Persistent Holes in the Universe

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Outline

• Cosmic web & CMB: description and challenges
• Why (multi-scale) topology?
• Persistence: multi-scale topology
• Gaussian random fields & LSS: results
• Morse geometry and filament detection
• Conclusion
Cosmic Web & Cosmic Microwave Background
Cosmic Web: complications

- Discretely sampled: galaxies (observation) & particles (N-body simulations)
- Complex structural connectivity
- Lack of structural symmetry
- Intrinsic multi-scale nature
- Wide range of densities
Cosmic Microwave Background: Edge of the visible universe

- Progenitor of LSS: quantum fluctuations in infant Universe (CMB)

- Earliest View of our Cosmos: the Universe 379,000 years after the Big Bang

- Described as near-perfect (?) gaussian random field

\[
P_N = \frac{\exp \left[ -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} f_i (M^{-1})_{ij} f_j \right]}{\left[ (2\pi)^N (\text{det } M) \right]^{1/2}} \prod_{k=1}^{N} df_k
\]
Correlation functions: Structural Insensitivity

2-pt correlation function is highly insensitive to the geometry & morphology of weblike patterns:

compare 2 distributions with same $\xi(r)$, cq. $P(k)$, but totally different phase distribution

$$\xi(r') = \left( \frac{r}{r_0} \right)^{-\gamma}$$

$\gamma \approx 1.8$

$r_0 \approx 5h^{-1}\text{Mpc}$
Topology, Manifolds & topological holes
Why Topology?

- Cosmic web: a complex pattern of connected structures
- Topology studies connectivity
- Intrinsic topology insensitive to trivial change in the shape of structure (stretching, compression) or trivial coordinate transformation (expansion/contraction, distortion, rotation)
- In the cosmic context: topology insensitive to gravitational amplification and redshift distortion
Manifolds

» Euclidean space \( \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3 \)
» Locally Euclidean \( M^1, M^2, M^3 \)
Functions on Manifolds

- Height of depth of the surface of earth with respect to sea level
- Primordial fluctuation field, CMB
- Grid-densities of N-body simulations
Topological holes

- Topology in terms of k-dimensional holes
- Betti numbers $\beta_k(k=0,\ldots,d)$: count the no. of k-dimensional holes

0 dimensional holes: connected objects
1 dimensional holes: loops/tunnels
2 dimensional holes: voids
Are all holes important?

- Need to distinguish structures at different scales (cosmic web is multi-scale)

- Some holes are *noise*: discrete sampling, instrumental errors etc

- A need to assign a relative weight
Persistence:
Multi-scale topology
Critical points: Change in topology

• Topological Structure of continuous field determined by singularities:
  - maxima
  - minima
  - saddle points

• Topology changes only while crossing a critical point
Persistence: Multi-scale topology

- Filtration: Study the change in topology as we sweep from highest to lowest function values.

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Persistence Diagrams

- Representation of multi-scale topology
- Dots in the diagram record birth and death
- A diagram for each ambient dimension of the manifold

- 0-dimensional diagram: representation of merger of isolated objects (merger trees?)
- 1-dimensional diagrams: formation and filling up of loops
- 2-dimensional diagrams: formation and destruction of topological voids
Persistence Diagrams: our preferred representation

- Rotation of co-ordinates

\[ D + B = \text{Mean density} \]

\[ D - B = \text{persistence} \]
A typical persistence diagram

Gaussian random field, power-law power spectrum, n=0 (all dimensions)
Statistics of persistence topology

• Topology connected to distribution of critical points

• Critical point distribution well defined for stochastic processes (GRF, CMB, LSS): probabilistic description

• Are the diagrams well defined and stable/convergent over many realization?

• Statistical description?
Discretize the diagram into regular grids

Empirical intensity function:

\[ \langle I_{ij} \rangle = \frac{\langle N_{ij} \rangle}{\langle N_{tot} \rangle} \]

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Persistence intensity maps: empirical probability description

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\]

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Statistical description?
Gaussian random field : Models

- LCDM, n=1
- Power-law, n=1
- Power-law, n=0
- Power-law, n=-1
- Power-law, n=-2
- Power-law, n=-3

Pranav et. al 2013
Gaussian random field: Persistence intensity maps

Pranav et. al 2013
Model comparison : Ratio intensity maps

- Local difference in topology

- Excess/deficit of topological objects in the neighborhood of the point defined by (mean density:pers)

Ratio function:

\[
\Delta_{ij}(f_1, f_2) = \frac{\left(\frac{\langle N_{ij} \rangle_{f_1}}{\langle N_{tot} \rangle_{f_1}}\right)}{\left(\frac{\langle N_{ij} \rangle_{f_2}}{\langle N_{tot} \rangle_{f_2}}\right)} \]

\[
= \left(\frac{\langle N_{ij} \rangle_{f_1}}{\langle N_{ij} \rangle_{f_2}}\right) \ast \frac{1}{\Omega_{(f_1, f_2)}}
\]

Pranav et. al 2013
• Novel and extremely detailed discrimination of topological differences between models

Pranav et. al 2013
Non-gaussianity: The untamable beast
Non-gaussianity: The untamable beast

- Constraining inflationary models
- Described as higher order terms in the expansion around a gaussian ($f_{NL}, g_{NL}$)
- Any function is potentially non-gaussian in nature!
- Cosmological principle: isotropy and homogeneity as constraints imply gaussian fields described strictly only by 2-point correlation function (a good constraint on separating gaussian from non-gaussian)
Ratio intensity maps: non-gaussian case

Pranav et. al 2013
Model comparison: Intensity ratio maps

- Local difference in topology
- Excess/deficit of topological objects in the local neighborhood of the point defined by (mean density: pers)

Ratio function:

\[
\Delta_i(j_{(f1,f2)}) = \frac{\langle N_{ij} \rangle_{f1}}{\langle N_{tot} \rangle_{f1}} / \frac{\langle N_{ij} \rangle_{f2}}{\langle N_{tot} \rangle_{f2}}
\]

Gaussian Random field, wrt n=0

Non-gaussian field, wrt n=0

Non-gaussian simulations, courtesy Licia Verde

Pranav et al. 2013b
Voronoi Kinematic Models:
Heuristic description of structural evolution in the Cosmos

- Persistence diagrams segregate structures remarkably well

Pranav et. al 2013a
Parameters of the model:
- Number of levels ($n$)
- Number of children ($\eta$)
- Ratio between the radius of parent and children spheres ($\lambda$)

Randomly place $\eta$ spheres inside the top-level and continue for all levels
Betti Numbers

- Topology of excursion sets in terms of k-dimensional holes
  \[ \beta_k = \# \text{k-dimensional holes} \]
  \[ \beta_0 = \text{connected components} \]
  \[ \beta_1 = \text{independent loops} \]
  \[ \beta_2 = \text{independent voids} \]

\[ \beta_0 = 1, \beta_1 = 2, \beta_2 = 1 \]
Betti numbers:
Euler Characteristic and genus extended

- Euler characteristic is a compression of topological information
  \[ \chi = \sum_i (-1)^i \beta_i = \sum_i (-1)^i c_i \]
  \[ \chi = \beta_0 - \beta_1 + \beta_2 - \cdots (-1)^d \beta_d \]

- Genus defined only for 2D surfaces (number of independent ways of cutting a 2D surface without leaving it disconnected): no satisfactory generalization in lower and higher dimensions(!)
Gaussian random fields: Betti numbers and genus

\[ g = -\frac{1}{8\pi^2} \left( \frac{\langle k^2 \rangle}{3} \right)^{3/2} (1 - \nu^2) e^{-\nu^2/2} \]
Morse geometry & Filament detection
Morse Functions and critical points

- Domain: $R, R^2, R^3$
- Gradient: $\nabla f$
- Critical points: $\nabla f = 0$
- Smooth functions: $\nabla^2 f$

(a) Minimum, 0, •
(b) Saddle, 1, ⊕
(d) Monkey Saddle, ★
(c) Maximum, 2, ○
Morse geometry:
Critical points, gradient lines & descending/ascending manifolds

Gradient lines
Morse geometry:
Filaments as ascending manifolds of 2-saddles
Morse geometry:
Filaments as ascending manifolds of 2-saddles

Shivshankar, Pranav et. al
2013
Spin alignment of DM haloes along filaments

Pranav et. al 2013c
Interactive filament detection:
New software
Universe in the Brain?

I think therefore I am - Descartes
Universe in the Brain?

I think therefore I am, therefore the Universe is - Pratyush
Conclusions

• Topology ideal tool for studying complex spatial patterns manifested in the universe

• Persistence topology: powerful tool for hierarchical topological description of LSS and CMB

• (Persistence) Homology offers new insights into structure, complexity and connectivity of the Cosmic Web

• Persistence provides rich language for description of multi-scale (hierarchical) topology of cosmic structure

• Persistence & ratio intensity maps: unprecedented detailed topological description (model discrimination)

• Ratio maps highly sensitive to deviations from gaussianity: Powerful probes for non-gaussianity

• Geometric connections: pattern/shape identification and description (cosmic filaments)
Holes and Homology

- Homology: algebraic formalization of topological holes in terms of chain, cycle and boundary groups

- $p^{\text{th}}$ homology group $H_p$: collection of independent $p$-dimensional cycles

- Homology groups: analogous to vector spaces

- $\beta_p$: $p$-dimensional Betti number (rank of the $p^{\text{th}}$ homology group $H_p$)
Morse Functions and critical points

- Topology changes only when crossing critical points

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» Gradient \( \nabla f \)

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