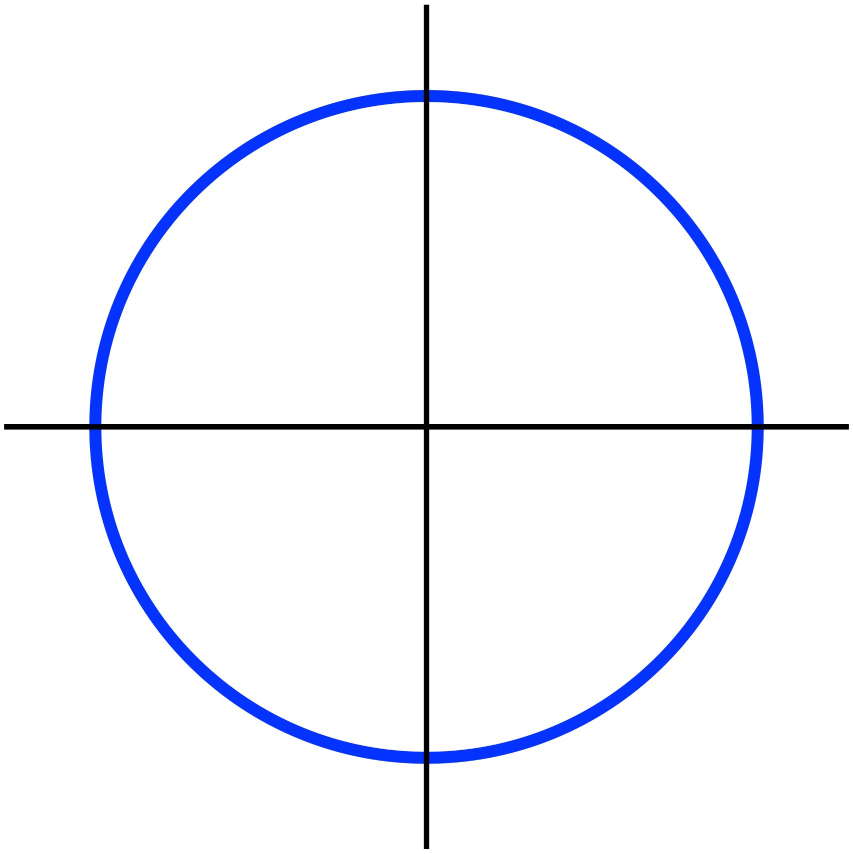


Goal

f_1: D → S^1, f_2: D → S^1, … f_n: D → S^1



Note S^1 is a ring:



Cohomology and circular coordinates:

Let X = finite simplicial complex.

X^0 = set of 0-simplices = vertices.

X^1 = set of 1-simplices = edges.

X^2 = set of 2-simplices = faces.

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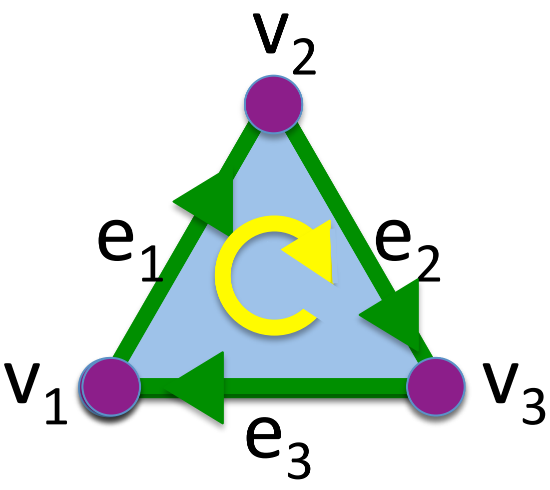

Homology

Let X = finite simplicial complex.

C_0 = set of 0-chains = linear combinations of vertices

C_1 = set of 1-chains = linear combinations of edges.

C_2 = set of 2-chains = linear combinations of faces.

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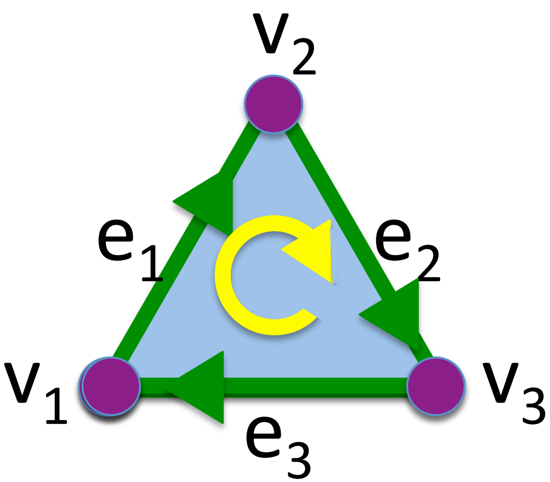

Cohomology and circular coordinates:

Let X = finite simplicial complex, R a ring.

C^0 = set of 0-cochains = $\{ f : C_0 \rightarrow R \mid f \text{ homomorphism}\}$

C^1 = set of 1-cochains = $\{ f : C_1 \rightarrow R \mid f \text{ homomorphism}\}$

C^2 = set of 2-cochains = $\{ f : C_2 \rightarrow R \mid f \text{ homomorphism}\}$

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Cohomology and circular coordinates:

Let X = finite simplicial complex, R a ring.

$C^0(X, R)$ = set of 0-cochains = $\{ f : X^0 \to R \}$

$C^1(X, R)$ = set of 1-cochains = $\{ f : X^1 \to R \}$

$C^2(X, R)$ = set of 2-cochains = $\{ f : X^2 \to R \}$

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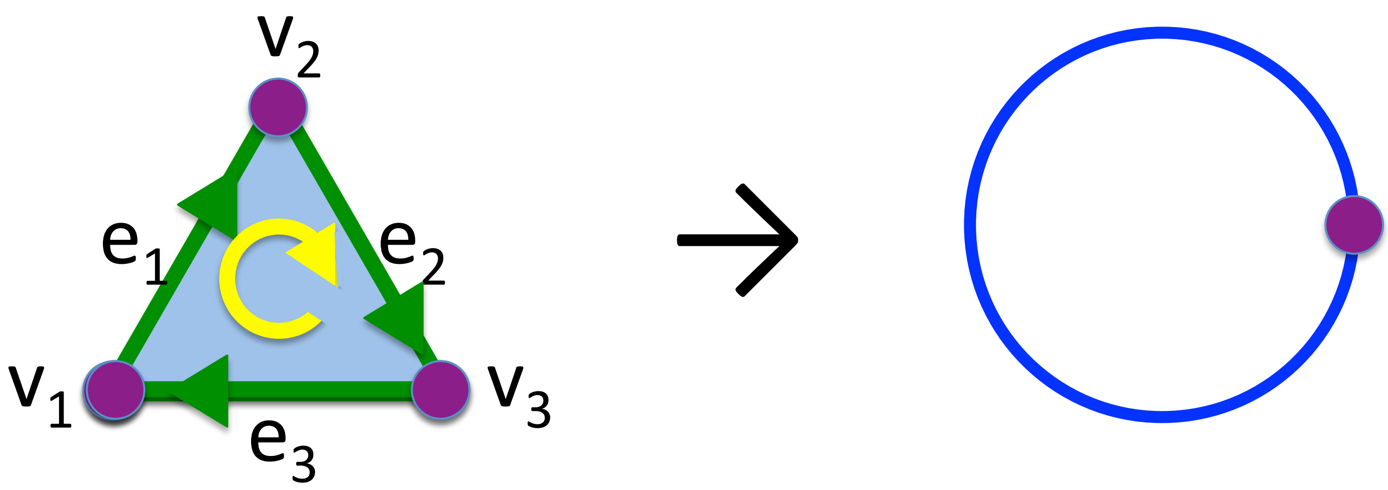
Proposition 1: Let $\alpha \in C^1(X;\mathbf{Z})$ be a cocycle. Then there exists a continuous function

$$\theta : X \rightarrow S^1$$

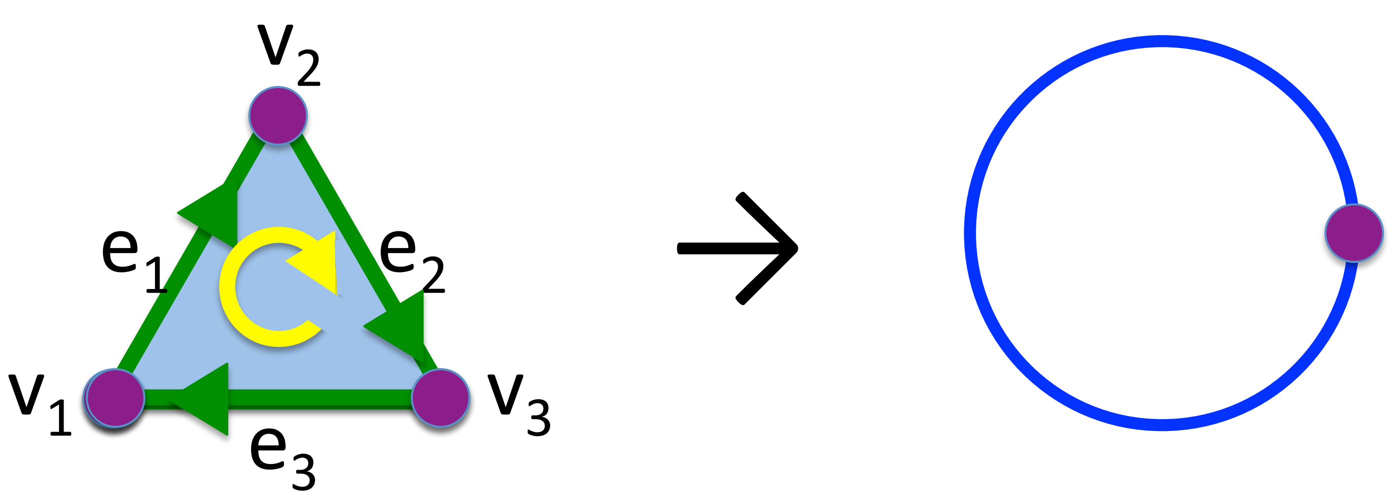
which maps each vertex to 0, and each edge ab around the entire circle with winding number $\alpha(ab)$.



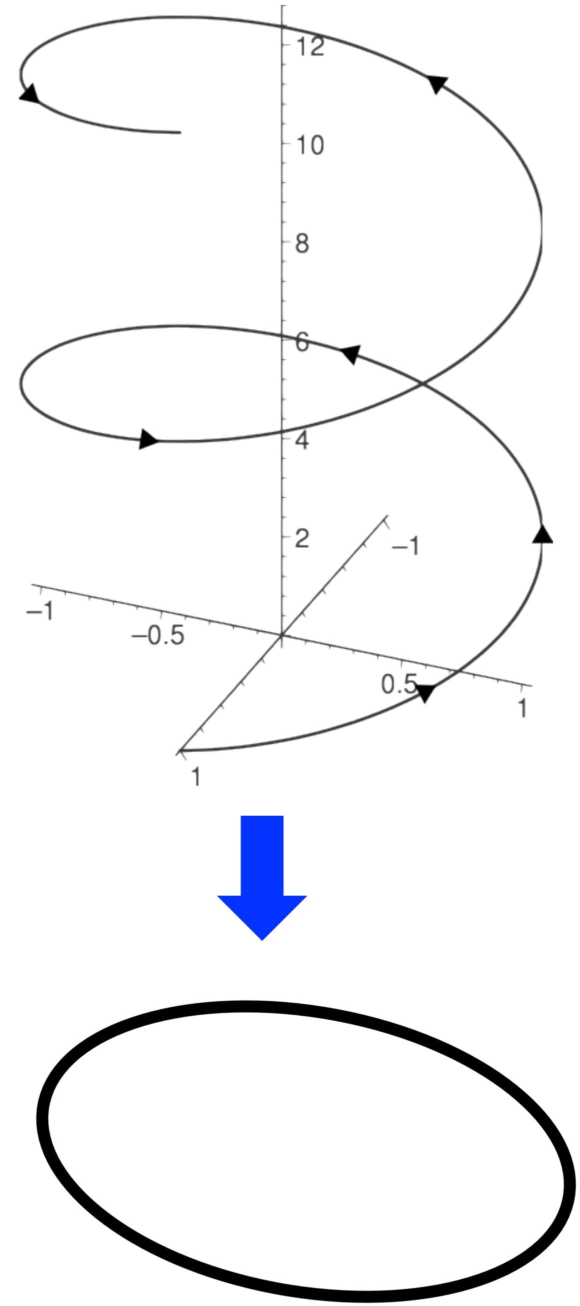
Proposition 2: 2 Let $\alpha \in C^1(X;\mathbf{R})$ be a cocycle. Suppose there exists $\alpha \in C^1(X;\mathbf{Z})$ and $f \in C^0(X;\mathbf{R})$ such that $\alpha = \alpha + d_0 f$. Then there exists a continuous function

$$\theta : X \rightarrow S^1$$

which maps each edge ab linearly to an interval of length $\alpha(ab)$, measured with sign.



$\theta : X \to \mathbf{R}/\mathbf{Z} = S^1$



coboundary maps

$d_0 : C^0 = \{ f : X^0 \to R \} \to C^1 = \{ f : X^1 \to R \}$

$$(d_0 f)(ab) = f(b) - f(a)$$

$d_1 : C^1 = \{ f : X^1 \to R \} \to C^2 = \{ f : X^2 \to R \}$

$$(d_1 \alpha)(abc) = \alpha(bc) - \alpha(ac) + \alpha(ab).$$

Let $\alpha \in C^i$,

$d_i : C^i = \{ f : X^i \rightarrow R \} \rightarrow C^{i+1} = \{ f : X^{i+1} \rightarrow R \}$

α is a *cocycle* if $d_i\alpha = 0$.

α is a *coboundary* if there exists f in C^{i-1} s.t. $d_{i-1}f = \alpha$

Note $d_1 d_0 f = 0$ for any $f \in C^0$.

$(d_0 f)(ab) = f(b) - f(a)$

$(d_1 d_0 f)(abc) = d_0 f(bc) - d_0 f(ac) + d_0 f(ab)$

$= d_0 f(c) - d_0 f(b) - d_0 f(c) + d_0 f(a) + d_0 f(b) - d_0 f(a)$

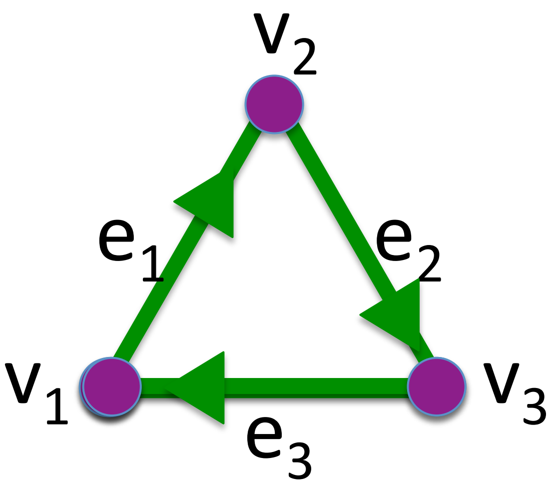
$d_1 \, d_0 \, f = 0$ implies $\mathrm{Im}\,(d_0) \subseteq \mathrm{Ker}\,(d_1)$.

$H^1(X;A) = \mathrm{Ker}\,(d_1) / \mathrm{Im}\,(d_0)$.

Two cocycles α, β are *cohomologous* if $\alpha - \beta$ is a coboundary.

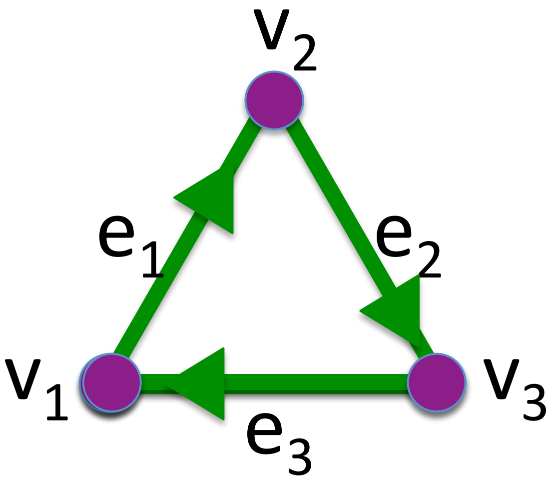
$d_1 \, d_0 \, f = 0$ implies $\mathrm{Im}(d_0) \subseteq \mathrm{Ker}(d_1)$.

$H^1(X;A) = \mathrm{Ker}(d_1)/\mathrm{Im}(d_0)$.



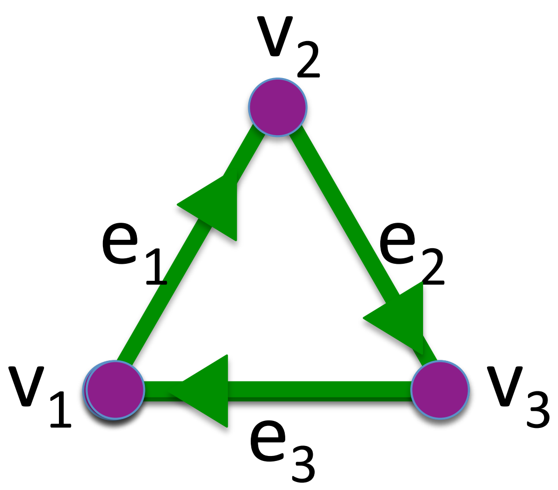
$d_1 d_0 f = 0$ implies $\mathrm{Im}(d_0) \subseteq \mathrm{Ker}(d_1)$.

$H^1(X;A) = \mathrm{Ker}(d_1)/\mathrm{Im}(d_0)$.



$0 \rightarrow C^0 \rightarrow C^1 \rightarrow 0$

$C^0(X, R)$ = set of 0-cochains = $\{ f : X^0 \rightarrow R \}$

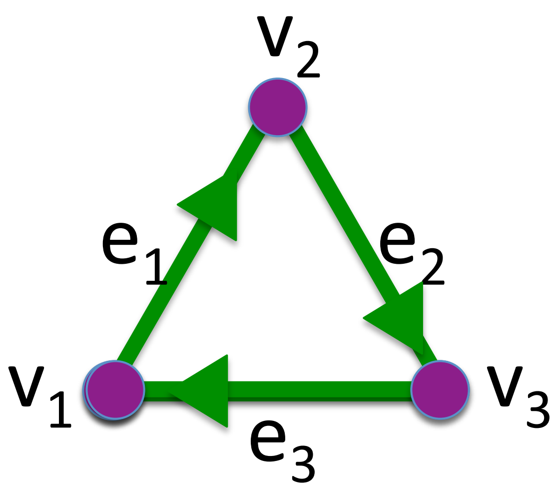


$C^0(X, R)$ = set of 0-cochains = $\{ f : X^0 \rightarrow R \}$

= $\{ f \mid f(v_1) = r_1, f(v_2) = r_2, f(v_3) = r_3; (r_1, r_2, r_3) \text{ in } R^3 \}$

$C^1(X, R)$ = set of 1-cochains = $\{ f : X^1 \rightarrow R \}$

= $\{ f \mid f(e_1) = r_1, f(e_2) = r_2, f(e_3) = r_3; (r_1, r_2, r_3) \text{ in } R^3 \}$

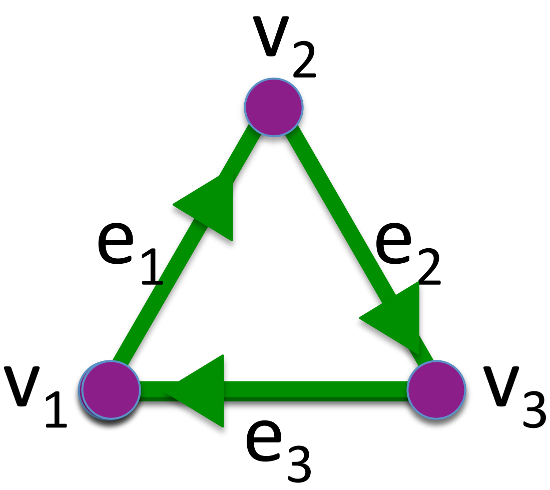


$C^0(X, R)$ = set of 0-cochains = $\{ f : X^0 \to R \}$

= $\{ f \mid f(v_1) = r_1, f(v_2) = r_2, f(v_3) = r_3;\ (r_1, r_2, r_3) \text{ in } R^3 \}$

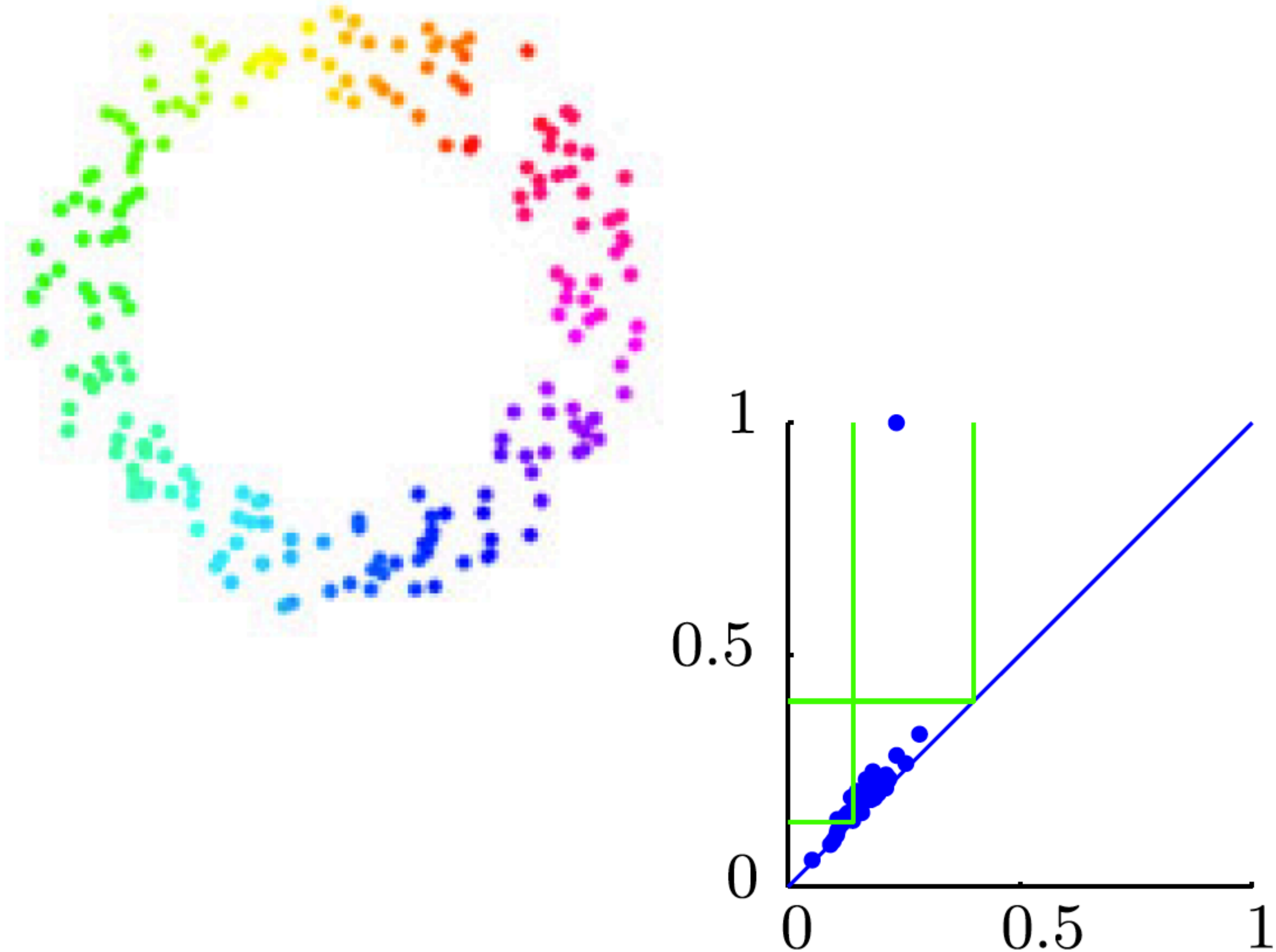
$C^1(X, R)$ = set of 1-cochains = $\{ f : X^1 \to R \}$

= $\{ f \mid f(e_1) = r_1, f(e_2) = r_2, f(e_3) = r_3;\ (r_1, r_2, r_3) \text{ in } R^3 \}$



$H^i(X;A) = \mathrm{Ker}(d_i) / \mathrm{Im}(d_{i-1})$.

$0 \to C^0 \to C^1 \to 0$



1
0.5
0
0
0.5
1