Persistent Holes in the Universe

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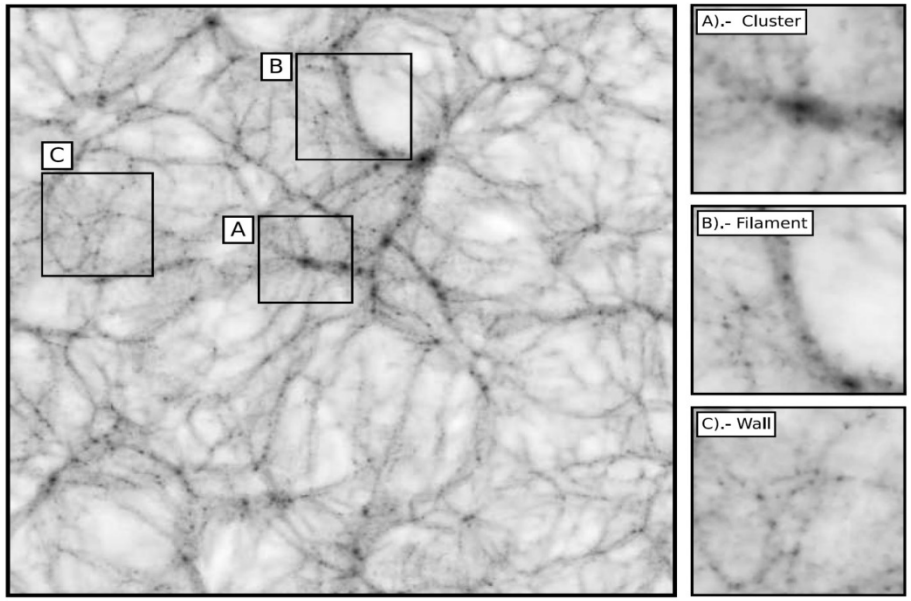
Outline

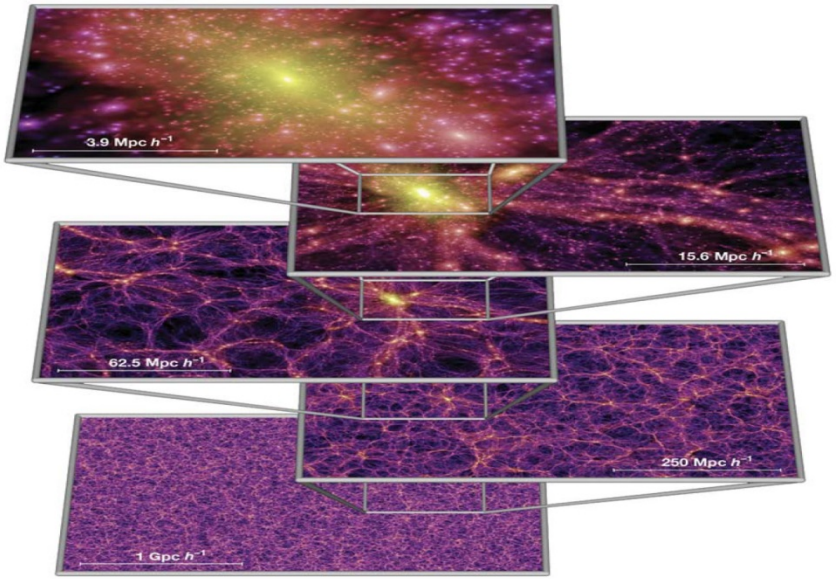
- Cosmic web & CMB : description and challenges
- Why (multi-scale) topology?
- Persistence : multi-scale topology
- Gaussian random fields & LSS: results
- Morse geometry and filament detection
- Conclusion


Cosmic Web
&
Cosmic Microwave Background

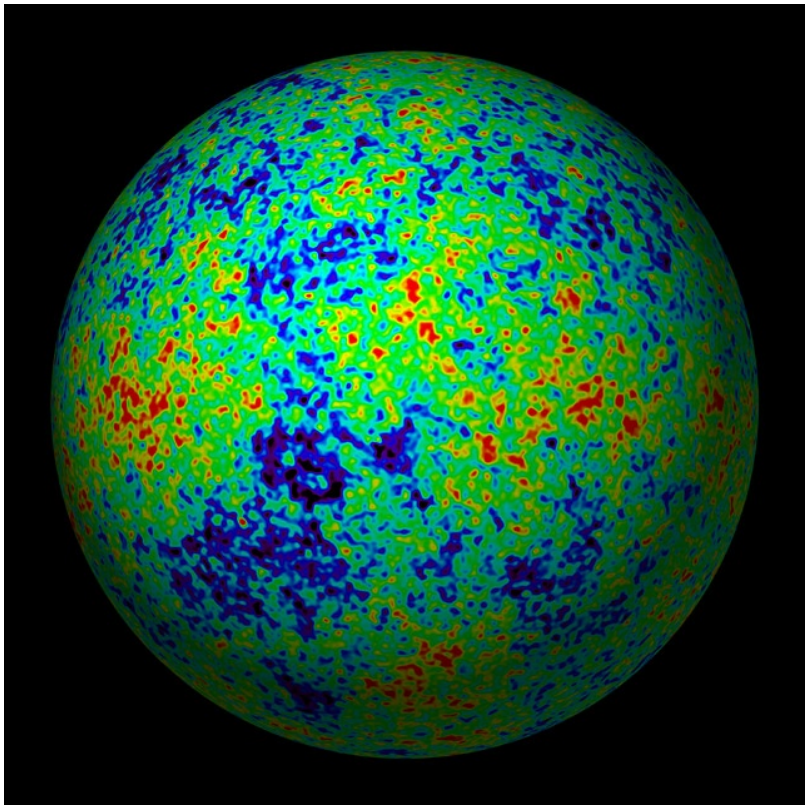
Cosmic Web : complications

- Discretely sampled : galaxies (observation) & particles (N-body simulatons)
- Complex structural connectivity
- Lack of structural symmetry
- Intrinsic multi-scale nature
- Wide range of densities





Cosmic Microwave Bacground : Edge of the visible universe

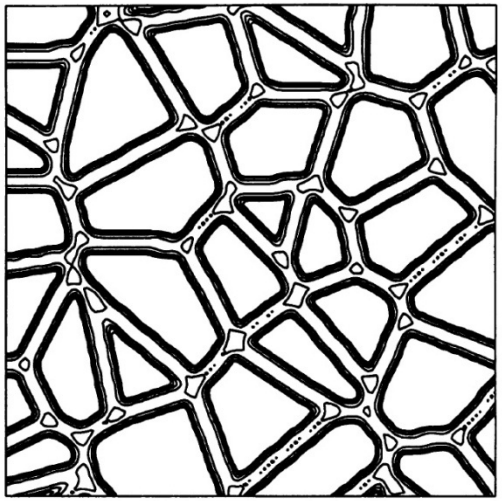


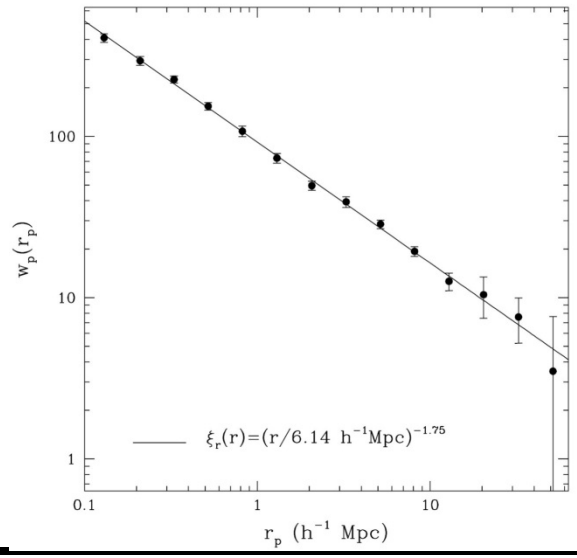
- Progenitor of LSS: quantum fluctuations in infant Universe (CMB)
- Earliest View of our Cosmos: the Universe 379,000 years after the Big Bang
- Described as near-perfect (?) gaussian random field

$$P_N = \frac{\exp\left[-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} f_i \left(M^{-1}\right)_{ij} f_j\right]}{\left[(2\pi)^N (\det M)\right]^{1/2}} \prod_{k=1}^{N} df_k$$

Correlation functions: Structural Insensitivity

Voronoi foam, R=1.6, smoothed original





$$\xi(r) = \left(\frac{r}{r_0}\right)^{-\gamma}$$

$$\gamma \approx 1.8$$

$$r_0 \approx 5h^{-1}Mpc$$

Voronoi foam, R=1.6, random phases

2-pt correlation function is highly insensitive to the geometry & morphology of weblike patterns:

compare 2 distributions with same ξ(r), cq. P(k), but totally different phase distribution

Topology, Manifolds & topological holes

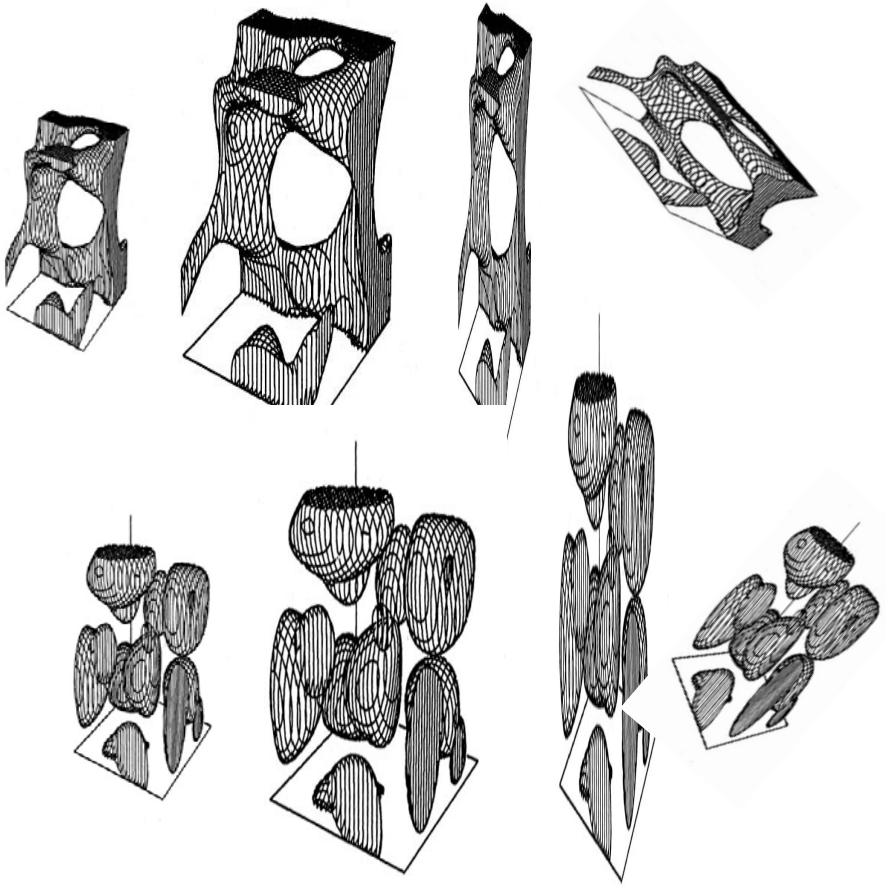
Why Topology?

- **Cosmic web : a complex pattern of connected structures**

- **Topology studies connectivity**

- **Intrinsic topology insensitive to trivial change in the shape of structure (stretching, compression) or trivial coordinate transformation (expansion/contraction, distortion, rotation)**

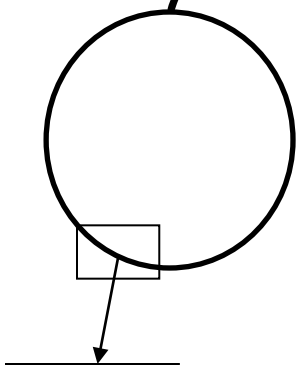
- **In the cosmic context : topology insensitive to gravitational amplification and redshift distortion**

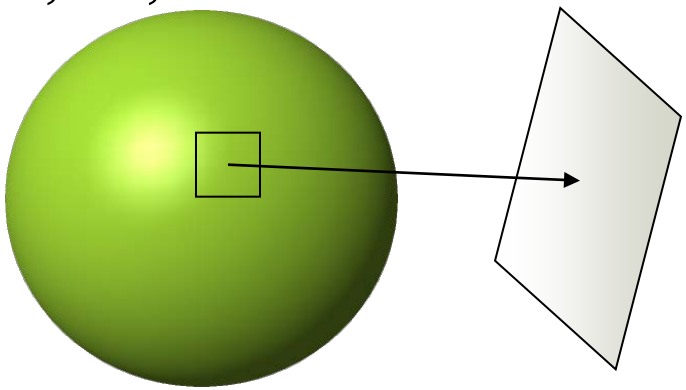


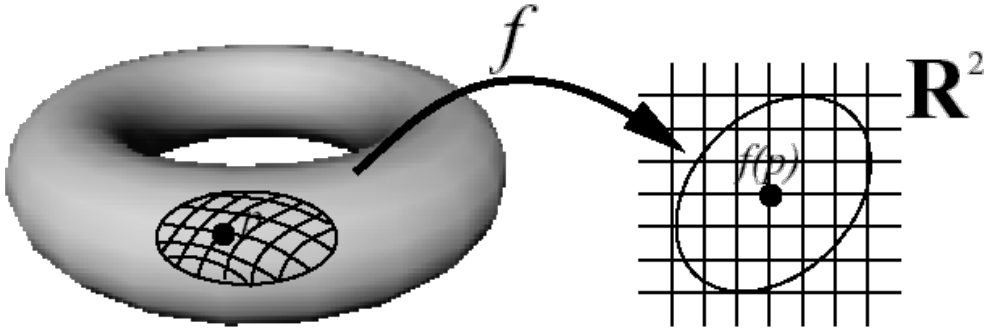
Manifolds

» Euclidean space R, R^2, R^3

» Locally Euclidean M^1, M^2, M^3

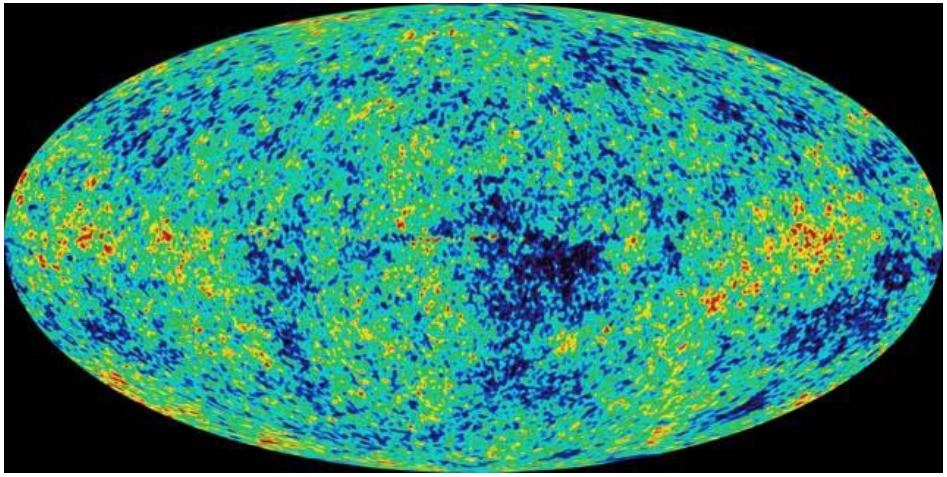


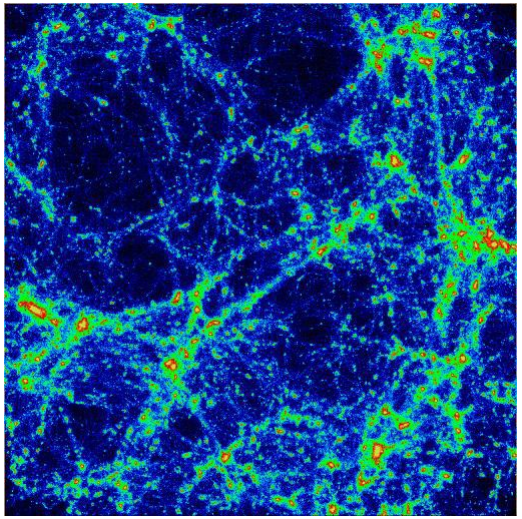




Functions on Manifolds

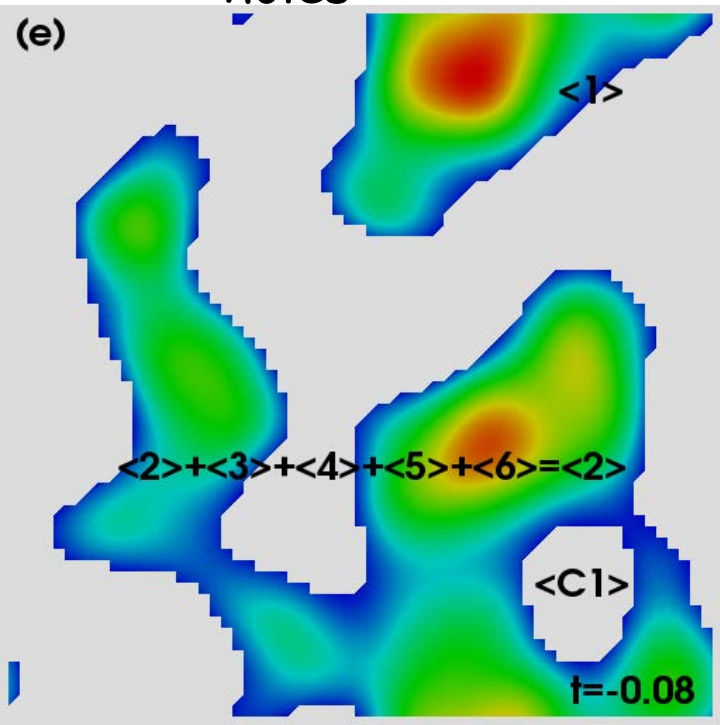
- Height of depth of the surface of earth with respect to sea level
- Primoridal fluctuation field, CMB
- Grid-densities of N-body simulations

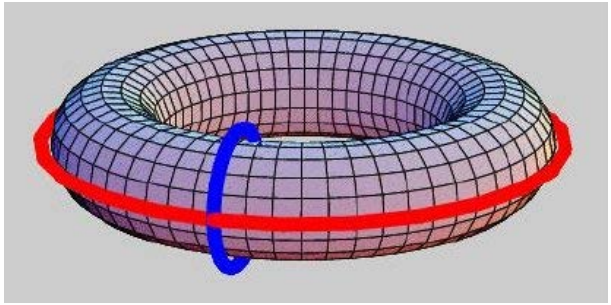




Topological holes

- Topology in terms of k-dimensional holes
- Betti numbers $\beta_k(k=0,\ldots d)$: count the no. of k-dimensional holes





0 dimensional holes
: connected objects

1 dimensional holes
: loops/tunnels

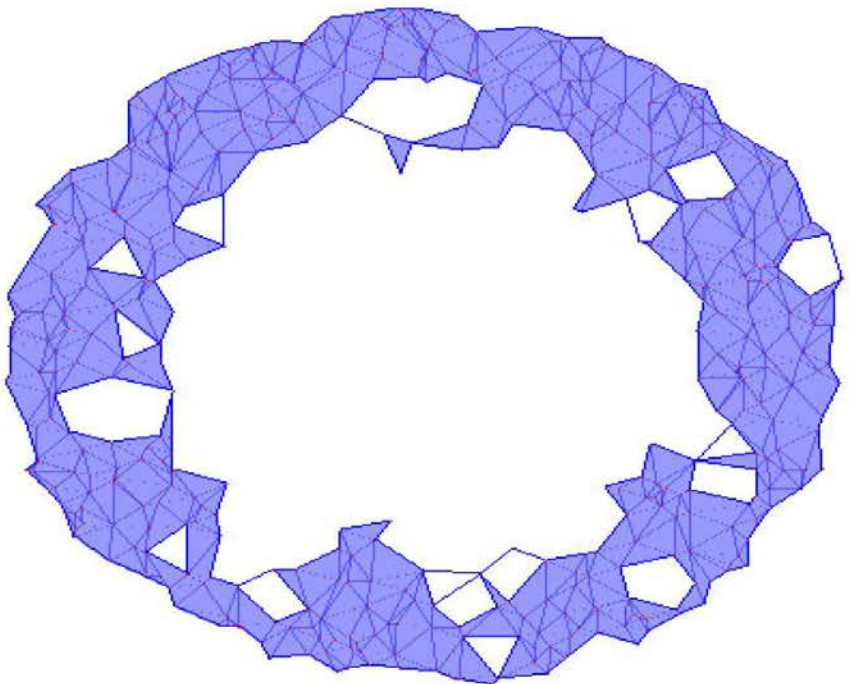
2 dimensional holes
: voids

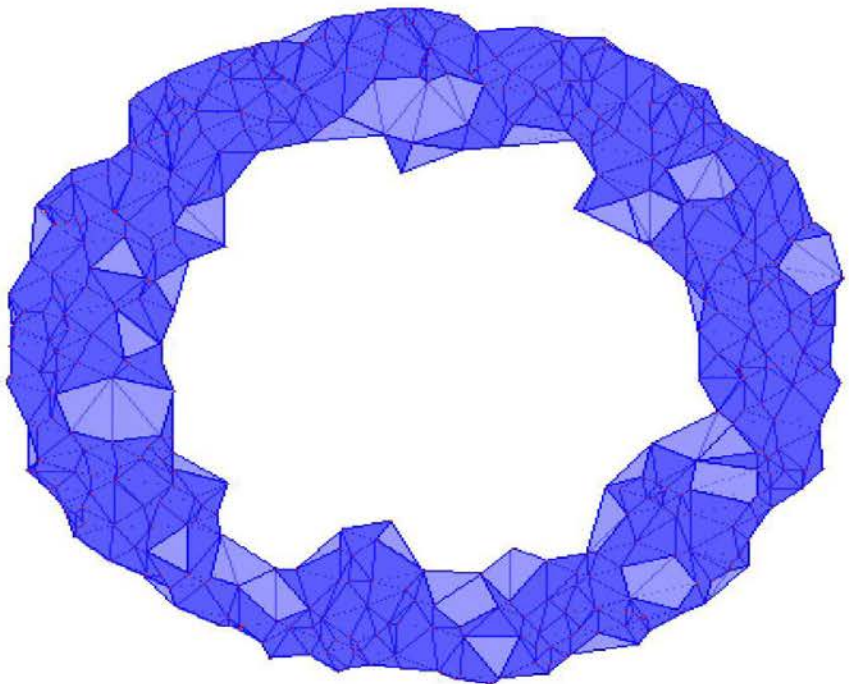
Are all holes important?

- Need to distinguish structures at different scales (cosmic web is multi-scale)

- Some holes are ***noise*** : discrete sampling, instrumental errors etc

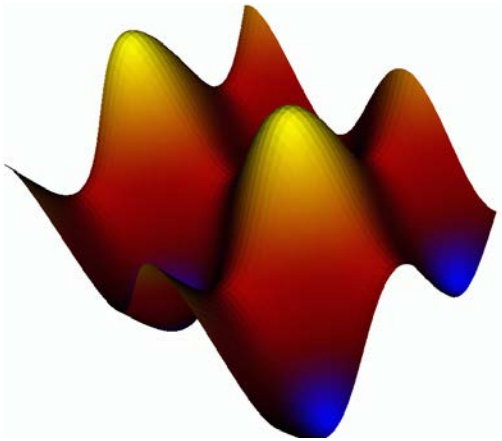
- A need to assign a relative weight



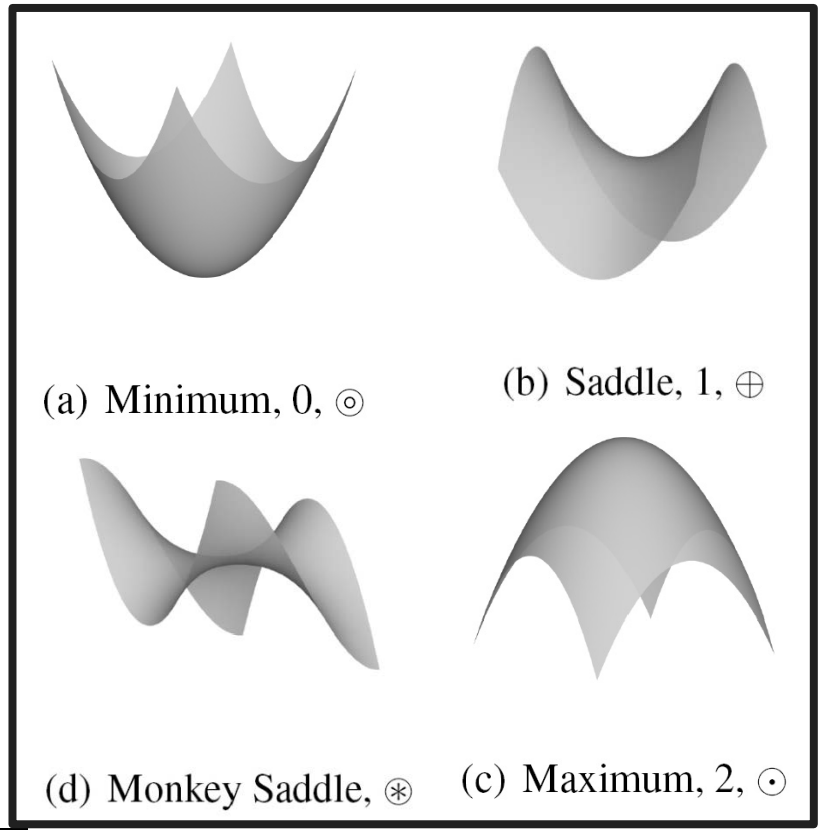


Persistence :
Multi-scale topology

Critical points : Change in topology



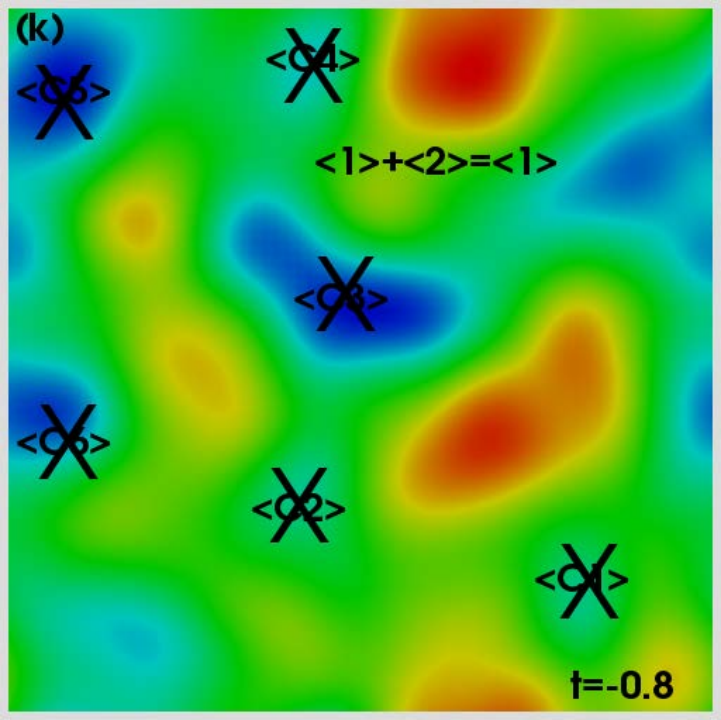
- Topological Structure of continuous field determined by singularities:
  - maxima
  - minima
  - saddle points
- Topology changes only while crossing a critical point



Persistence :
Multi-scale topology

- Filtration : Study the change in topology as we sweep from highest to lowest function values

- Topology changes only at critical points (maxima, minima, saddle points): **birth** and **death** process



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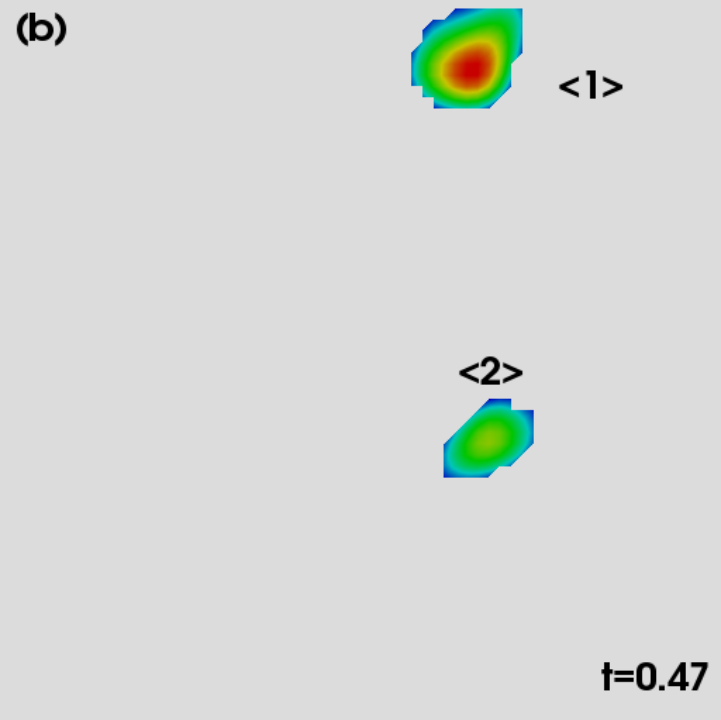

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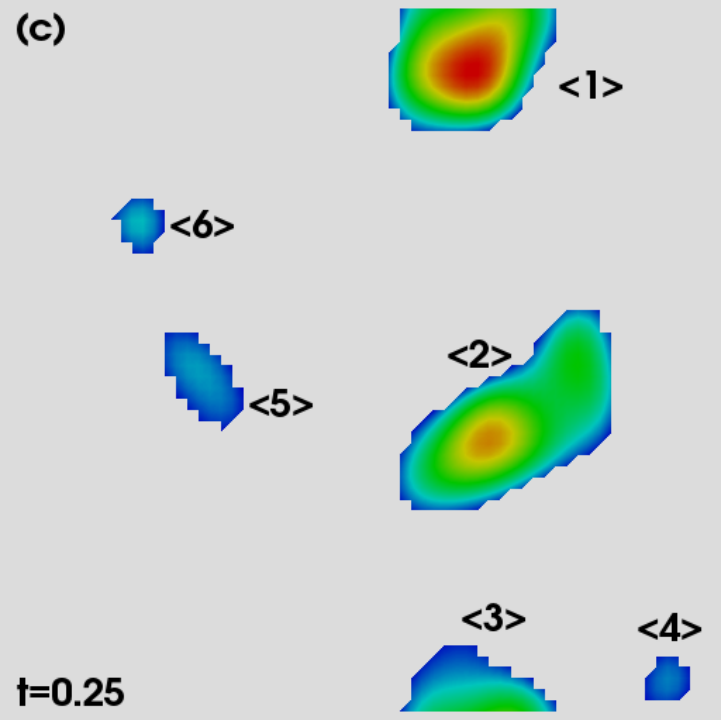

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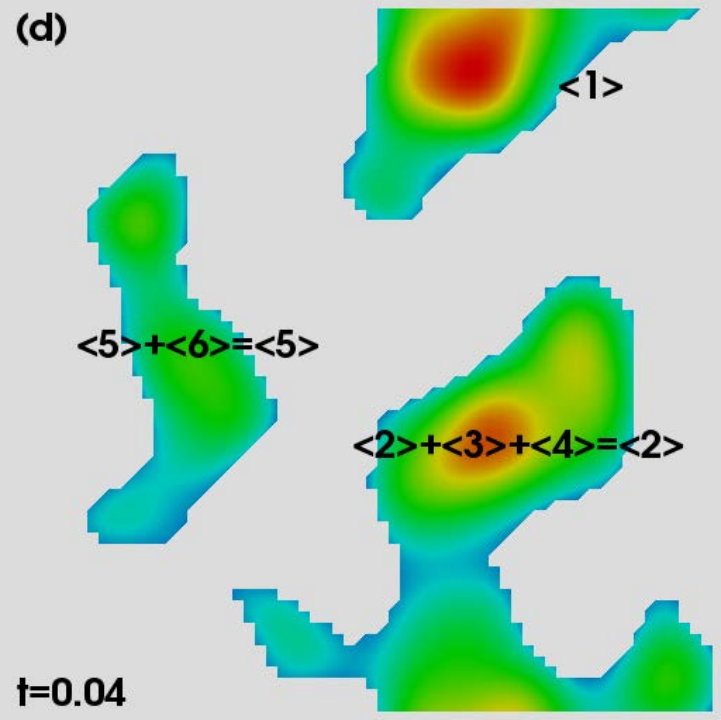

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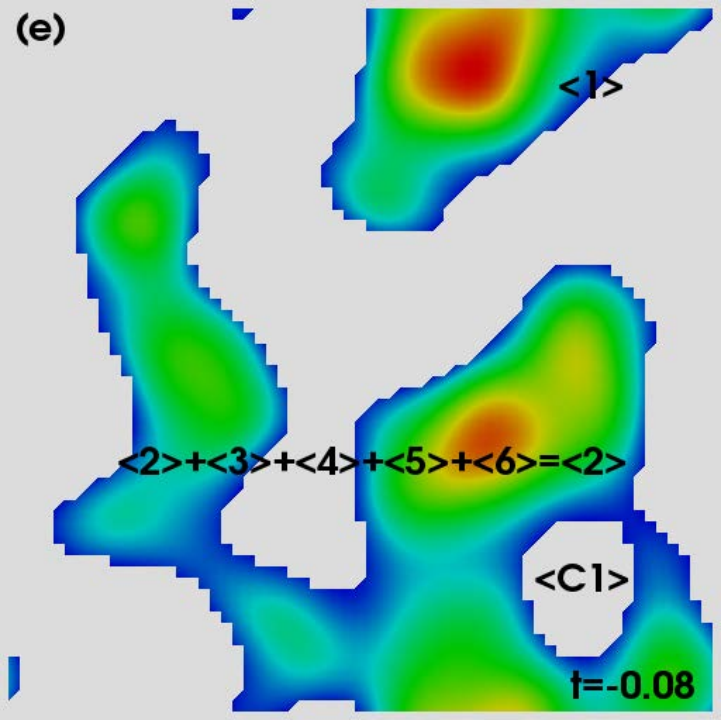

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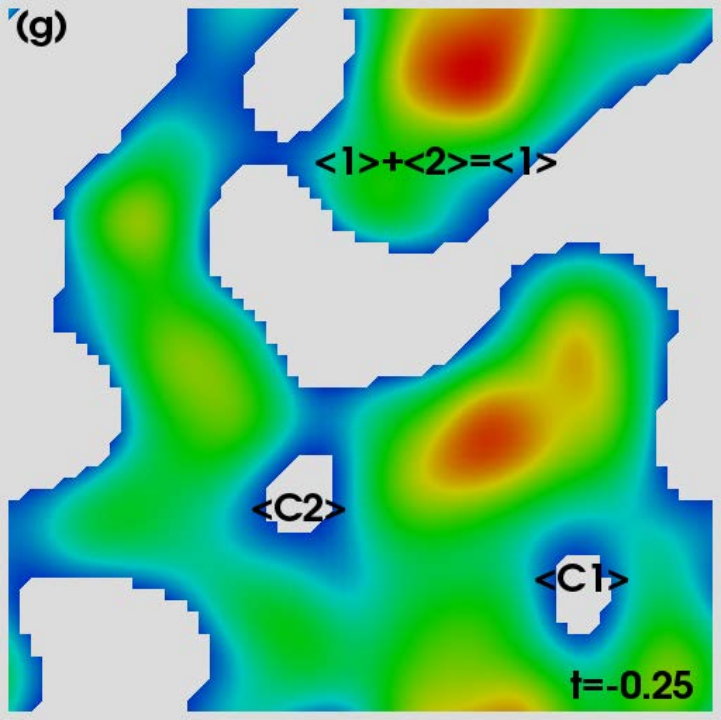

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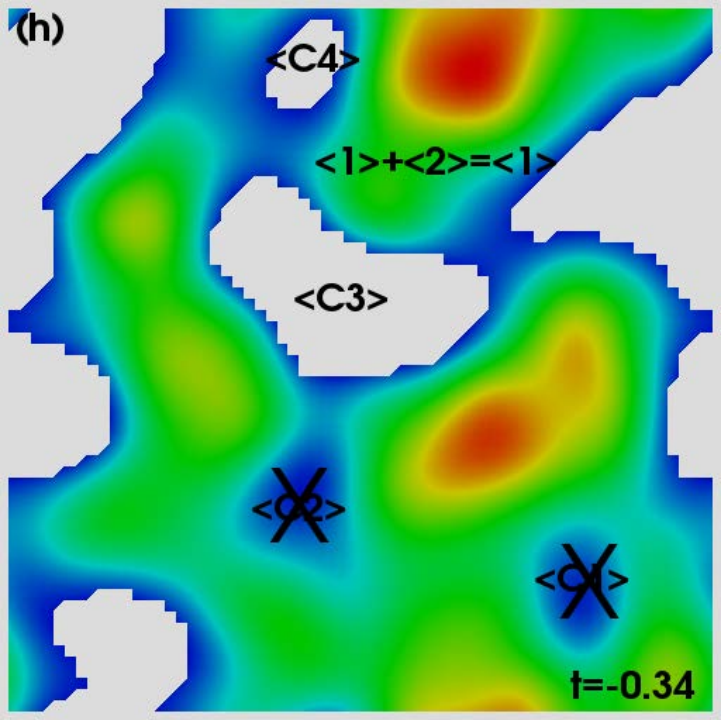

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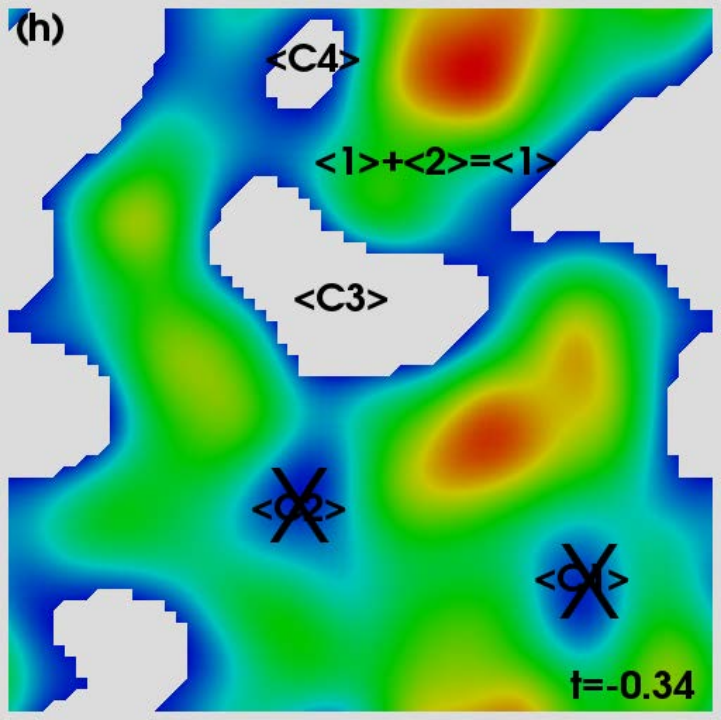

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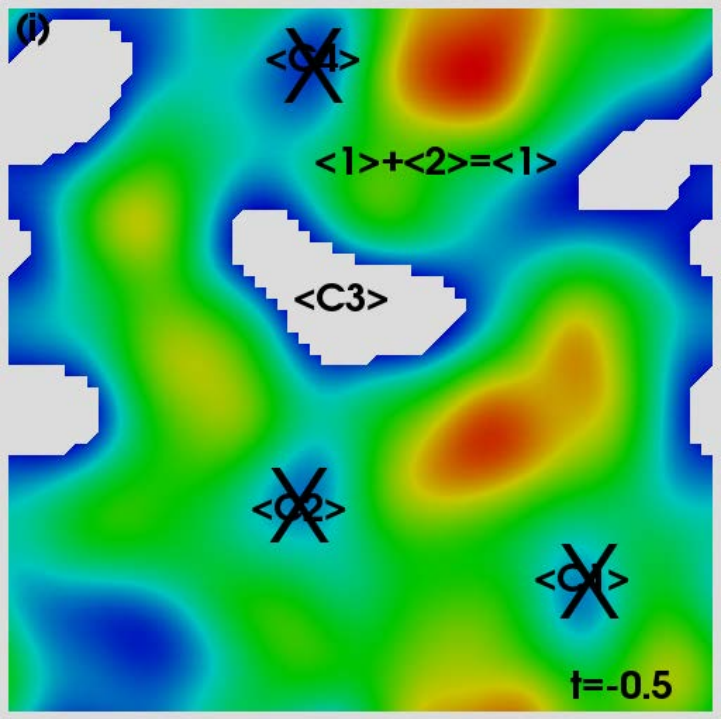

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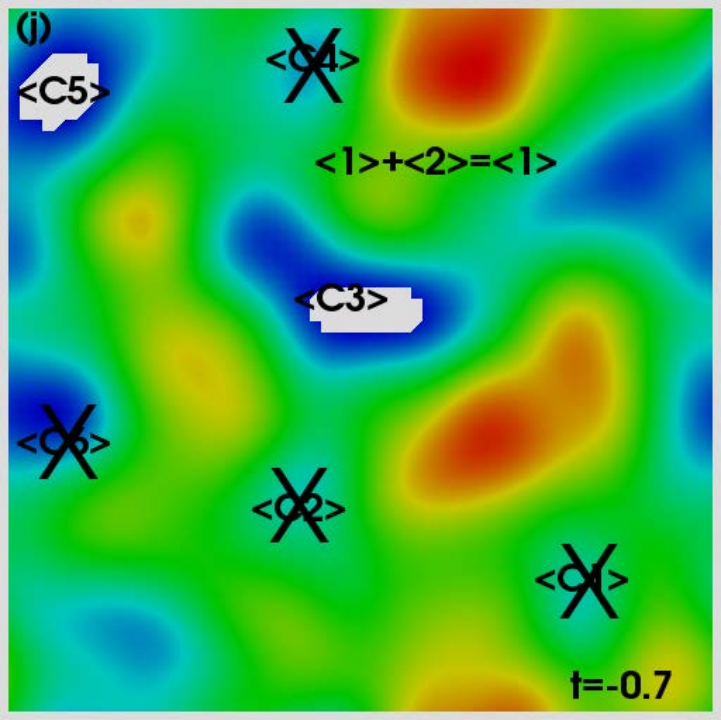

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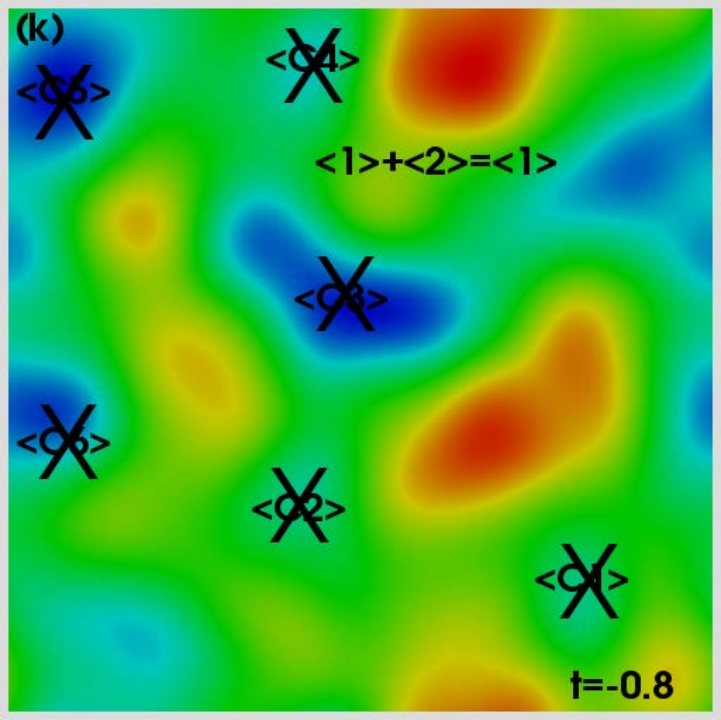
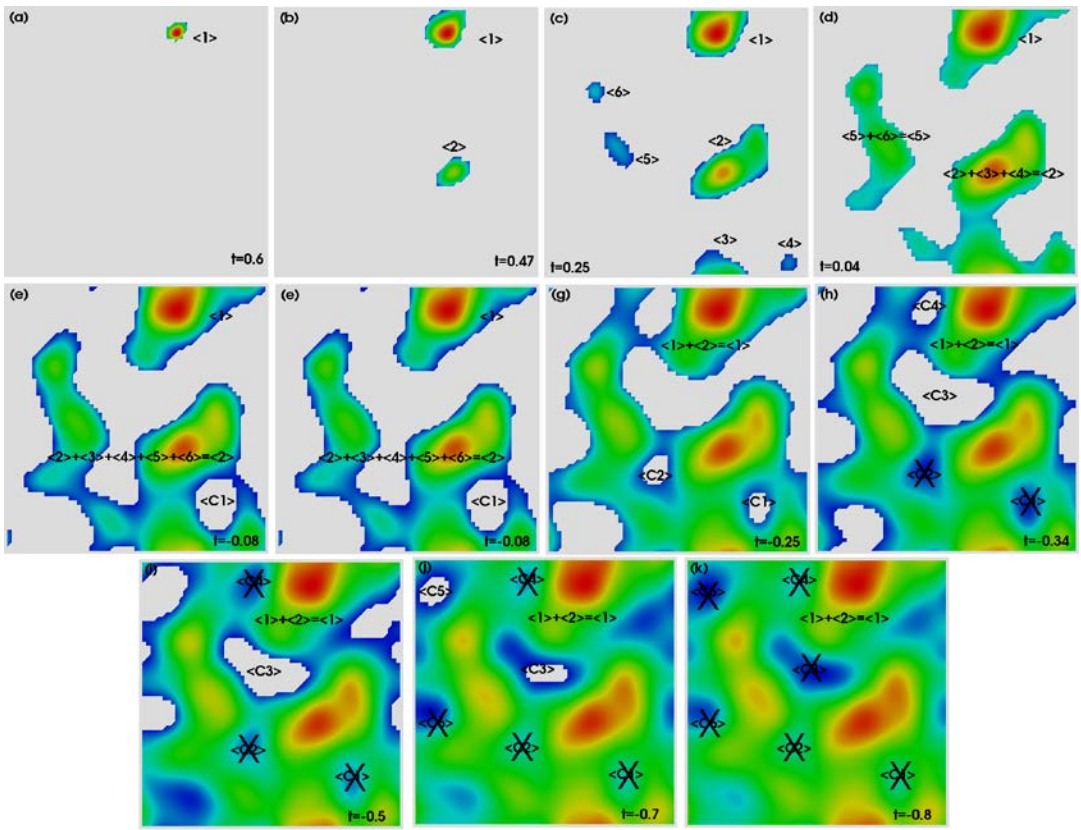
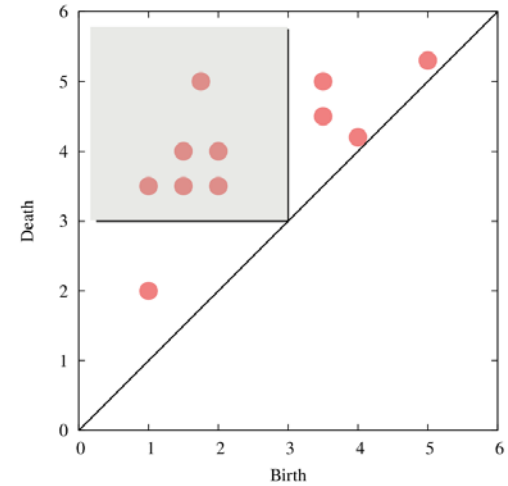


Persistence Diagrams

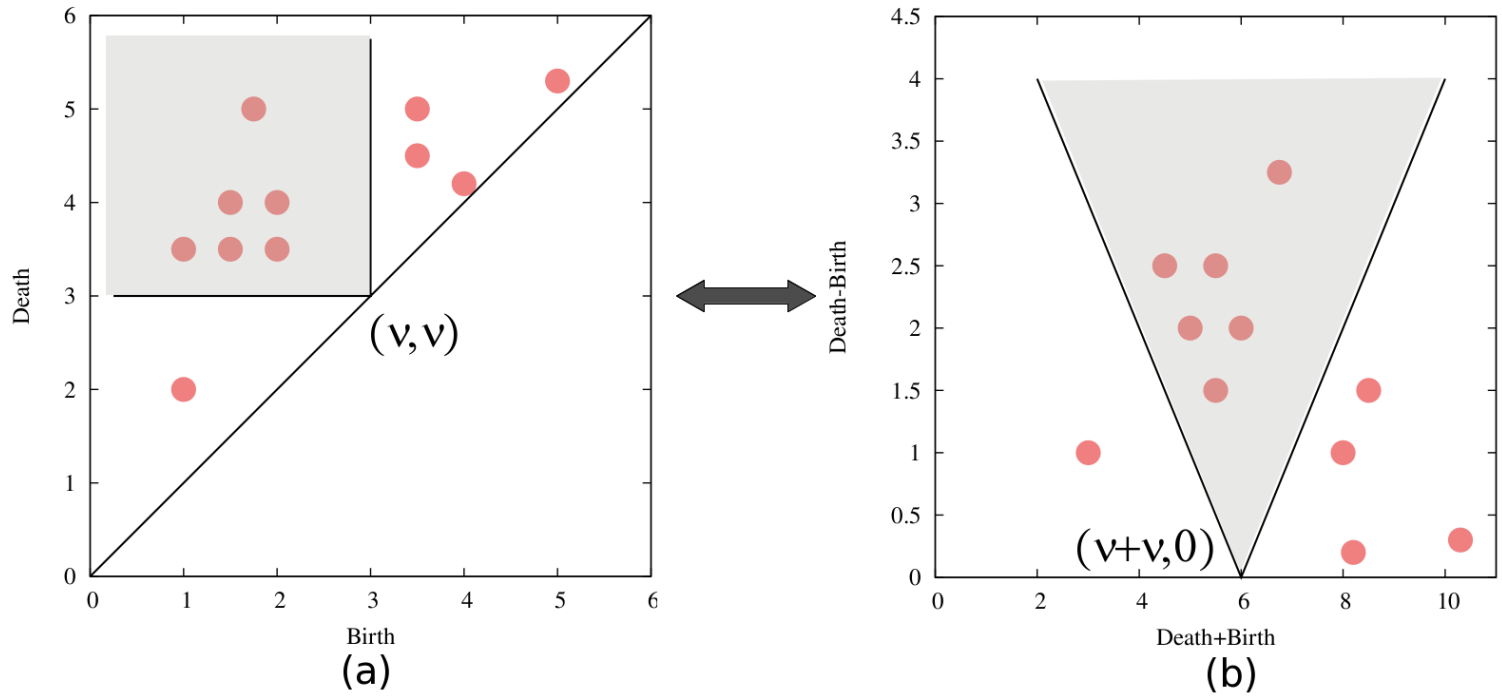




- Representation of multi-scale topology
- Dots in the diagram record birth and death
- A diagram for each ambient dimension of the manifold

- 0-dimensional diagram: representation of merger of isolated objects (merger trees?)
- 1-dimensional diagrams: formation and filling up of loops
- 2-dimensional diagrams: formation and destruction of topological voids

Persistence Diagrams : our preferred representation

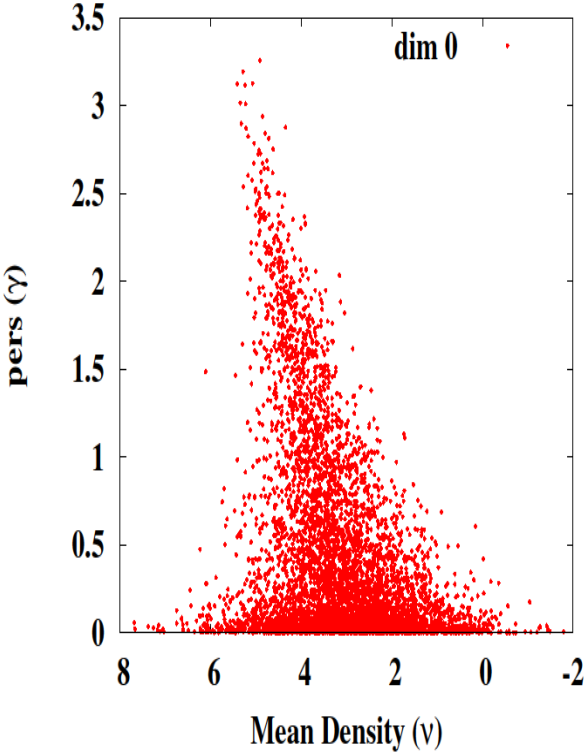


- Rotation of co-ordinates

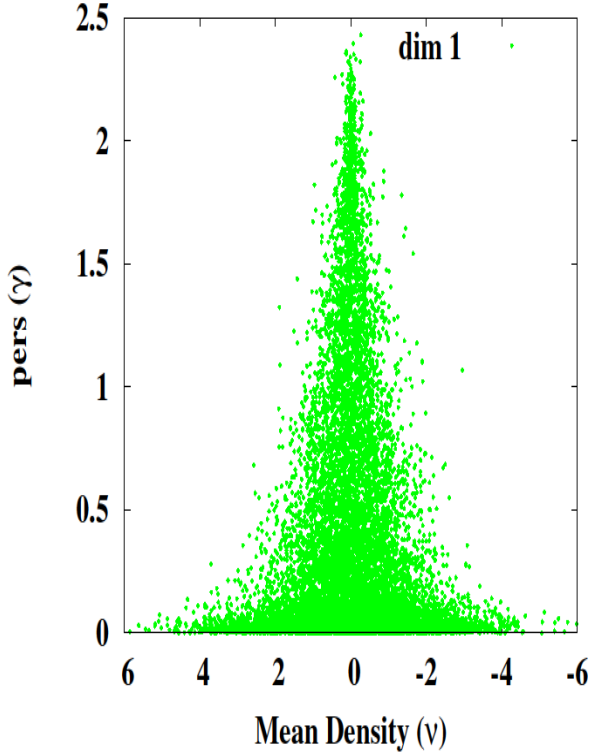
$$\frac{D+B}{2} = Mean\ density$$

$$D - B = persistence$$

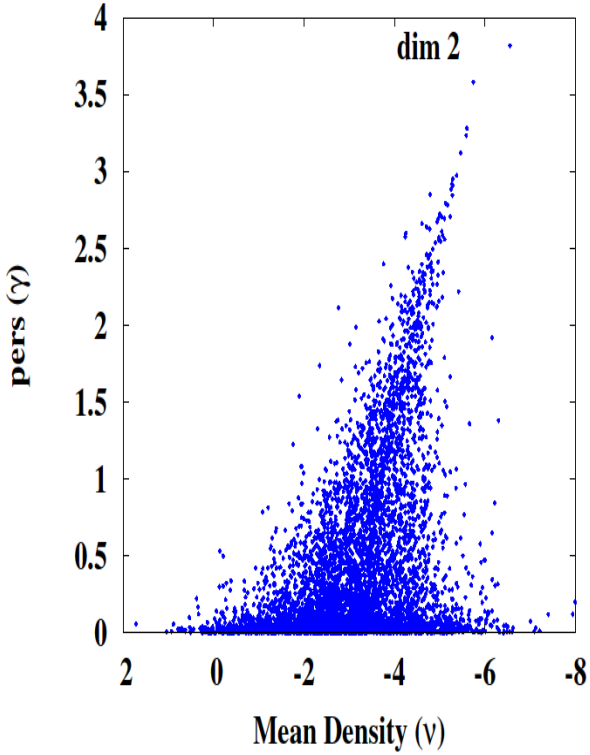
A typical persistence diagram



Islands
(0-dimension)



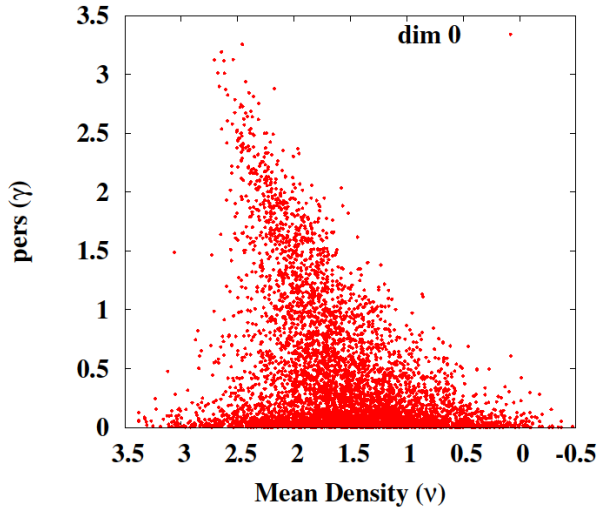
Loops
(1-dimension)



Voids
(2-dimension)

Gaussian random field, power-law power spectrum, n=0
(all dimensions)

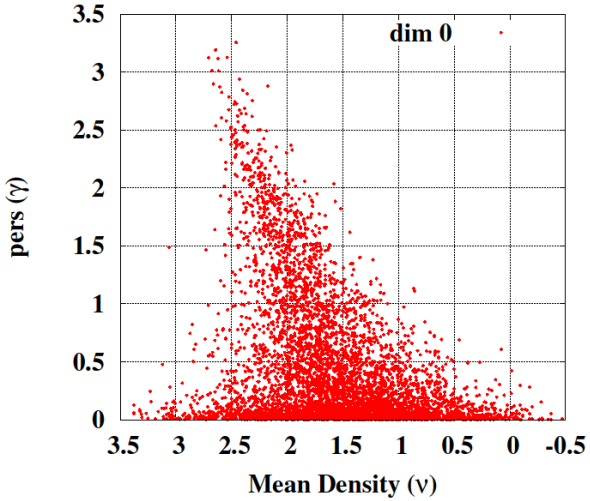
Statistics of persistence topology



- Topology connected to distribution of critical points
- Critical point distribution well defined for stochatic processes (GRF,CMB, LSS) : probabilistic description

- Are the diagrams well defined and stable/convergent over many realization?
- Statistical description?

Persistence landscape : emperical probability description



Discretize the diagram into regular grids

Empirical intensity function :

$$\langle I_{ij} \rangle = \frac{\langle N_{ij} \rangle}{\langle N_{tot} \rangle}$$

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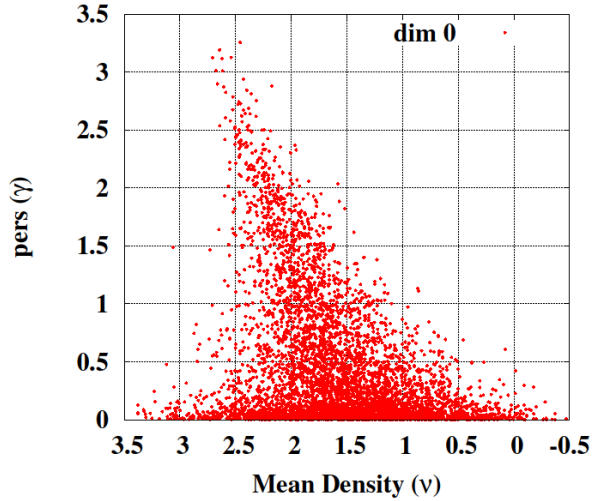
- Statistical description?

Persistence intensity maps : empirical probability description



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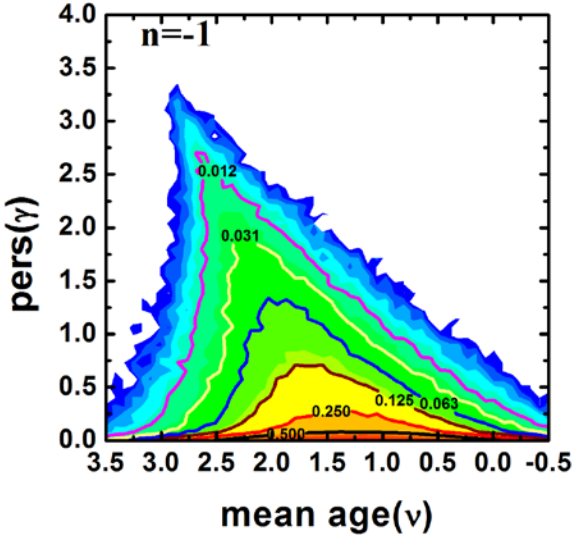

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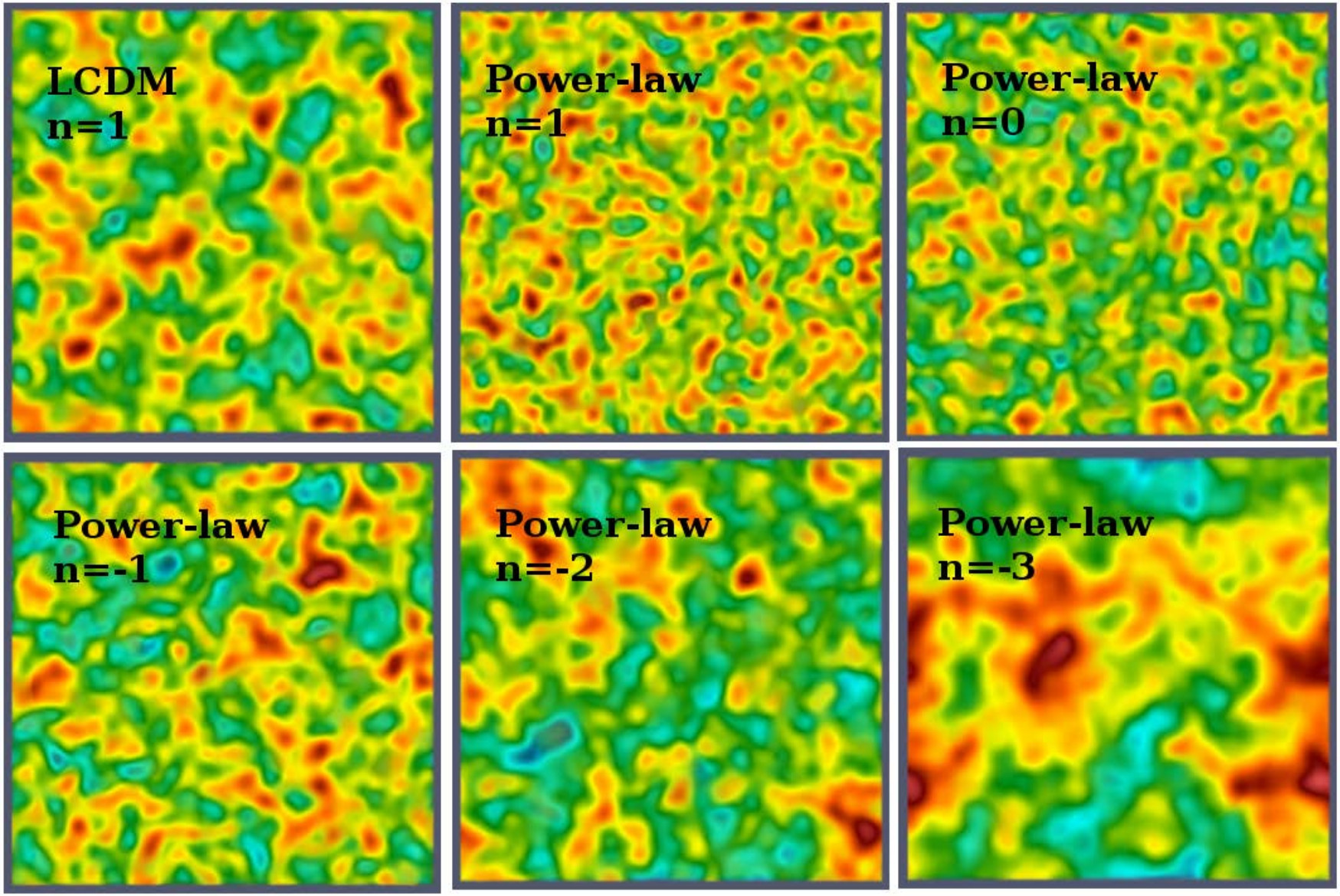
Statistical description?

Gaussian random field : Models



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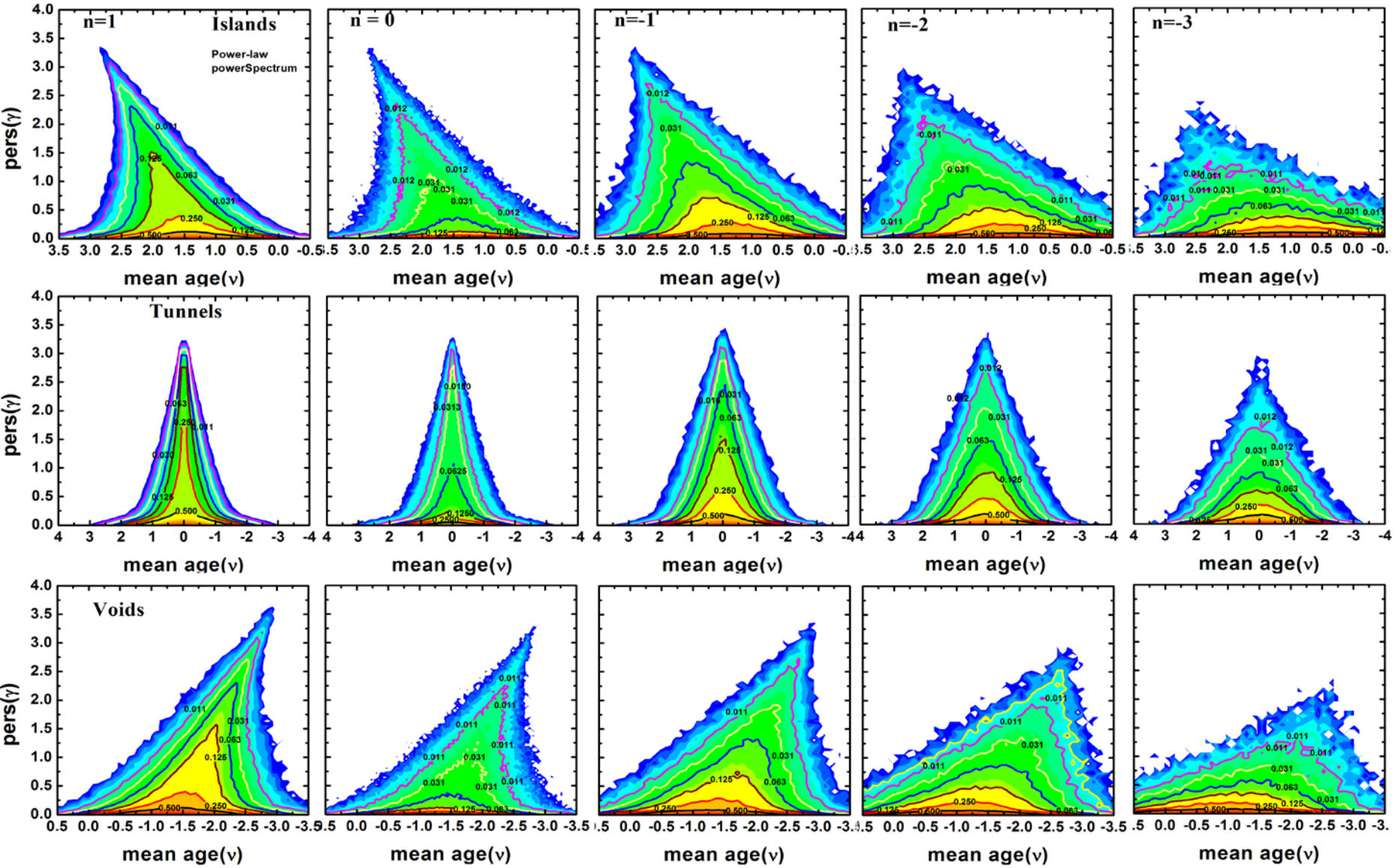
Gaussian random field : Persistence intensity maps



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Model comparison : Ratio intensity maps

- Local difference in topology
- Excess/deficit of topological objects in the neighborhood of the point defined by (mean density:pers)

Ratio function:

$$\Delta_{ij(f1,f2)} = \left(\frac{\langle N_{ij} \rangle_{f1}}{\langle N_{tot} \rangle_{f1}} \right) / \left(\frac{\langle N_{ij} \rangle_{f2}}{\langle N_{tot} \rangle_{f2}} \right)$$

$$= \left(\frac{\langle N_{ij} \rangle_{f1}}{\langle N_{ij} \rangle_{f2}} \right) * \frac{1}{\Omega_{(f1,f2)}}$$

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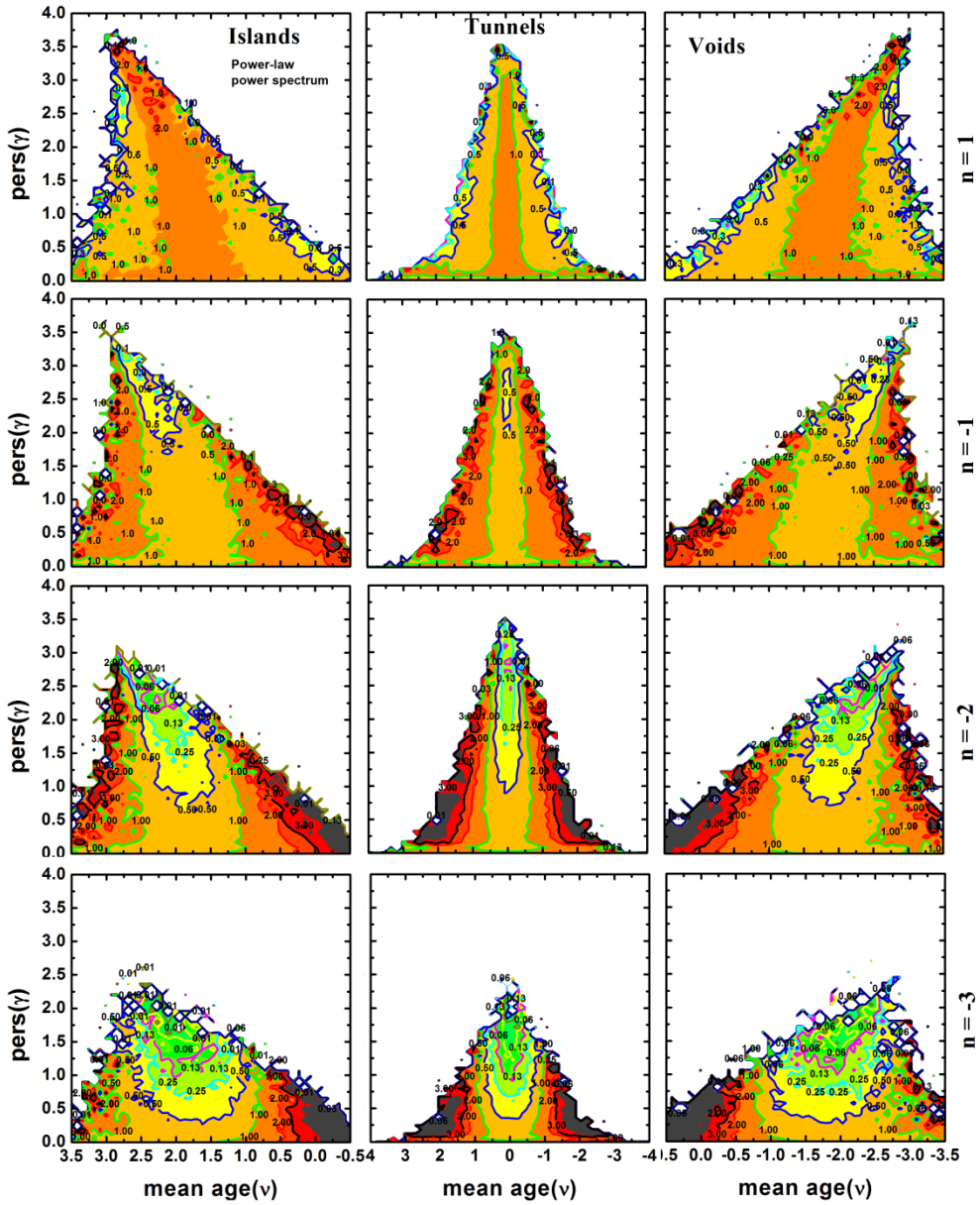
Ratio intensity maps : Gaussian Random fields



- Novel and extremely detailed descrimination of topological differences between models

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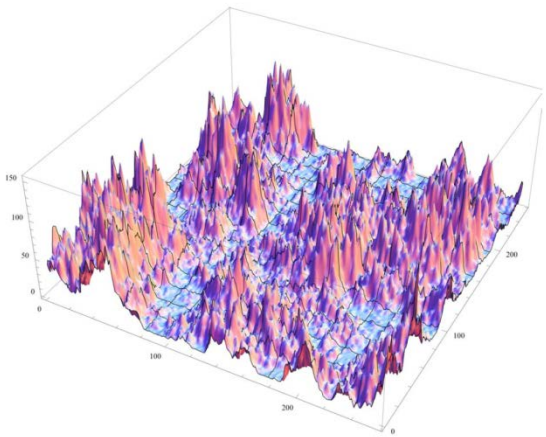
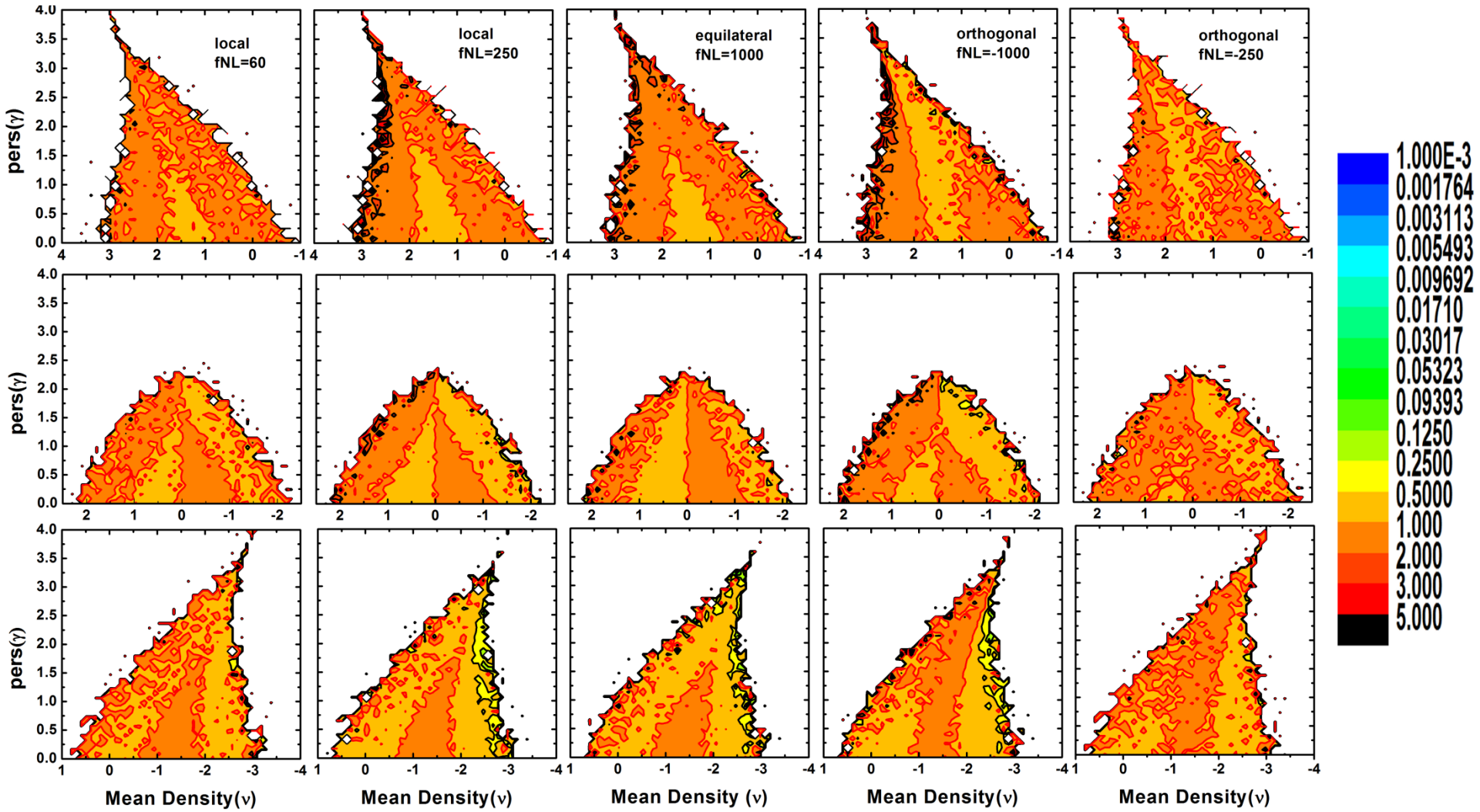
Non-gaussianity: The untamable beast

Non-gaussianity: The untamable beast

- Constraining inflationary models
- Described as higher order terms in the expansion around a gaussian (fNL, gNL)
- Any function is potentially non-gaussian in nature!
- Cosmological principle: isotropy and homogeneity as constraints imply gaussian fields described strictly only by 2-point correlation function (a good constraint on separating gaussian from non-gaussian)



Ratio intensity maps: non-gaussian case



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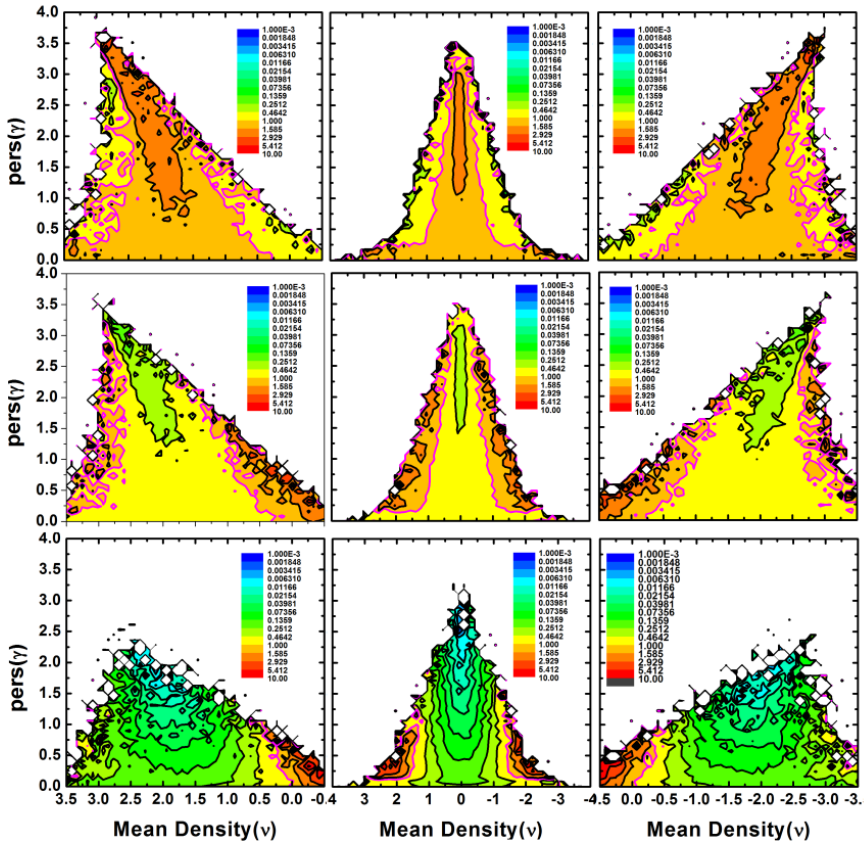
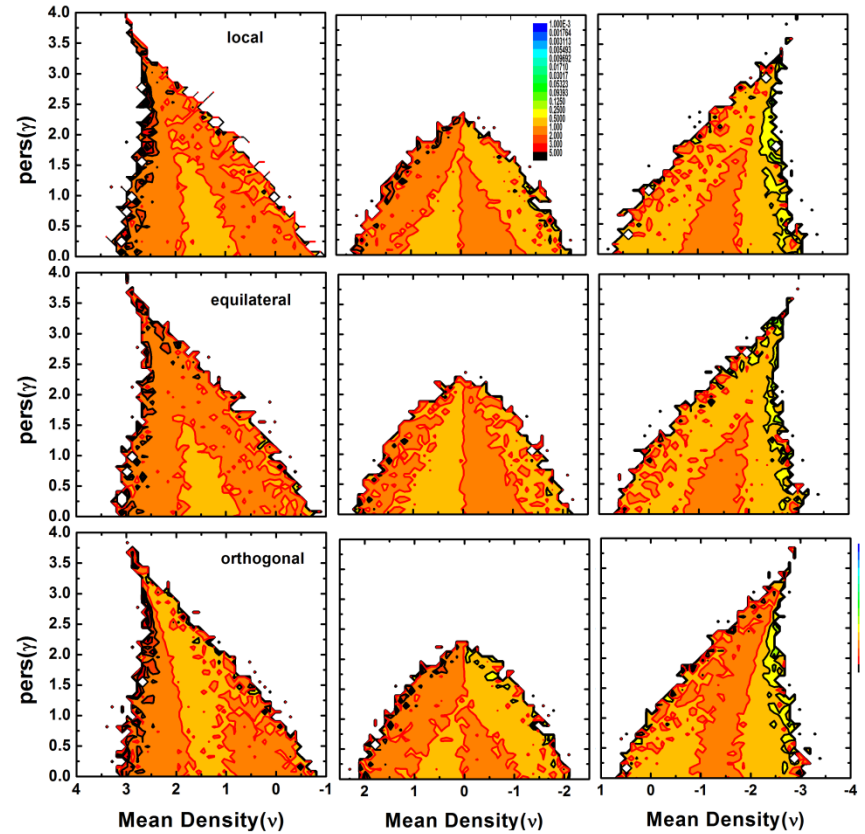
Ratio function:

$$\Delta_{ij(f1,f2)} = \left(\frac{\langle N_{ij}\rangle_{f1}}{\langle N_{tot}\rangle_{f1}}\right) / \left(\frac{\langle N_{ij}\rangle_{f2}}{\langle N_{tot}\rangle_{f2}}\right)$$



Gaussian Random field, wrt n=0

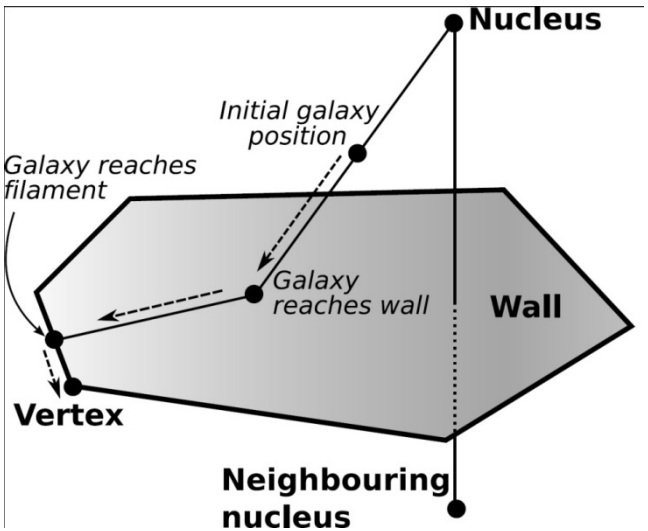


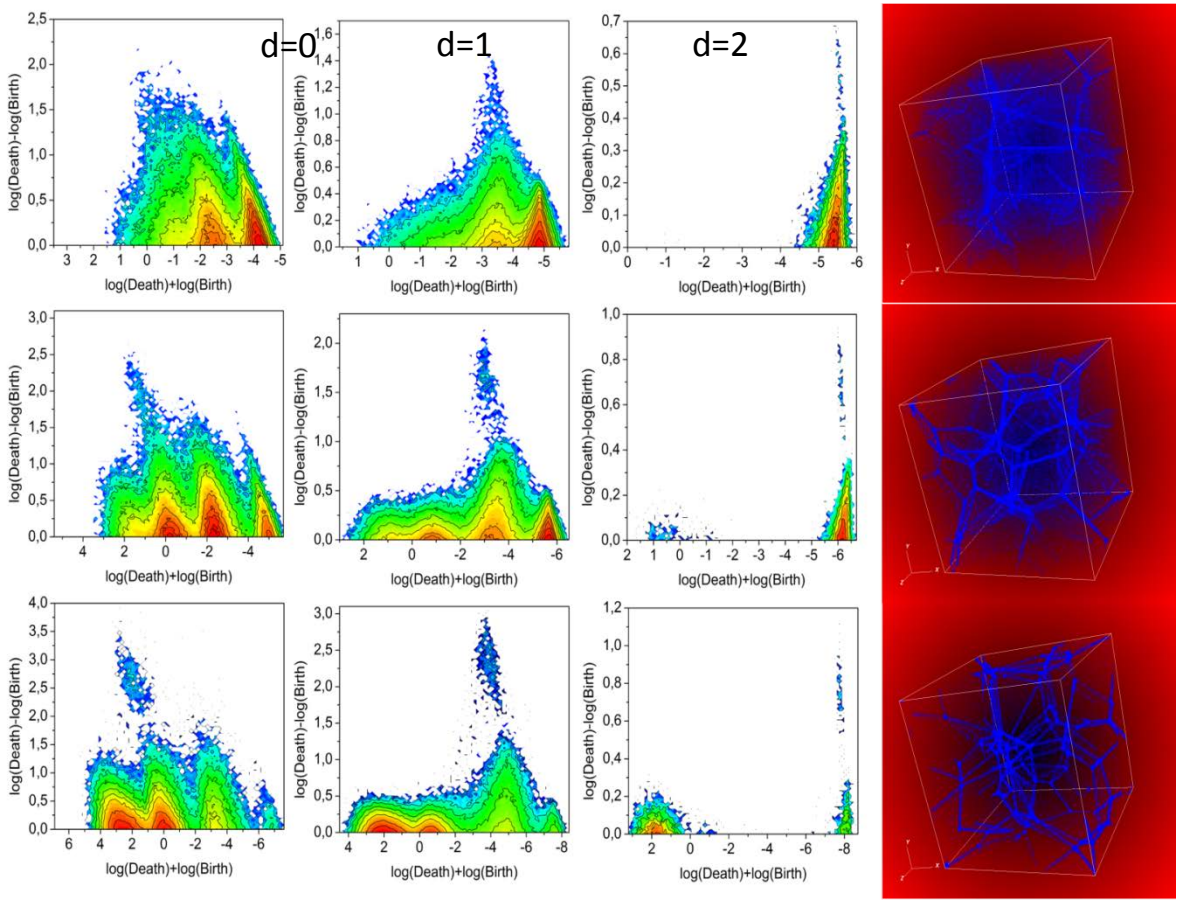
Non-gaussian field, wrt n=0

Non-gaussian simulations, courtesy Licia Verde

Pranav et. al 2013b

Voronoi Kinematic Models:
Heuristic description of structural evolution in the Cosmos

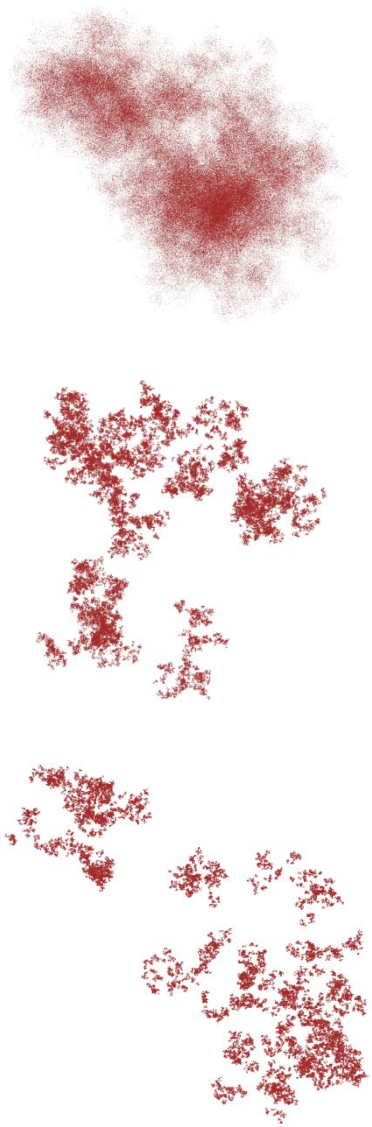


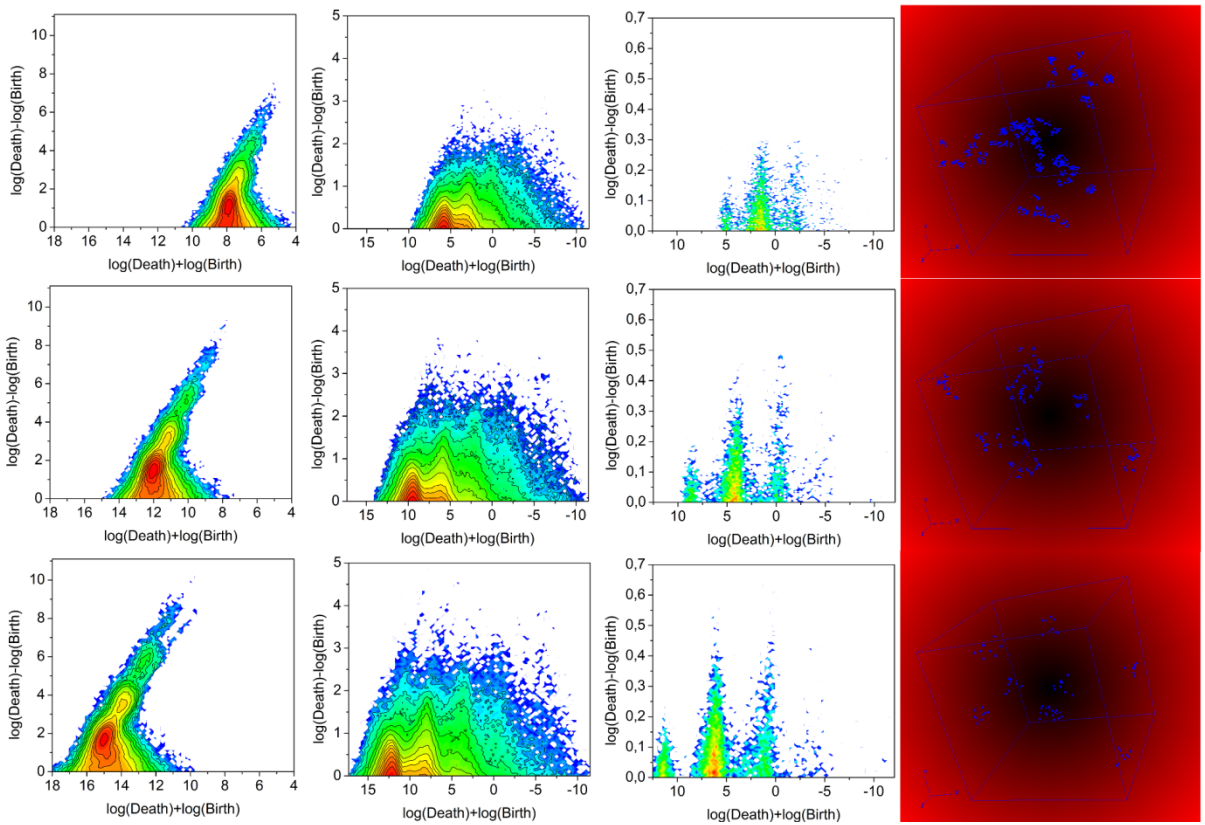


- Persistence diagrams segregate structures remarkably well

Pranav et. al 2013a

Soneira-Peebles Models: Heuristic description Galaxy clustering





Parameters of the model :

- Number of levels (n)
- Number of children (η)
- Ratio between the radius of parent and children spheres(λ)

Randomly place η spheres inside the top-level and continue for all levels

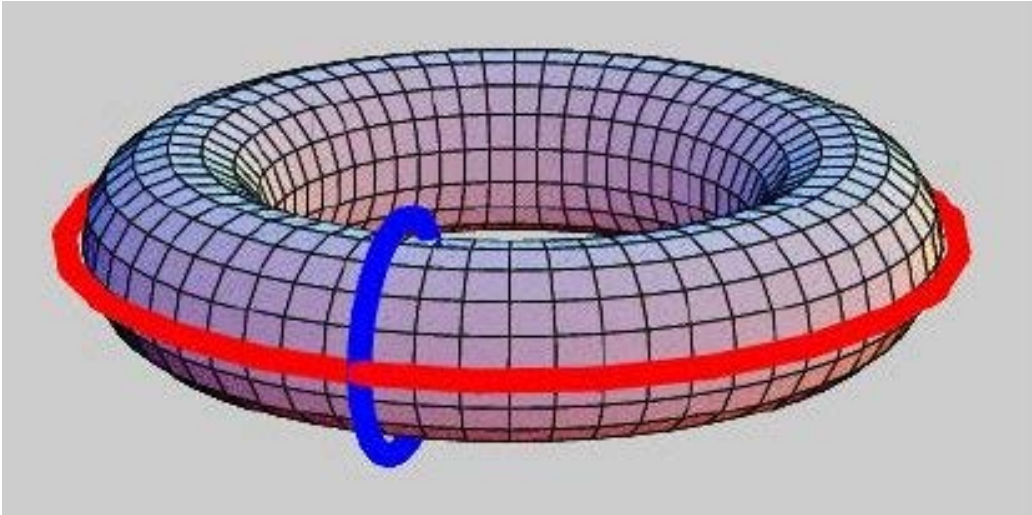
Pranav et. al 2013a

Betti Numbers

- Topology of excursion sets in terms of k-dimensional holes

β_k = # k-dimensional holes

β_0 = connected components
β_1 = independent loops
β_2 = independent voids



$\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 1$

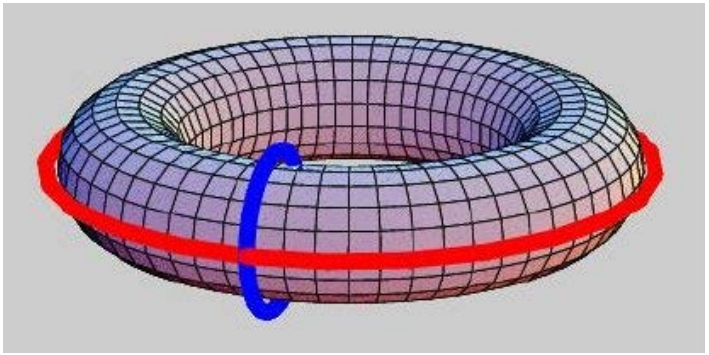
Betti numbers :
Euler Characteristic and genus extended

- Euler characteristic is a compression of topological information

$$\chi = \sum_i (-1)^{i`} \beta_i = \sum_i (-1)^i c_i$$

$$\chi = \beta_0 - \beta_1 + \beta_2 - \ldots (-1)^d \beta_d$$

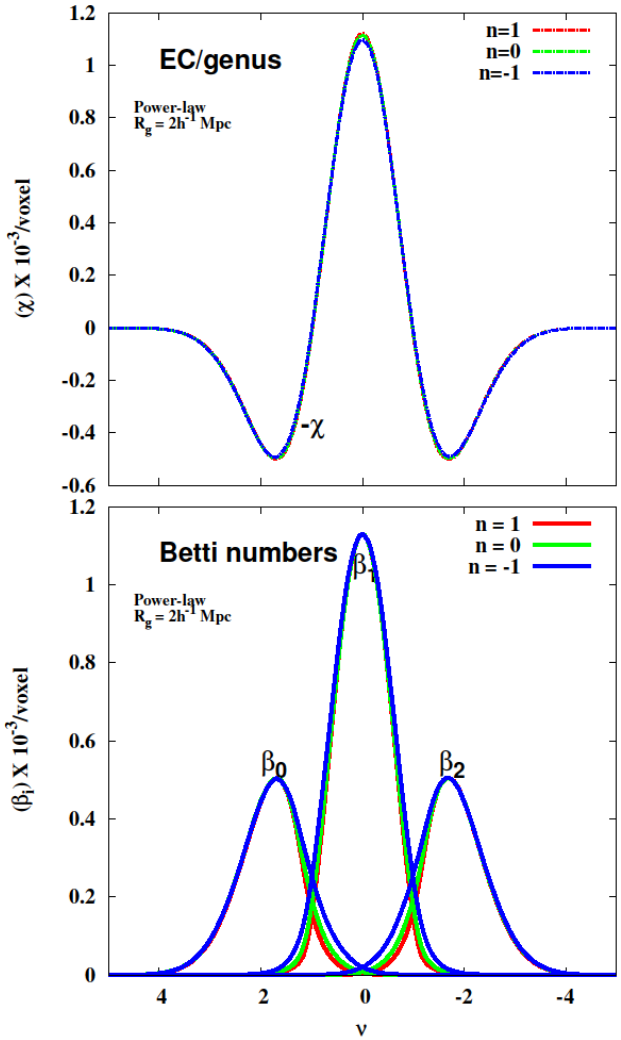
- Genus defined only for *2D surfaces* (number of independent ways of cutting a *2D surface* without leaving it disconnected) : no satisfactory generalization in lower and higher dimensions *(!)*



Pranav et. al 2013

Gaussian random fields : Betti numbers and genus

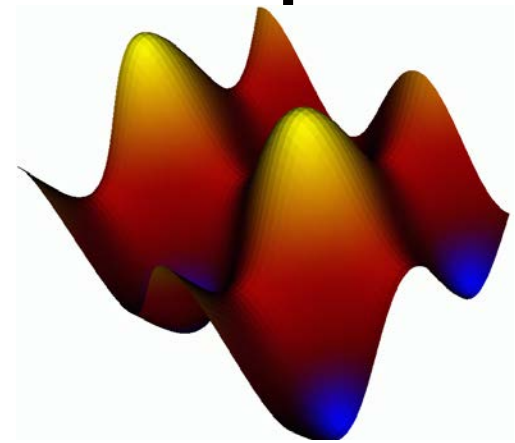
$$g = -\frac{1}{8\pi^2}\left(\frac{\langle k^2 \rangle}{3}\right)^{3/2}(1-\upsilon^2)e^{-\upsilon^2/2}$$

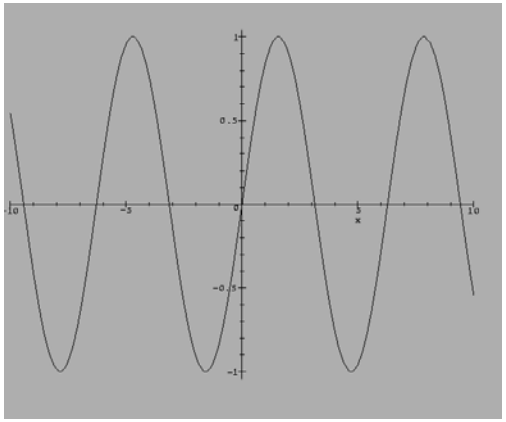


Pranav et. al 2013
Park, Pranav et. al (2012)

Morse geometry & Filament detection

Morse Functions and critical points





- » Domain R, R^2, R^3
- » Gradient ∇f
- » Critical points $\nabla f = 0$
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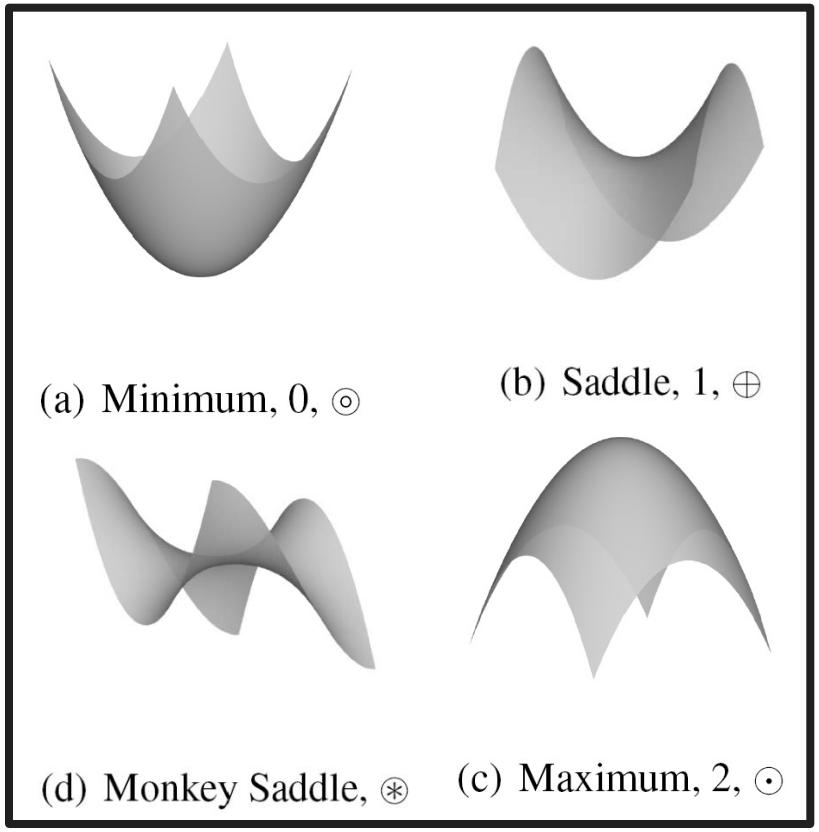
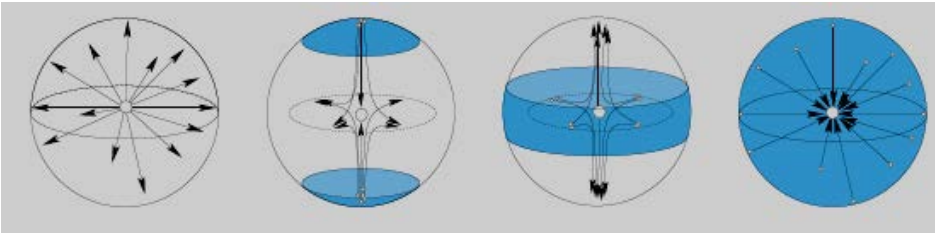
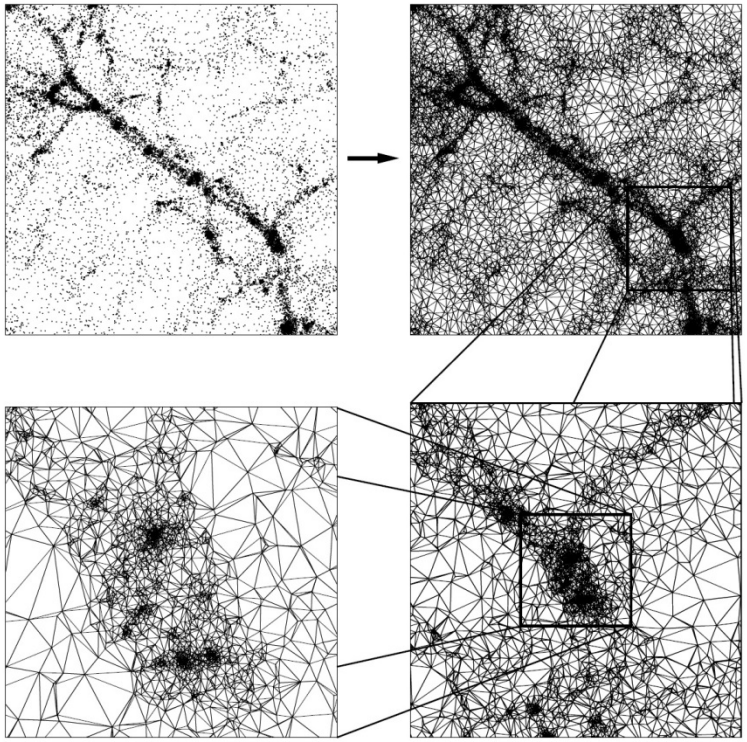
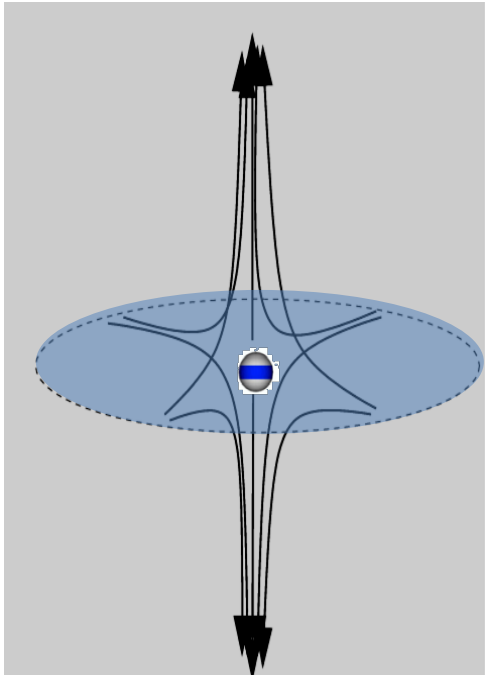


Morse geometry :
Critical points, gradient lines & descending/ascending manifolds

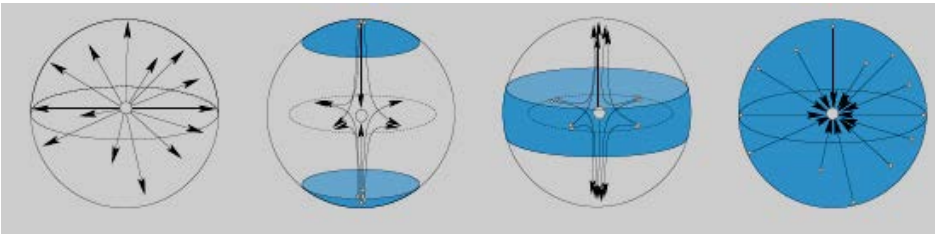


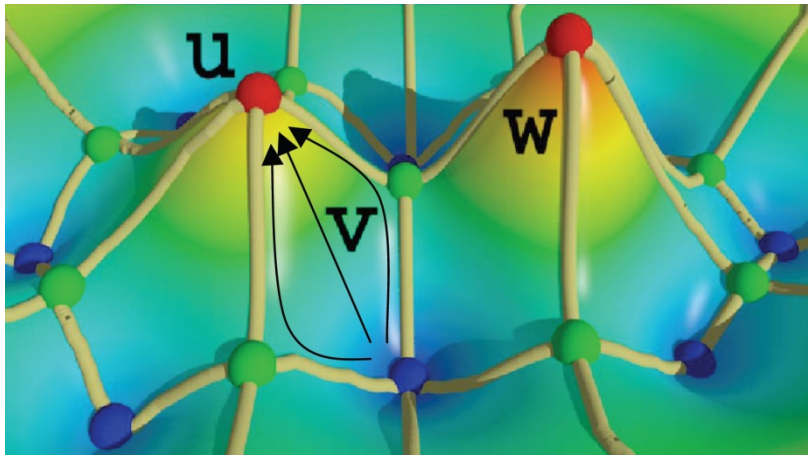


Gradient lines

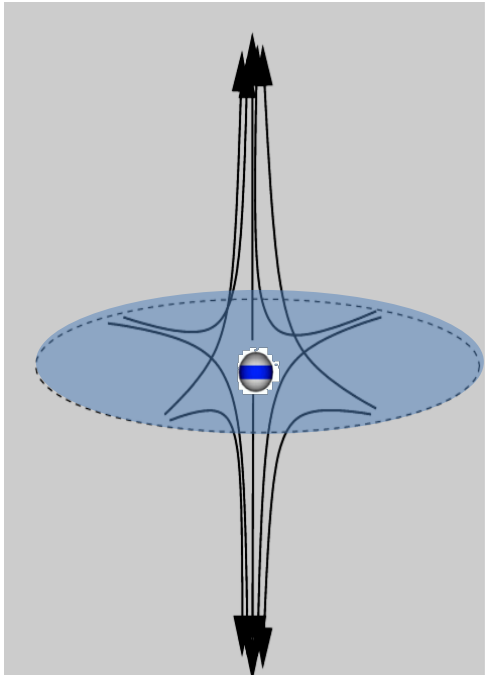


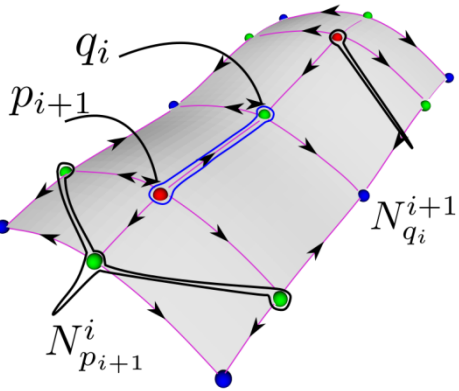
Morse geometry :
Filaments as ascending manifolds of 2-saddles

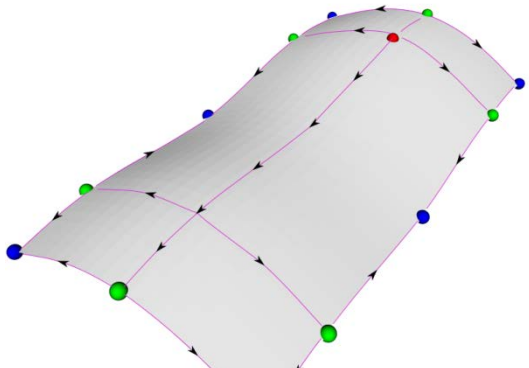




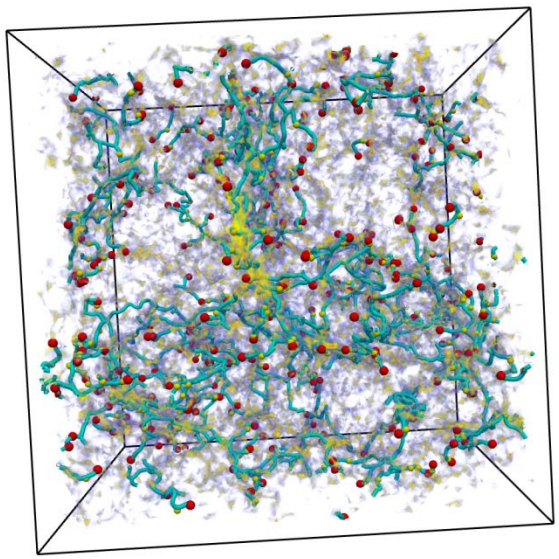
Gradient lines

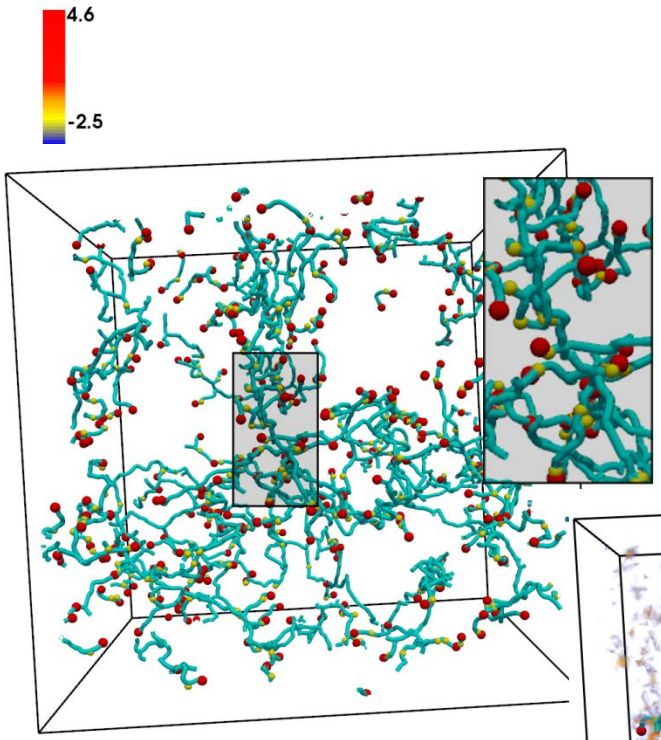


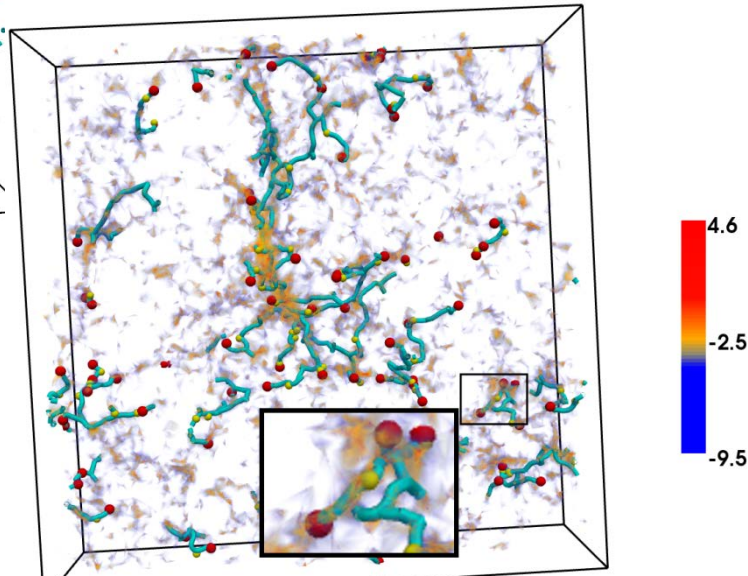




Morse geometry :
Filaments as ascending manifolds of 2-saddles

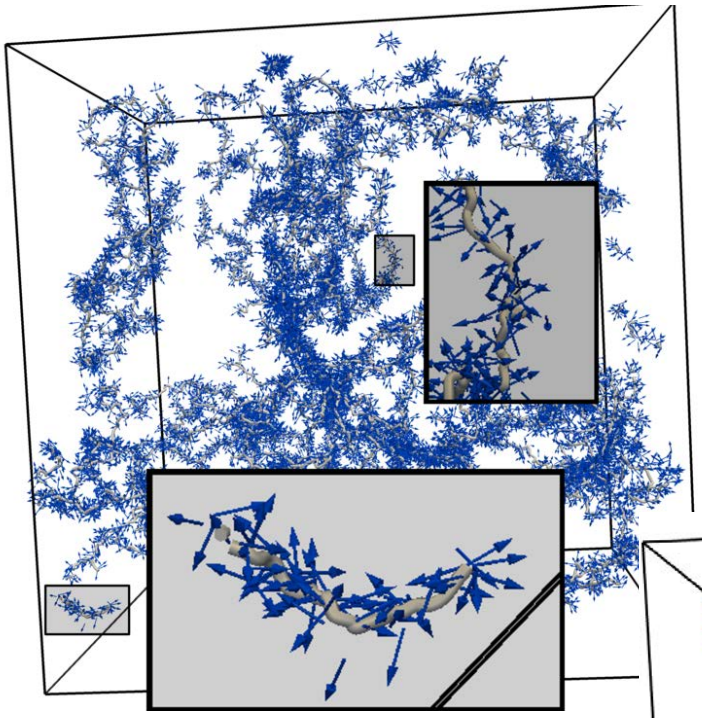


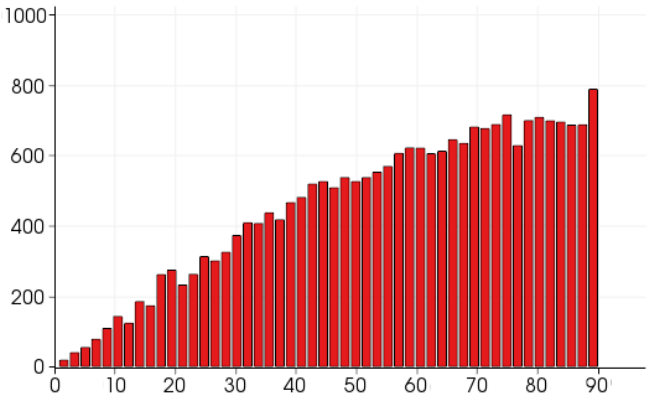


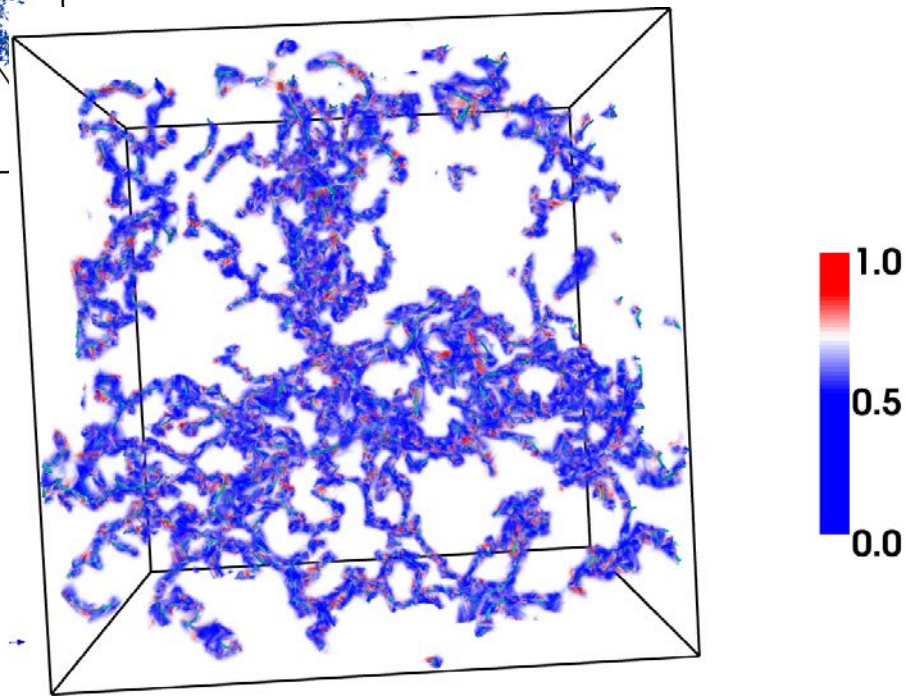


Shivshankar, Pranav et. al
2013

Spin alignment of DM haloes along filaments

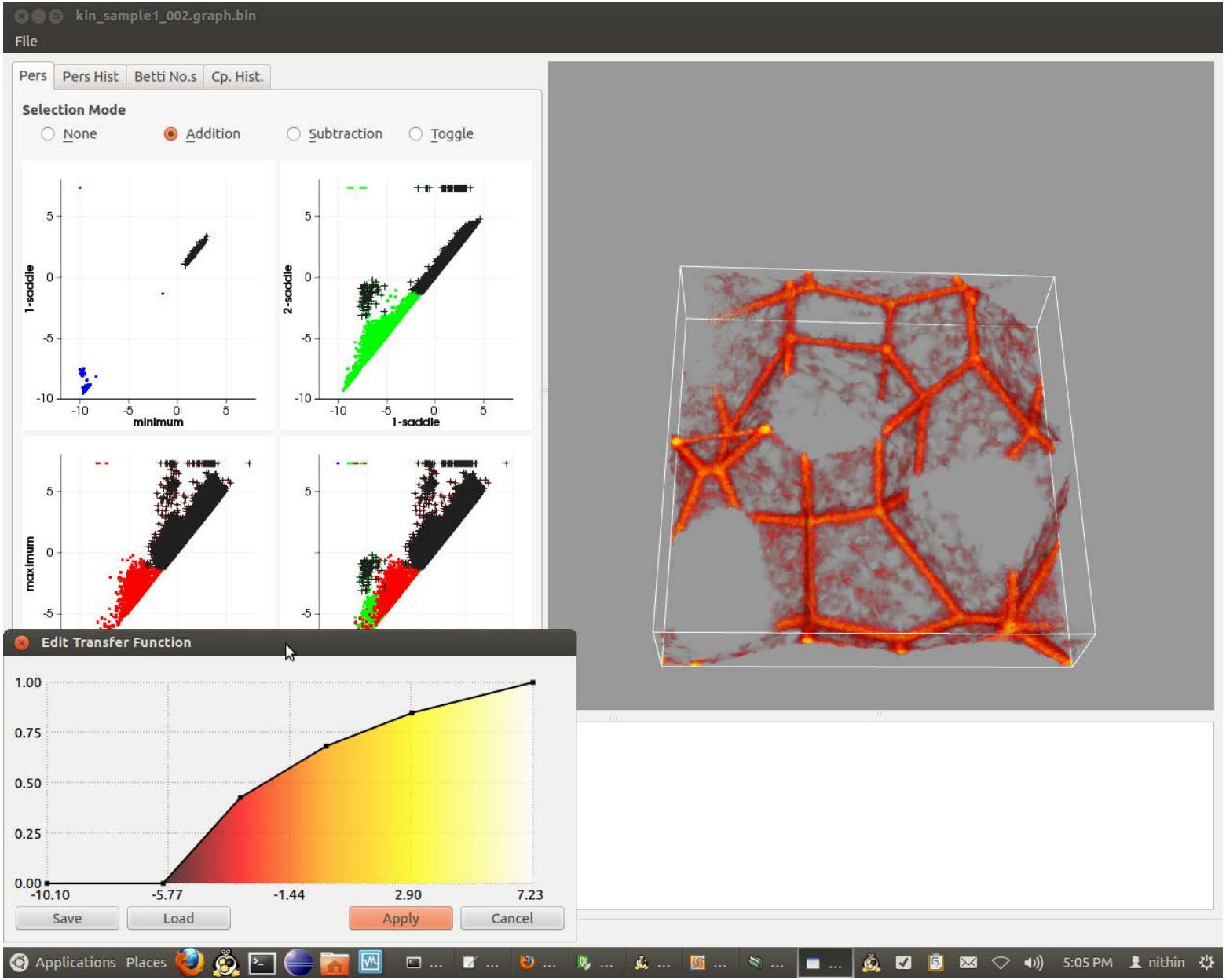




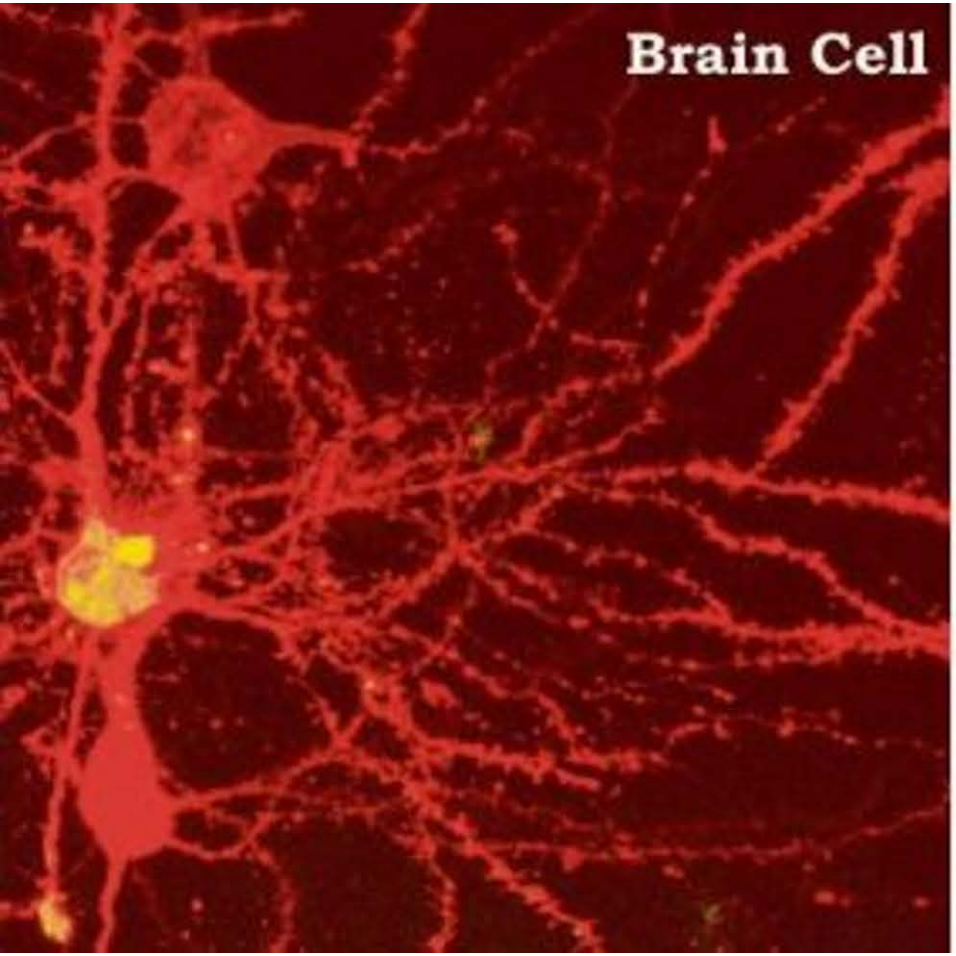


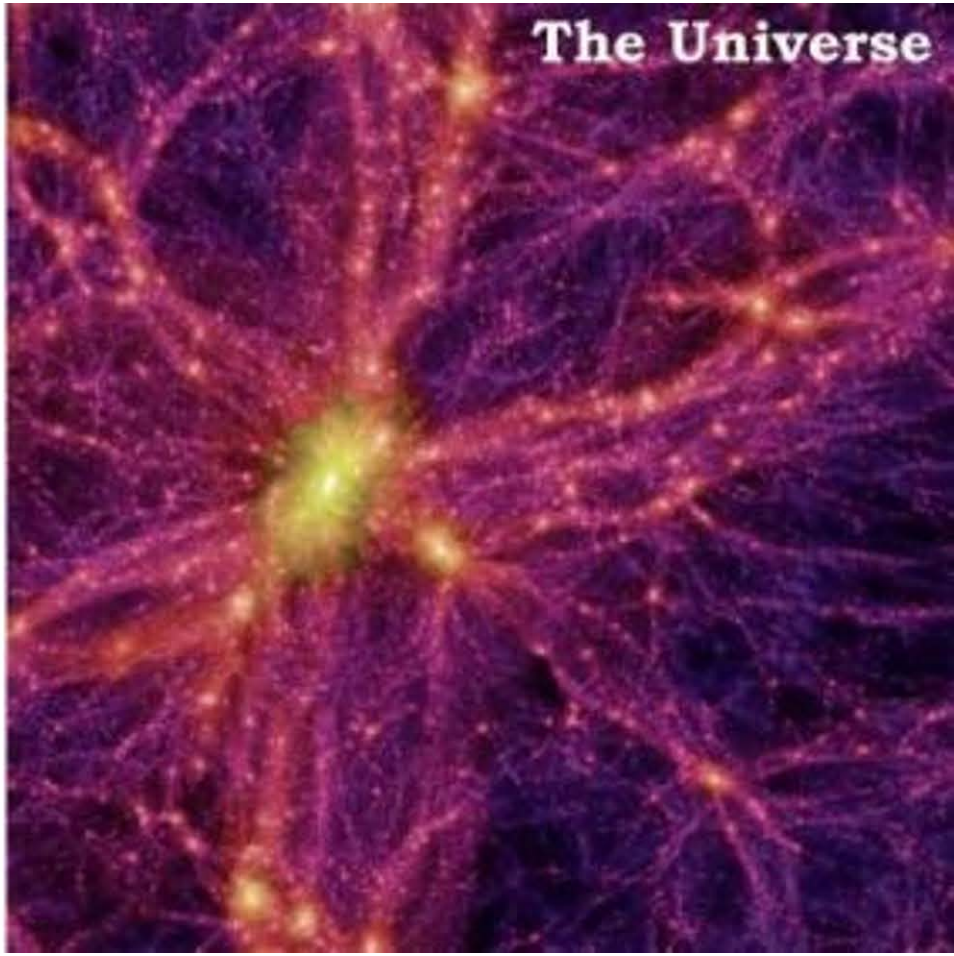
Pranav et. al 2013c

Interactive filament detection : New software



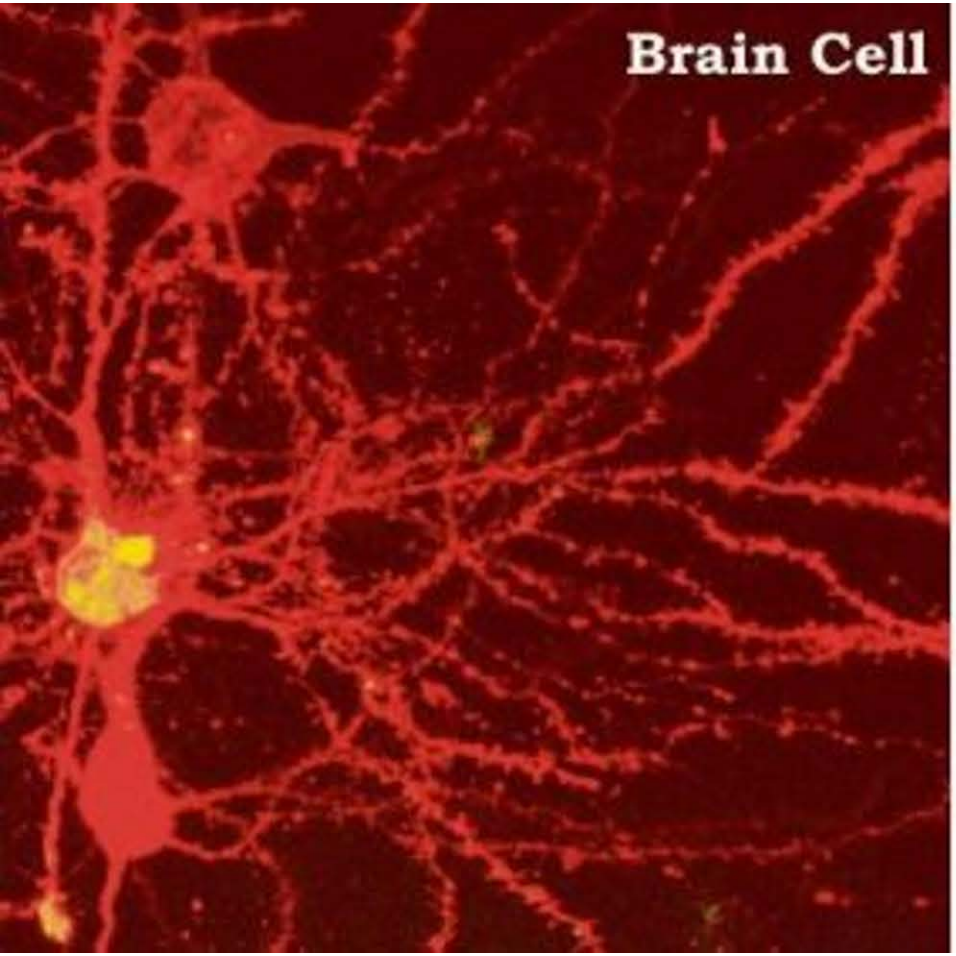
Universe in the Brain?

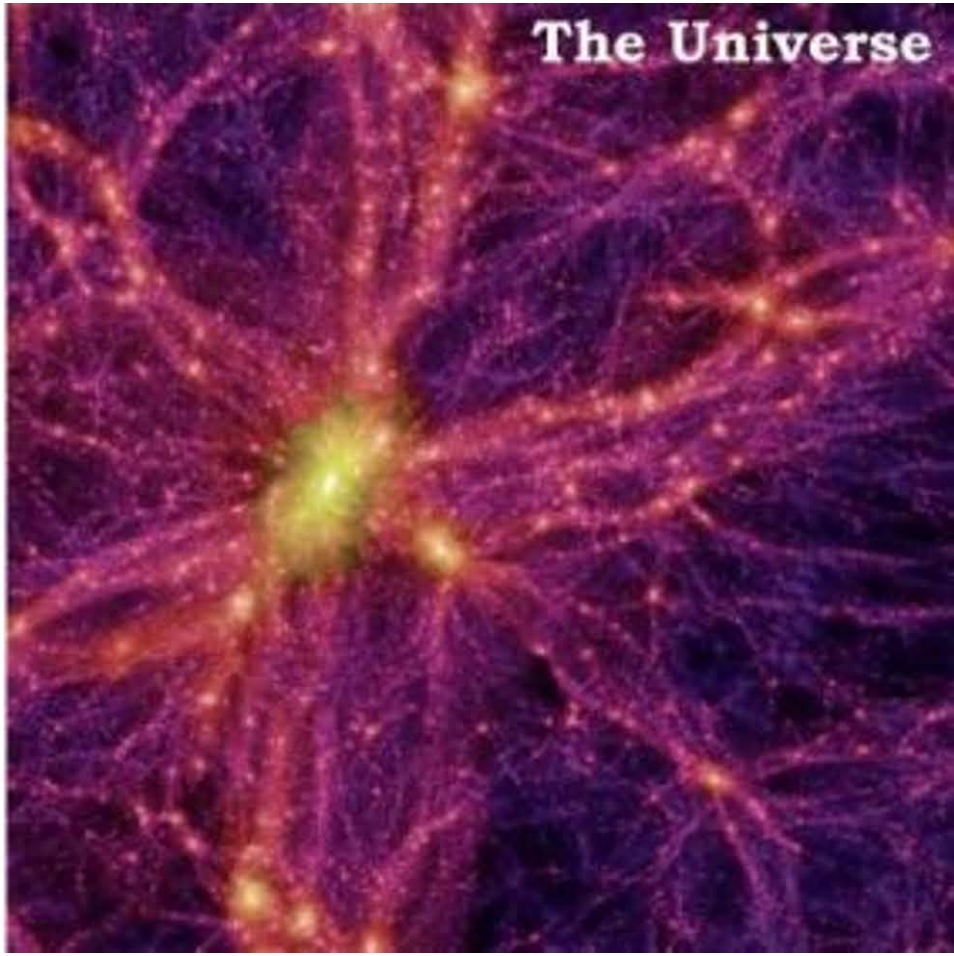




I think therefore I am - Descartes

Universe in the Brain?





I think therefore I am, therefore the Universe is - Pratyush

Conclusions

- Topology ideal tool for studying complex spatial patterns manifested in the universe
- Persistence topology: powerful tool for hierarchical topological description of LSS and CMB
- (Persistence) Homology offers new insights into structure, complexity and connectivity of the Cosmic Web
- Persistence provides rich language for description of multi-scale (hierarchical) topology of cosmic structure
- Persistence & ratio intensity maps: unprecedented detailed topological description (model discrimination)
- Ratio maps highly sensitive to deviations from gaussianity : Powerful probes for non-gaussianity
- Geometric connections : pattern/shape identification and description (cosmic filaments)

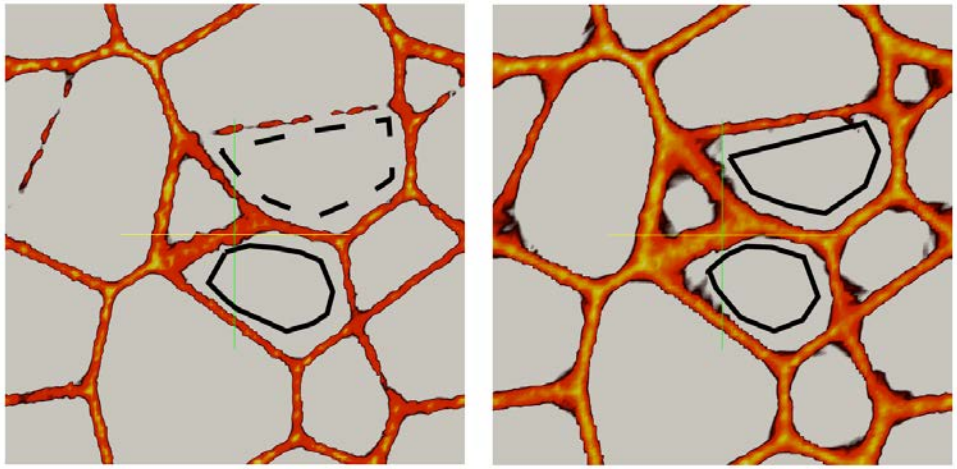
Holes and Homology

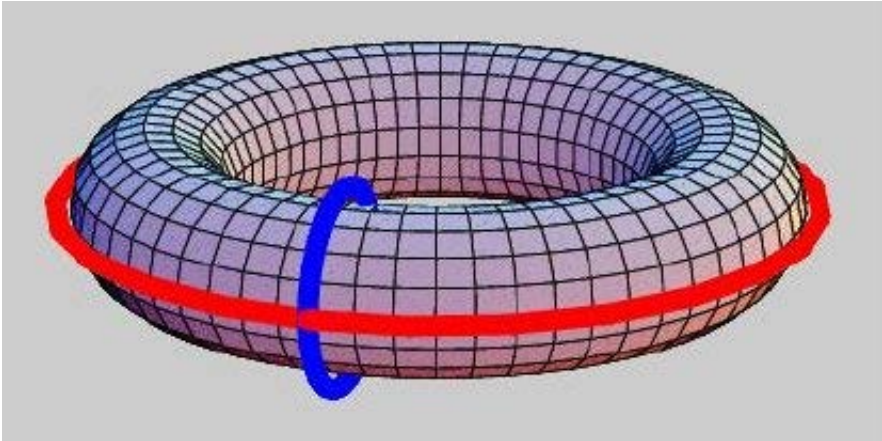
- **Homology : algebraic formalization of topological holes in terms of chain, cycle and boundary groups**

- **p^{th} homology group H_p : collection of independent p-dimensional cycles**

- **Homology groups : analogous to vector spaces**

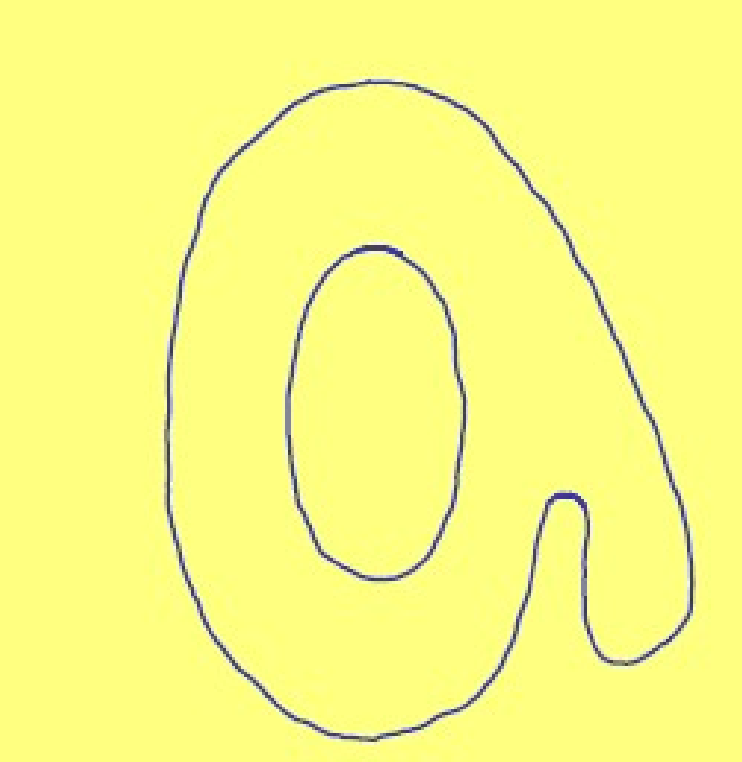
- **β_p : p-dimensional Betti number (rank of the pth homology group H_p)**





Morse Functions and critical points

- **Topology changes oly when crossing critical points**



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» Gradient ∇f

» Critical points $\nabla f = 0$

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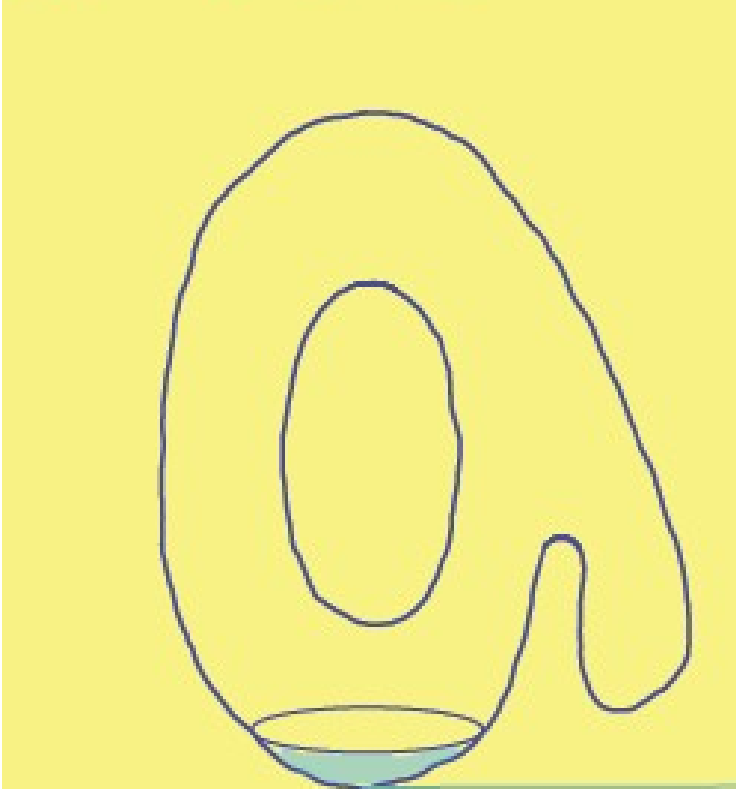

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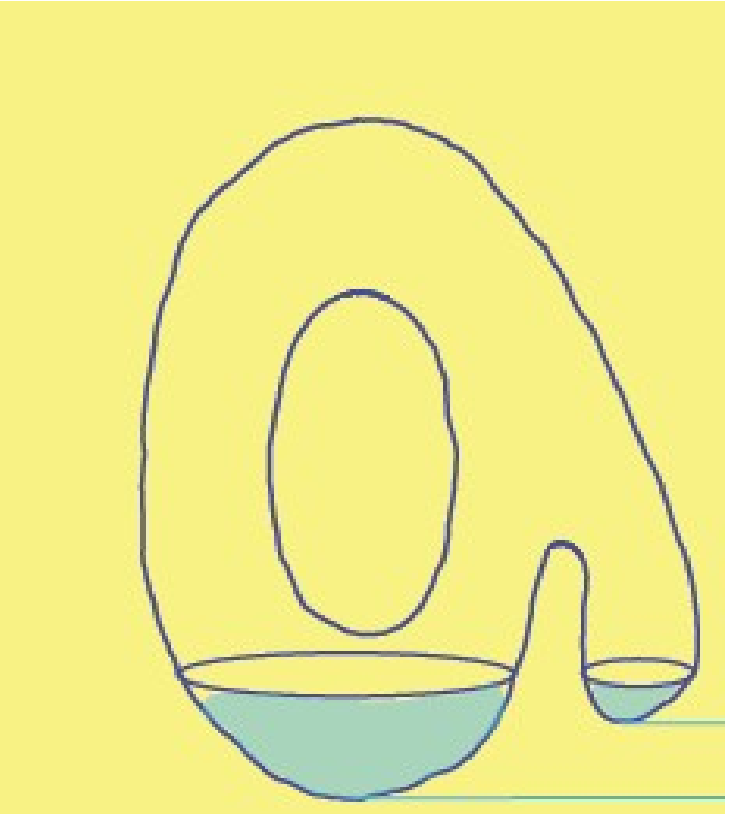

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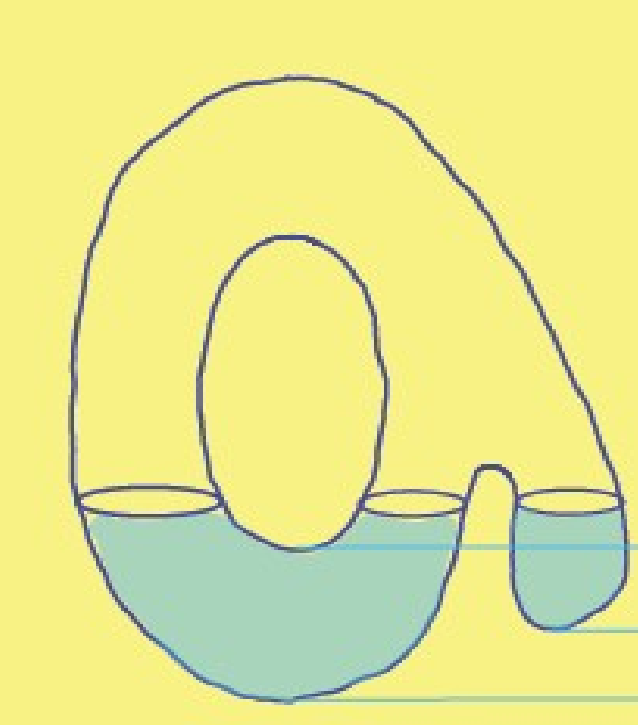

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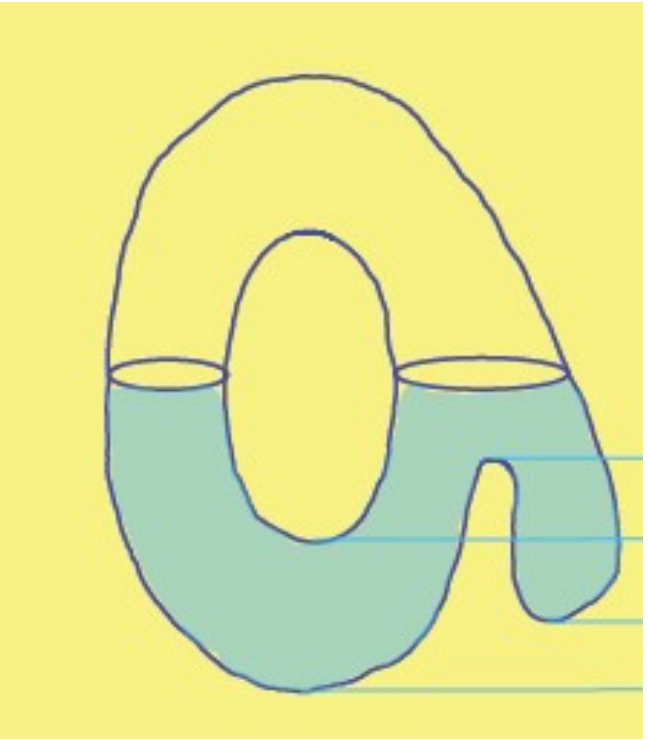

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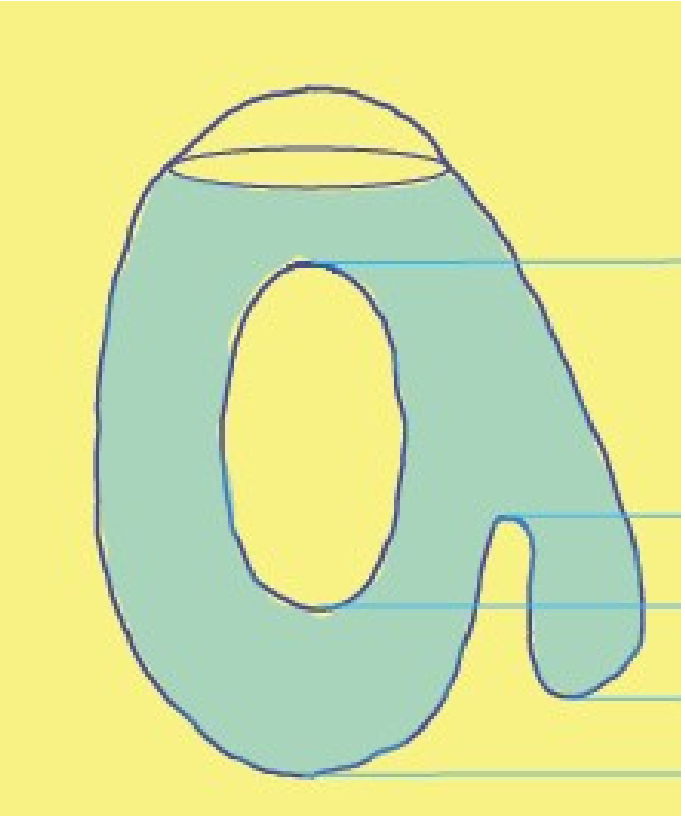

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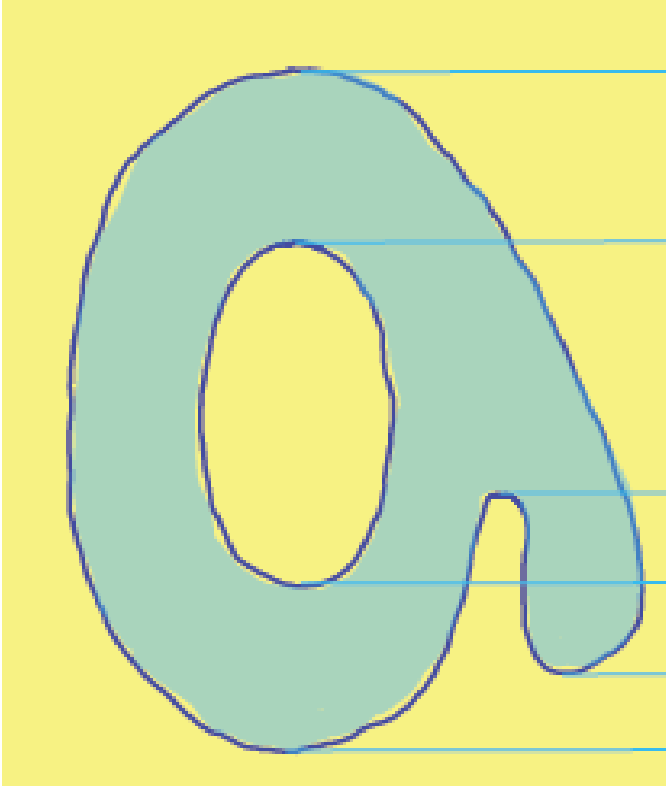

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