MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 9, 2013: Create your own homology.

Fall 2013 course offered through the
University of Iowa Division of Continuing Education

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http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

Homology

3 ingredients:

1.) Objects

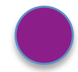
2.) Grading

3.) Boundary map

1.) OBJECTS

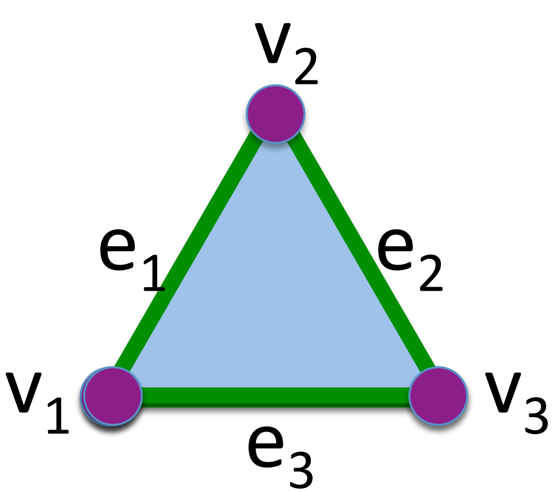
Building blocks for a simplicial complex

0-simplex = vertex = v



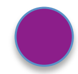
1-simplex = edge = $\{v_1, v_2\}$

2-simplex = triangle = $\{v_1, v_2, v_3\}$



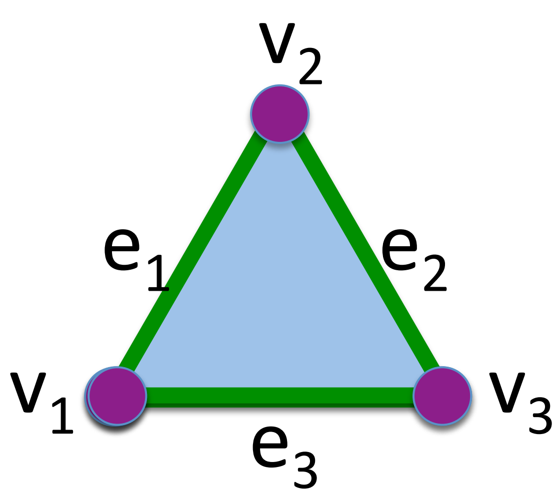
Building blocks for a simplicial complex

0-simplex = vertex = v



1-simplex = edge

2-simplex = triangle



Building blocks for a simplicial complex

0-simplex = vertex = v

1-simplex = edge = $\{v_1, v_2\}$

2-simplex = triangle = $\{v_1, v_2, v_3\}$

Objects = generators

Generators = X = { x_α | α in A }

Let **R** be a ring (or field)

$\mathbf{R}[X]$ = { $n_1x_1 + n_2x_2 + \ldots + n_kx_k$: x_i in **R** }

Ex: $\mathbf{Z_2}[X]$ = { $n_1x_1 + n_2x_2 + \ldots + n_kx_k$: x_i in $\mathbf{Z_2}$ }

A free vector space over the field **F** generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

where n_i in **F**.

Examples of a field: **R** = set of real numbers

Q = set of rational numbers

$\mathbf{Z_2} = \{0, 1\}$

Addition:

$$(n_1x_1 + n_2x_2 + \ldots + n_kx_k) + (m_1x_1 + m_2x_2 + \ldots + m_kx_k)$$

$$= (n_1 + m_1)\, x_1 + (n_2 + m_2)x_2 + \ldots + (n_k + m_k)x_k$$

A free vector space over the field **F** generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

where n_i in **F**.

Examples of a field:

R = set of real numbers:

$$\pi x + \sqrt{2}\, y + 3z \text{ is in } \mathbf{R}[x, y, z]$$

Q = set of rational numbers (i.e. fractions):

$$(½)x + 4y \text{ is in } \mathbf{Q}[x, y]$$

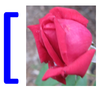
$\mathbf{Z_2}$ = {0, 1}: $0x + 1y + 1w + 0z$ is in $\mathbf{Z_2}[x, y, z, w]$

Slide from preparatory lecture 4: Addition (and free vector spaces)

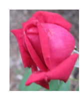
A free abelian group generated by the elements $x_1, x_2, \ldots, x_k$ consists of all elements of the form

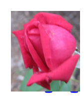
$$n_1x_1 + n_2x_2 + \ldots + n_kx_k$$

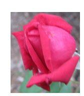
where n_i are integers.

Example: **Z**[, ]

4  + 2 

 – 2 

-3 

k  + n 

Z = The set of integers = { …, -2, -1, 0, 1, 2, …}

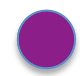
2.) GRADING

Grading

Grading: Each object is assigned a unique grade

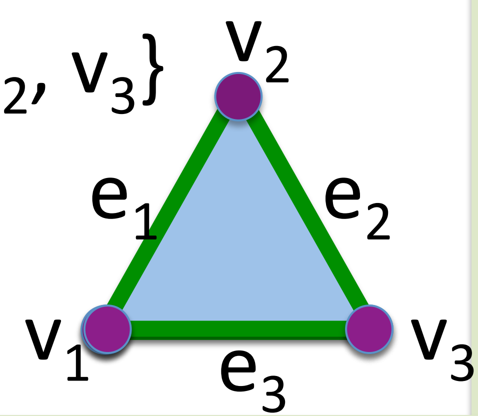
Grading = Partition of R[x]

<u>Ex: Grade = dimension</u>

Grade 0: **0-simplex = vertex** = v 

Grade 1: **1-simplex = edge** = $\{v_1, v_2\}$

Grade 2: **2-simplex = triangle** = $\{v_1, v_2, v_3\}$



Grading

Grading: Each object is assigned a unique grade

Grading = Partition of R[x]

Ex: Grade: Cardinality

Grade 0: **0-simplex = vertex** = {v}

Grade 1: **1-simplex = edge** = $\{v_1, v_2\}$

Grade 2: **2-simplex = triangle** = $\{v_1, v_2, v_3\}$

Grade 3: **3-simplex = tetrahedron** = $\{v_1, v_2, v_3, v_4\}$

Grading

Grading: Each object is assigned a unique grade.

Grading = Partition of R[x].

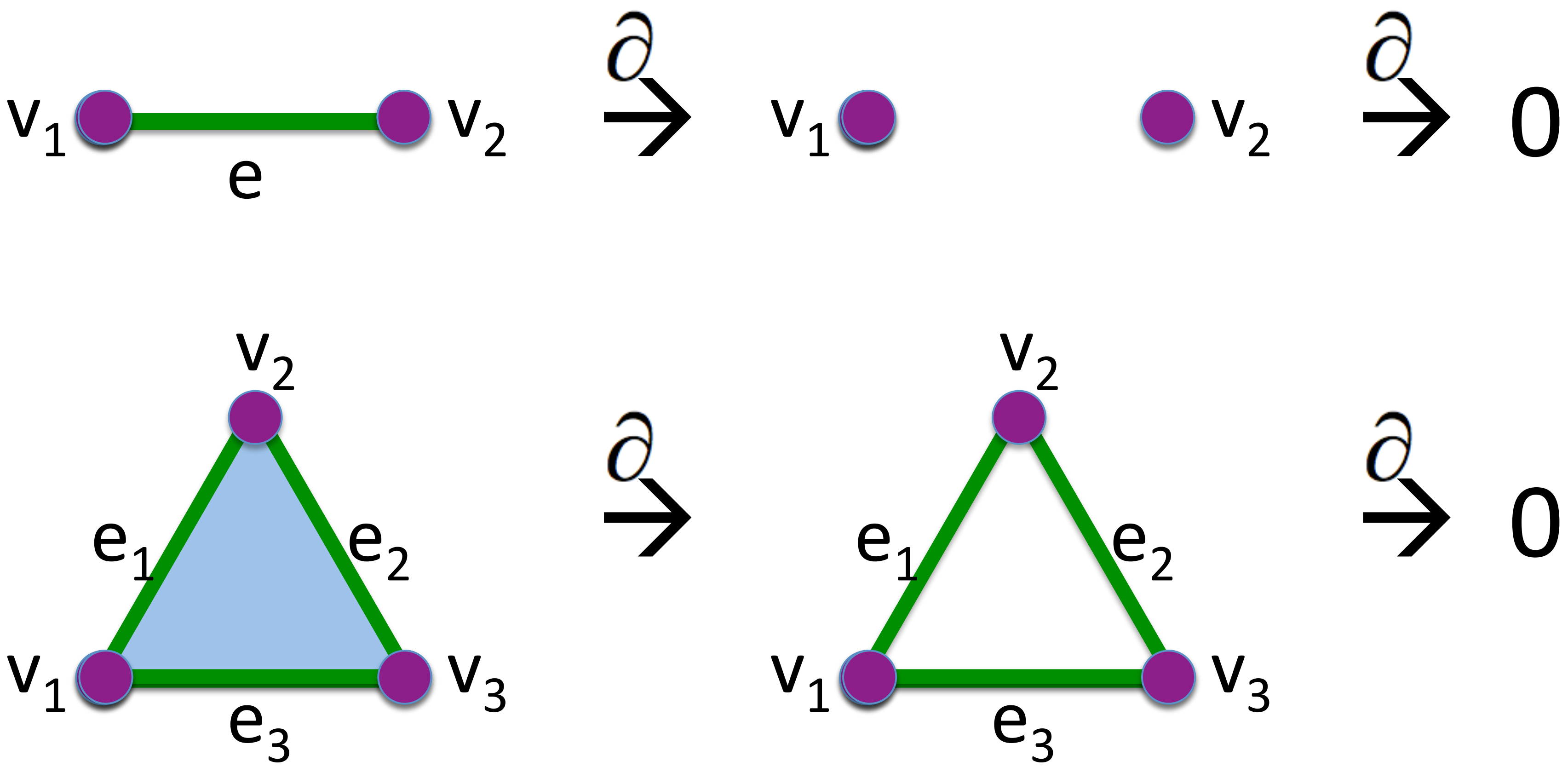
Let $X_n = \{x_1, \ldots, x_k\}$ = generators of grade n.

C_n = set of n-chains = $R[X_n]$

3.) BOUNDARY MAP

Boundary Map

$$\partial_n : C_n \rightarrow C_{n-1} \text{ such that } \partial^2 = 0$$



Boundary Map

$$\partial_n : C_n \rightarrow C_{n-1} \text{ such that } \partial^2 = 0$$

$$\{v_1, v_2\} \xrightarrow{\partial} \{v_1\} + \{v_2\} \xrightarrow{\partial} 0$$

$$\{v_1, v_2, v_3\} \xrightarrow{\partial} \{v_1, v_2\} + \{v_1, v_3\} + \{v_2, v_3\} \xrightarrow{\partial} 0$$

$$C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \rightarrow . . . \rightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$H_n = Z_n/B_n = (\text{kernel of } \partial_n)/\ (\text{image of } \partial_{n+1})$$

$$= \frac{\text{null space of } M_n}{\text{column space of } M_{n+1}}$$

$$\text{Rank } H_n = \text{Rank } Z_n - \text{Rank } B_n$$

Your name homology

3 ingredients:

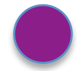
1.) Objects

2.) Grading

3.) Boundary map

Unoriented simplicial complex using $\mathbf{Z_2}$ coefficients

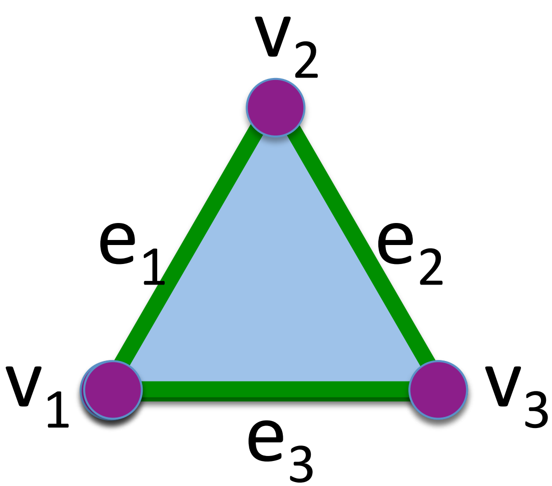
Grading = dimension

0-simplex = vertex = v 

1-simplex = edge = $\{v_1, v_2\}$

Note that the boundary of this edge is $v_2 + v_1$

2-simplex = face = $\{v_1, v_2, v_3\}$

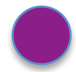


Note that the boundary of this face is the cycle

$$e_1 + e_2 + e_3$$

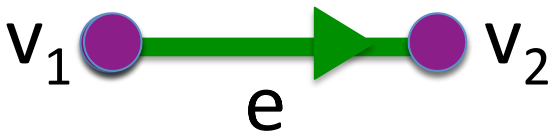
$$= \{v_1, v_2\} + \{v_2, v_3\} + \{v_1, v_3\}$$

Oriented simplicial complex

0-simplex = vertex = v 

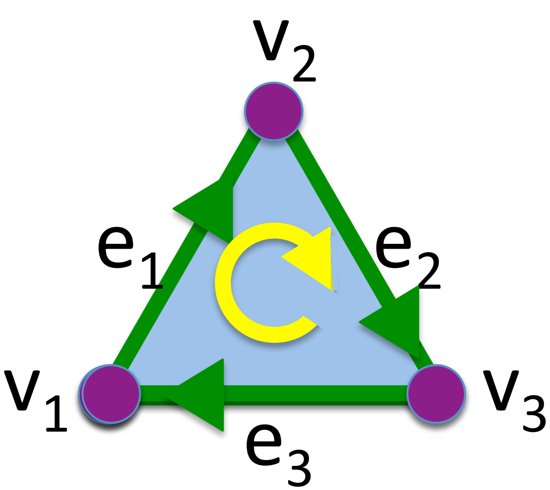
Grading = dimension

1-simplex = oriented edge = (v_1, v_2)



Note that the boundary of this edge is $v_2 - v_1$

2-simplex = oriented face = (v_1, v_2, v_3)



Note that the boundary of this face is the cycle

$$e_1 + e_2 + e_3$$

$$= (v_1, v_2) + (v_2, v_3) - (v_1, v_3)$$

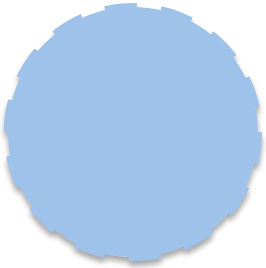
$$= (v_1, v_2) - (v_1, v_3) + (v_2, v_3)$$

Cell complex

Building block: n-cells = { x in R^n : || x || ≤ 1 }

Examples: 0-cell = { x in R^0 : ||x || < 1 } 

1-cell =open interval ={ x in R : ||x || < 1 } 

2-cell = open disk = { x in R^2 : ||x || < 1 } 

$\boxed{\text{Grading = dimension}}$

∂_n(n-cells) = { x in R^n : || x || = 1 }

Čech homology

Given $\bigcup_{\alpha \text{ in } A} V_\alpha$ where V_α open for all α in A.

Objects = finite intersections = $\{ \bigcap_{i=1}^{n} V_{\alpha_i} : \alpha_i \text{ in } A \}$

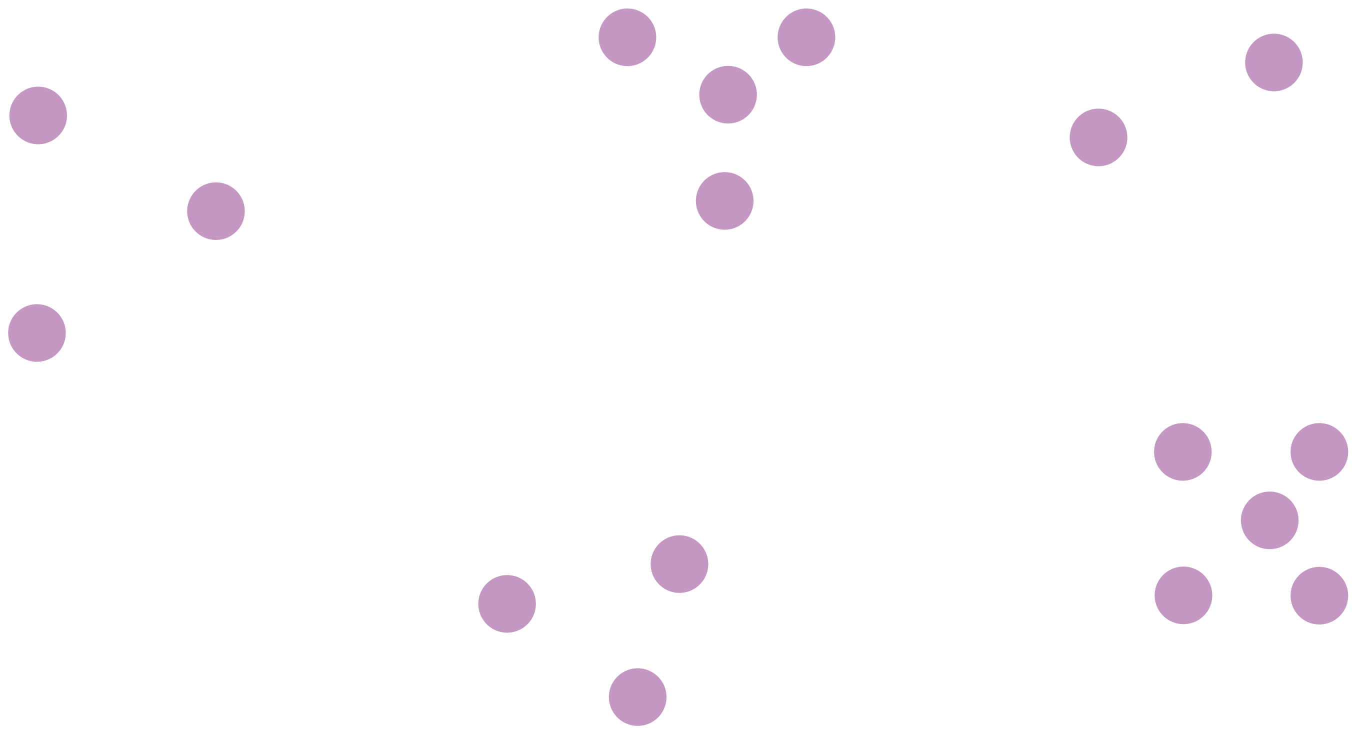
Grading = n = depth of intersection.

$$\partial_{n+1} \left(\bigcap_{i=1}^{n} V_{\alpha_i} \right) = \sum_{j=1}^{n} \left(\bigcap_{\substack{i=1 \\ i \neq j}}^{n} V_{i\alpha} \right)$$

Ex: $\partial_0(V_\alpha) = 0$, $\partial_1(V_\alpha \cap V_\beta) = V_\alpha + V_\beta$

$$\partial_2(V_\alpha \cap V_\beta \cap V_\gamma) = (V_\alpha \cap V_\beta) + (V_\alpha \cap V_\gamma) + (V_\beta \cap V_\gamma)$$

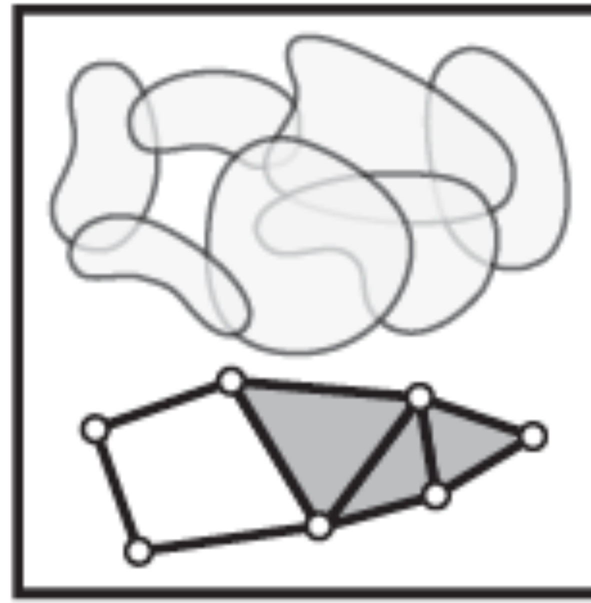
Creating the Čech simplicial complex



Consider X an arbitrary topological space.

Let $V = \{V_i \mid i = 1, \ldots, n\}$ where $V_i \quad X$,

The nerve of $V = N(V)$ where



The k -simplices of $N(V)$ =

nonempty intersections of

k +1 distinct elements of V.

For example,

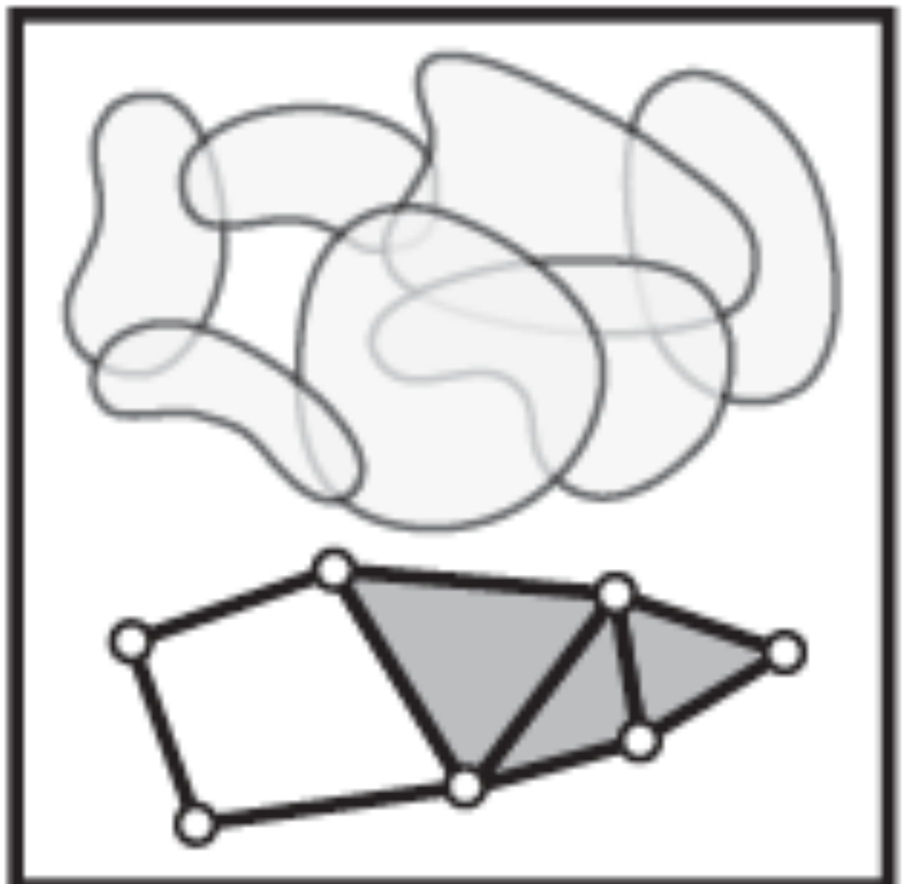
Vertices = elements of V

Edges = pairs in V which intersect nontrivially.

Triangle = triples in V which intersect nontrivially.

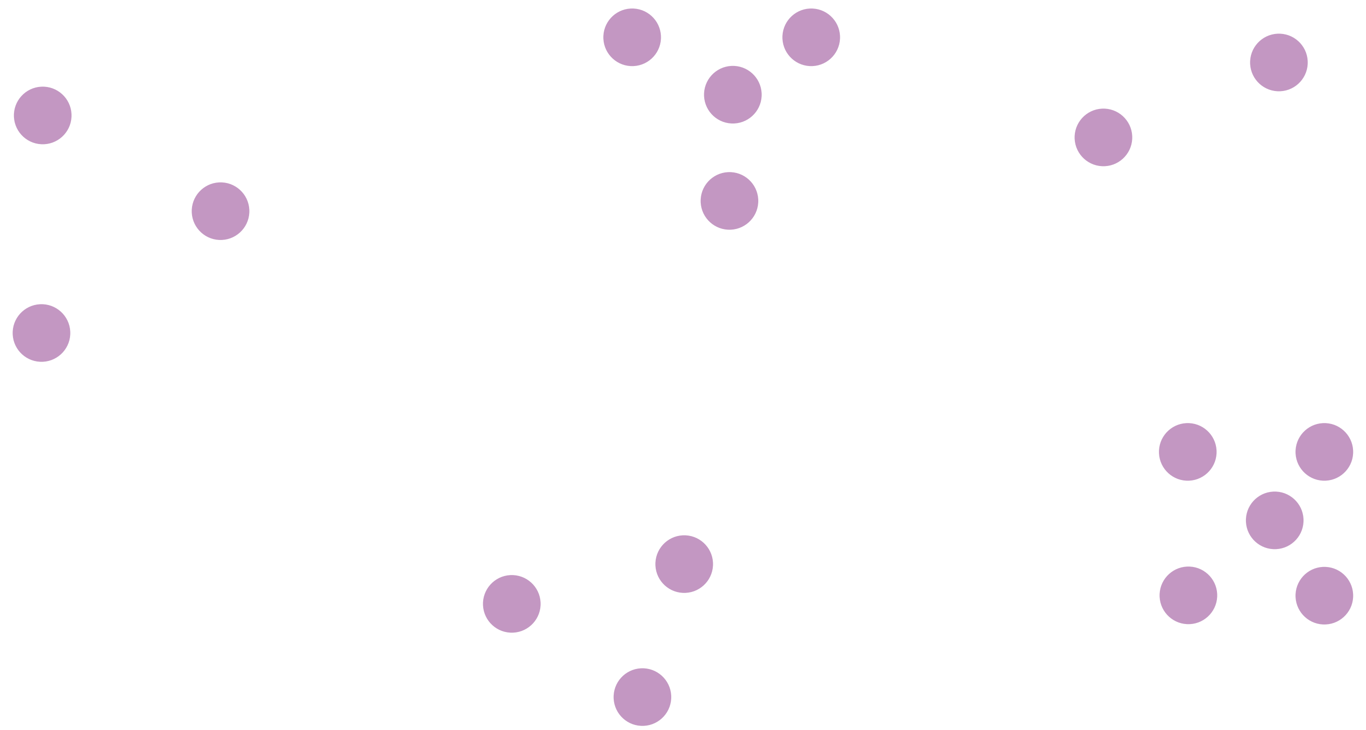
http://www.math.upenn.edu/~ghrist/EAT/EATchapter2.pdf

Nerve Lemma: If *V* is a finite collection of subsets of X with all non-empty intersections of subcollections of *V* contractible, then N(V) is homotopic to the union of elements of *V*.



http://www.math.upenn.edu/~ghrist/EAT/EATchapter2.pdf

Creating the Vietoris Rips simplicial complex



Journal of Vision (2008) 8(8):11, 1–18 http://journalofvision.org/8/8/11/

Topological analysis of population activity in visual cortex

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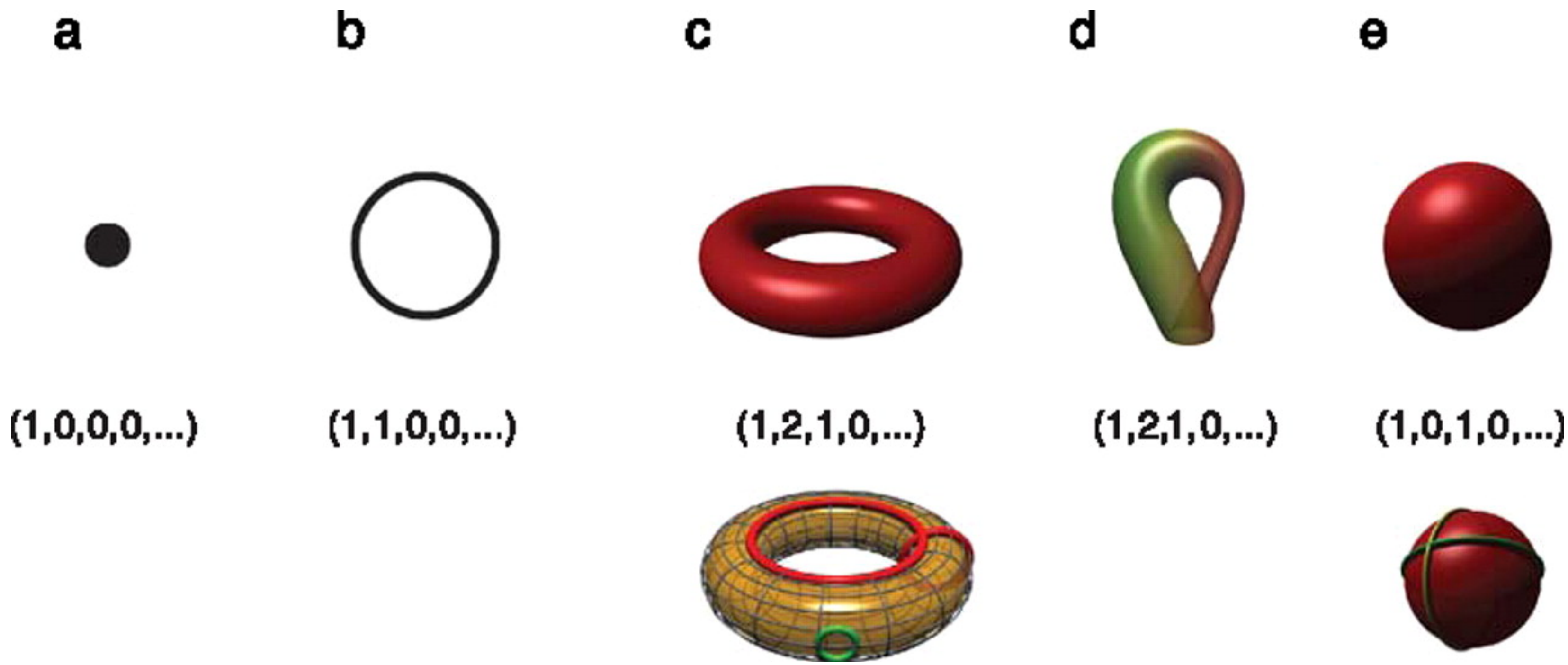
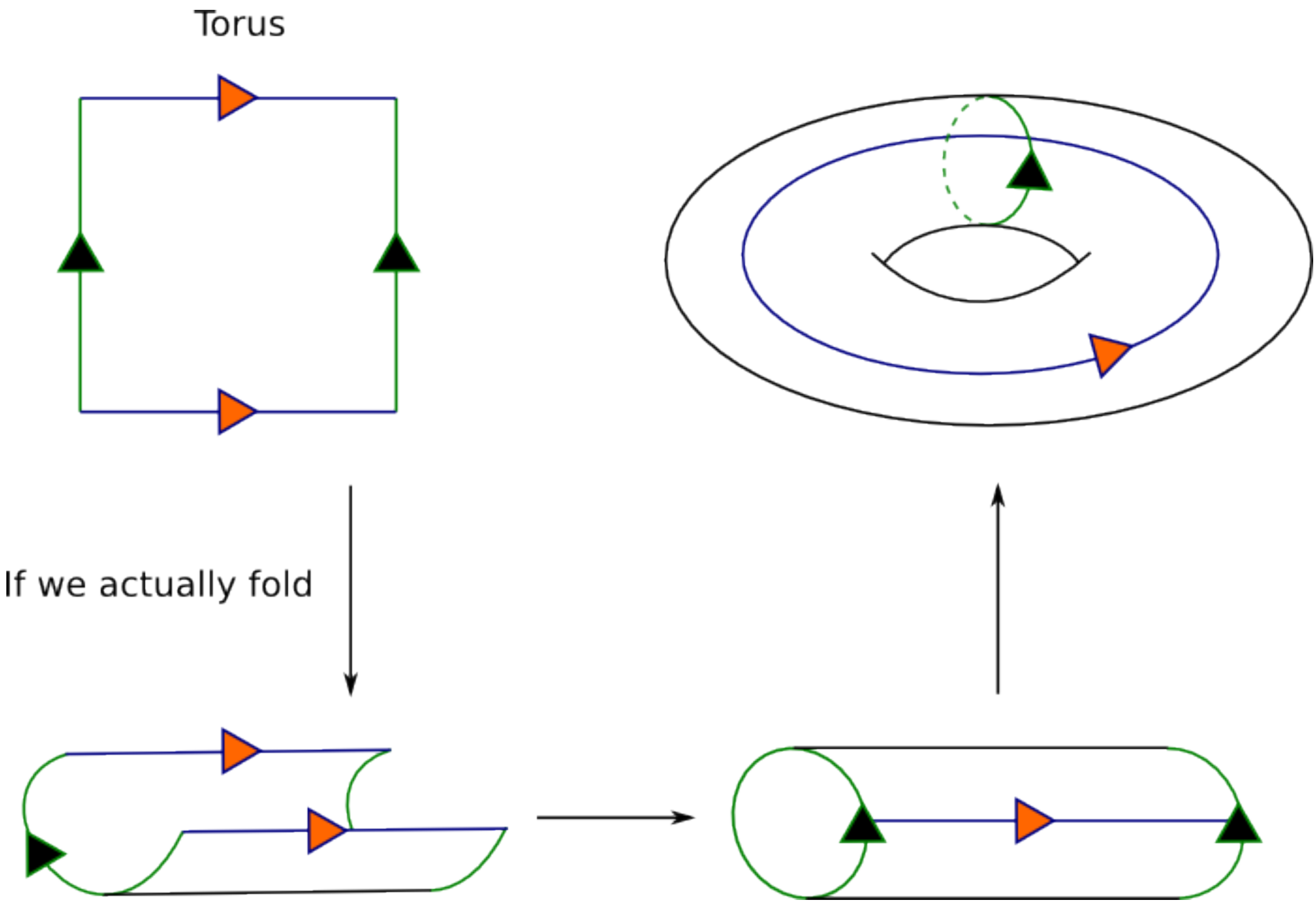
Betti numbers provide a signature of the underlying topology.

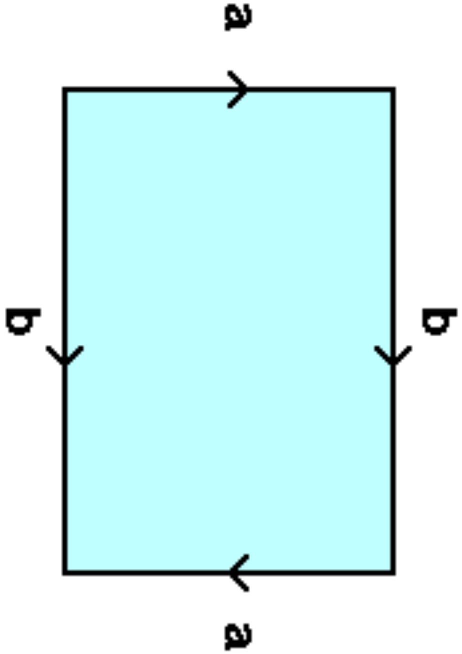


Singh G et al. J Vis 2008;8:11

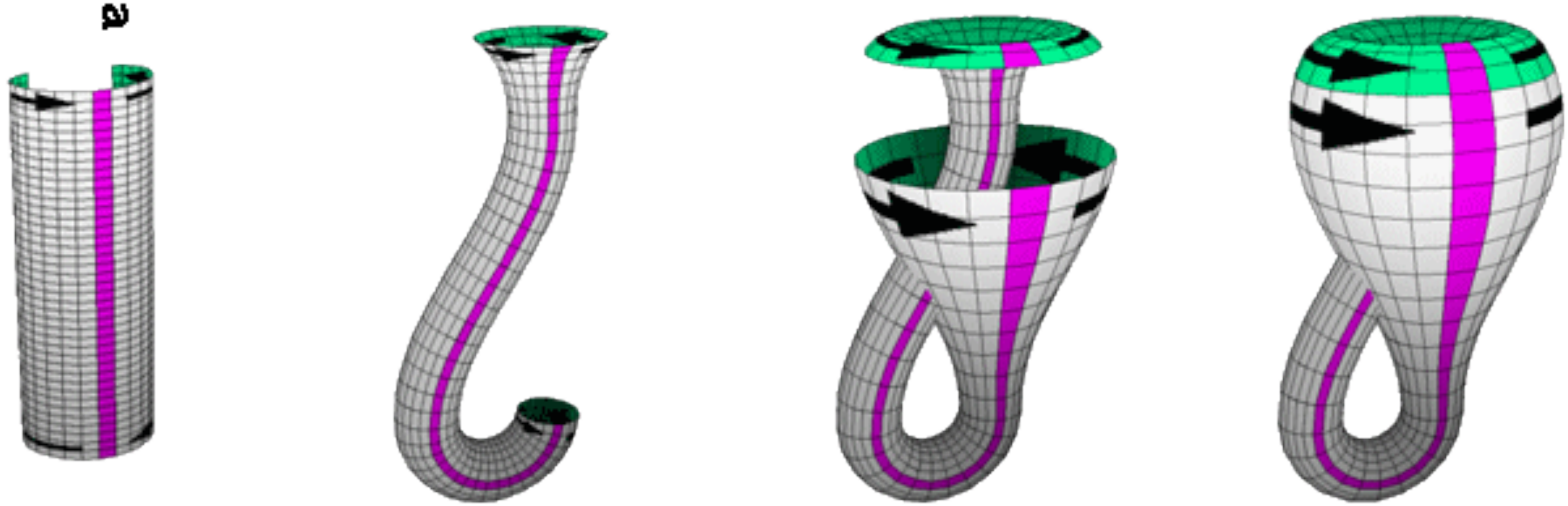


From: http://www.math.cornell.edu/~mec/Winter2009/Victor/part1.htm

From: http://www.math.osu.edu/~fiedorowicz.1/math655/Klein2.html



Klein Bottle



From:
http://plus.maths.org/content/imaging-maths-inside-klein-bottle

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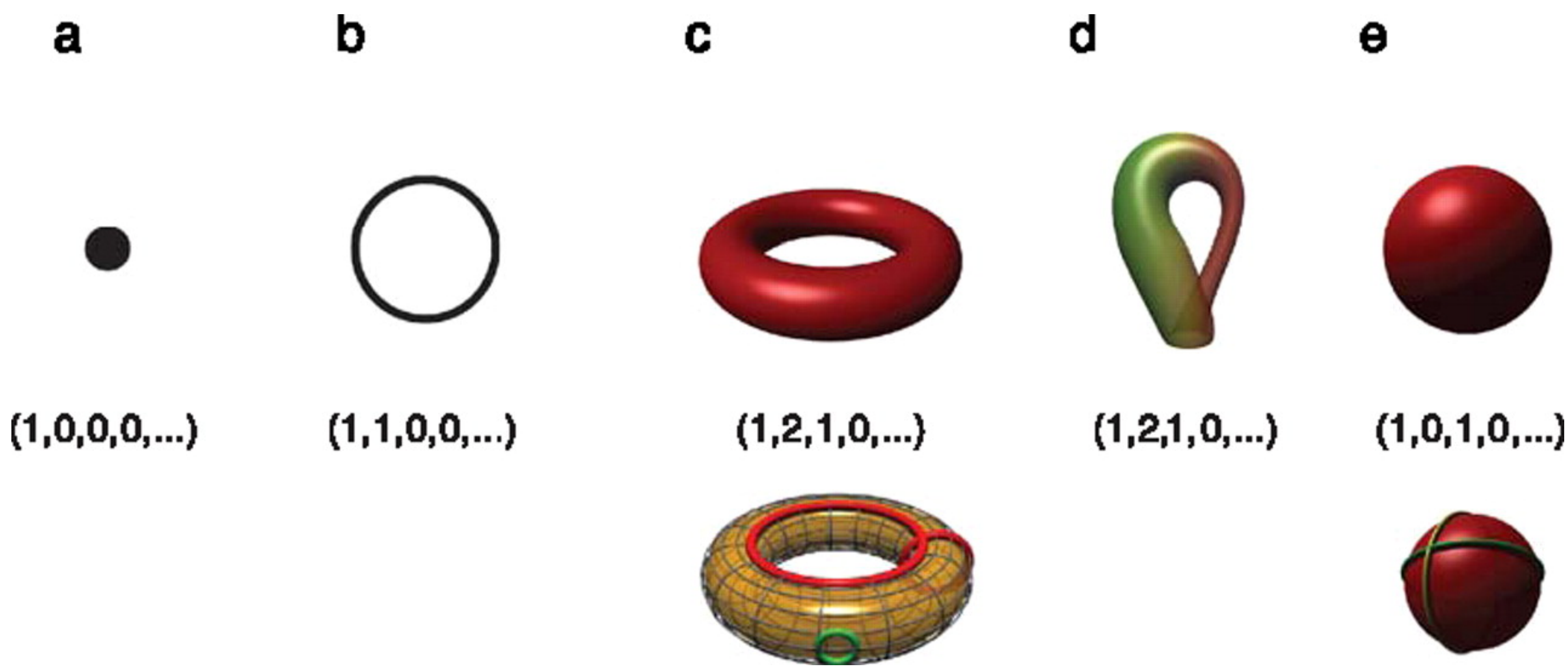
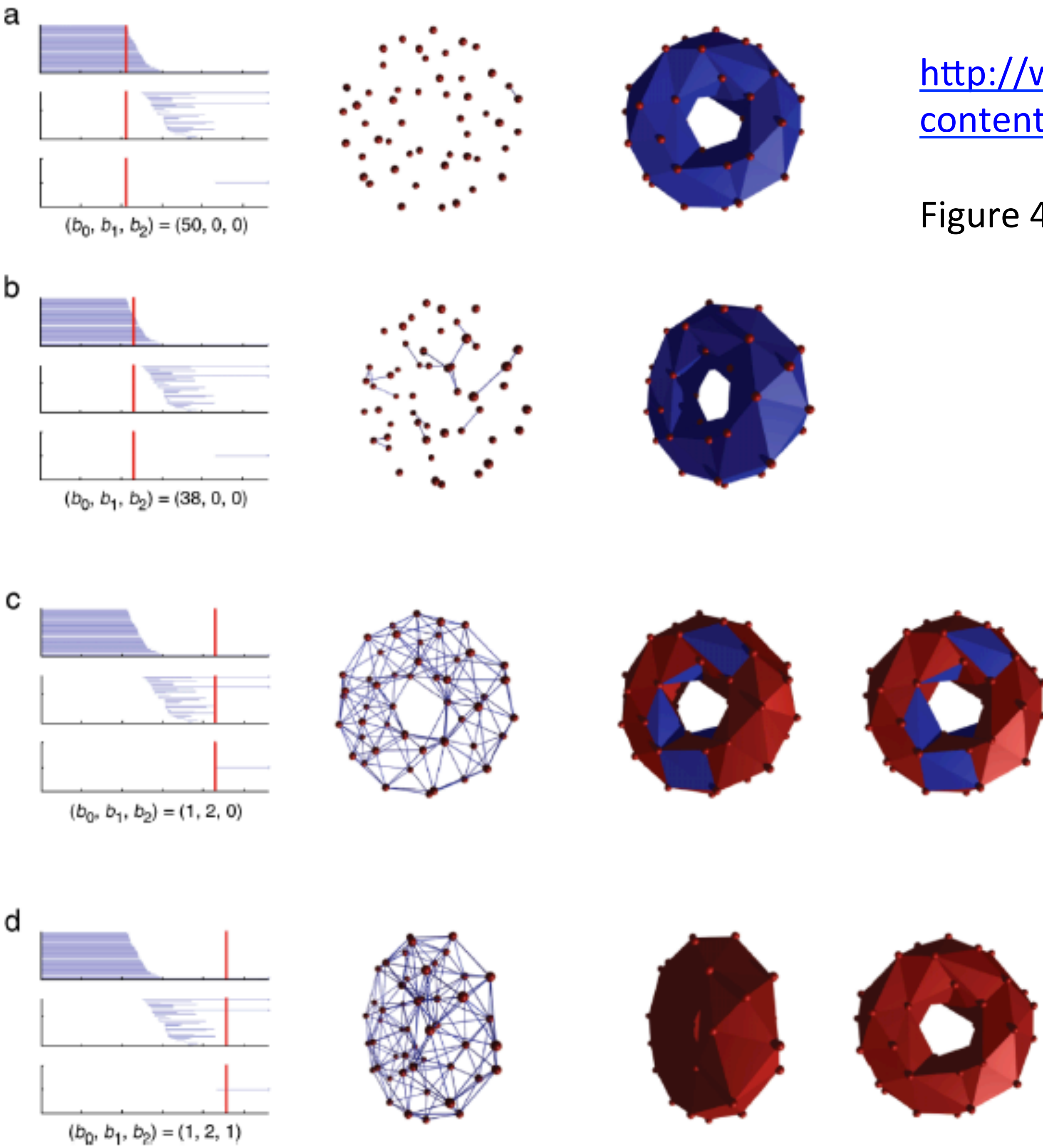


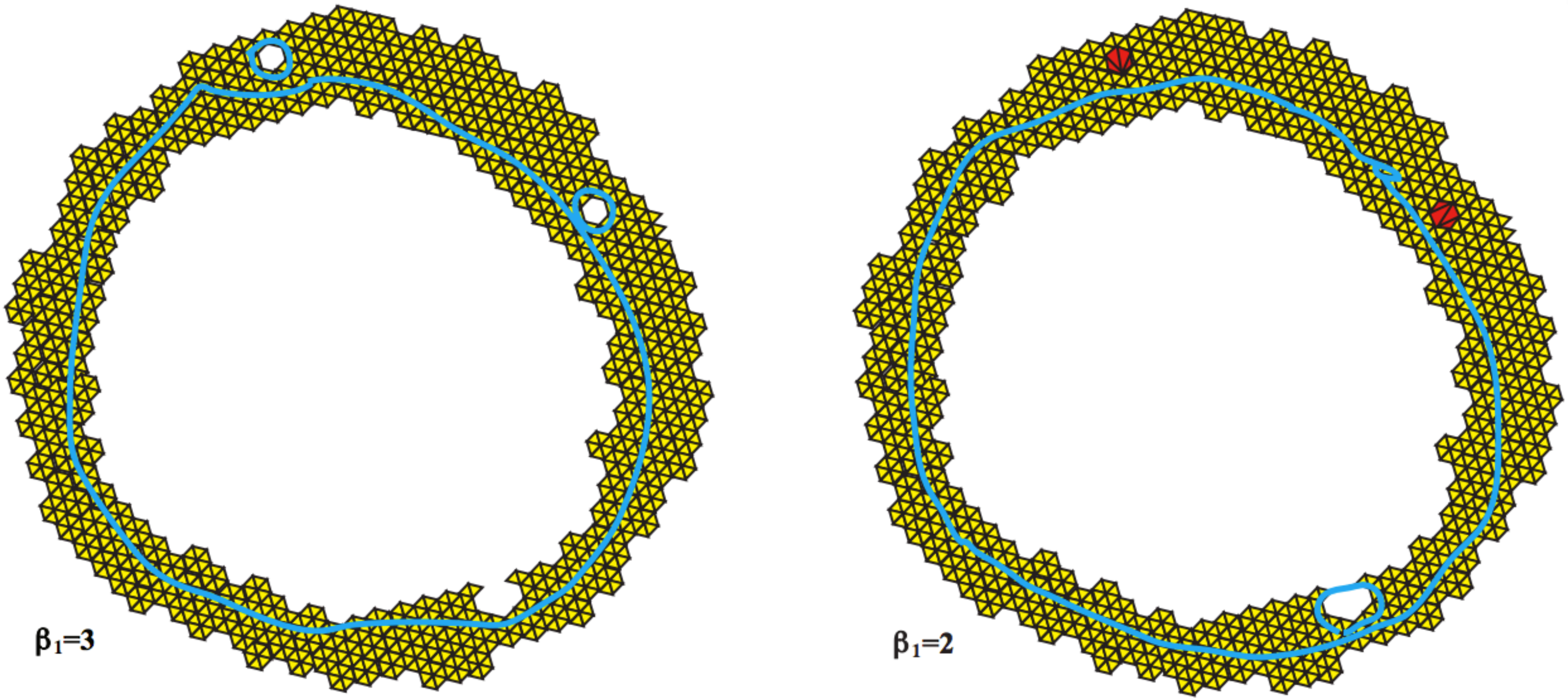
Singh G et al. J Vis 2008;8:11

Journal of VISION



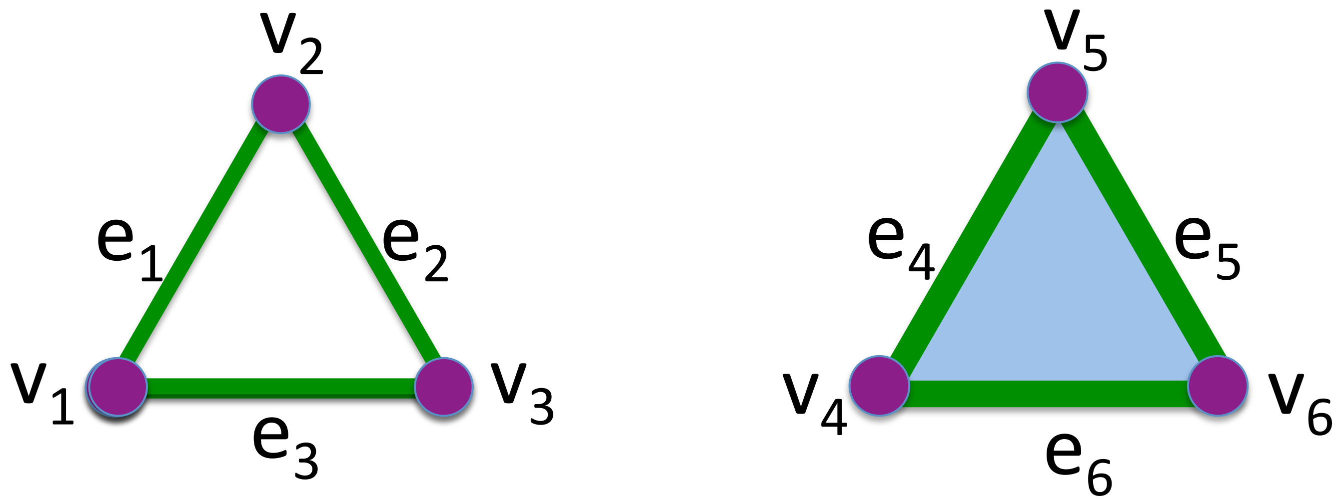
http://www.journalofvision.org/content/8/8/11.full

Figure 4 animation

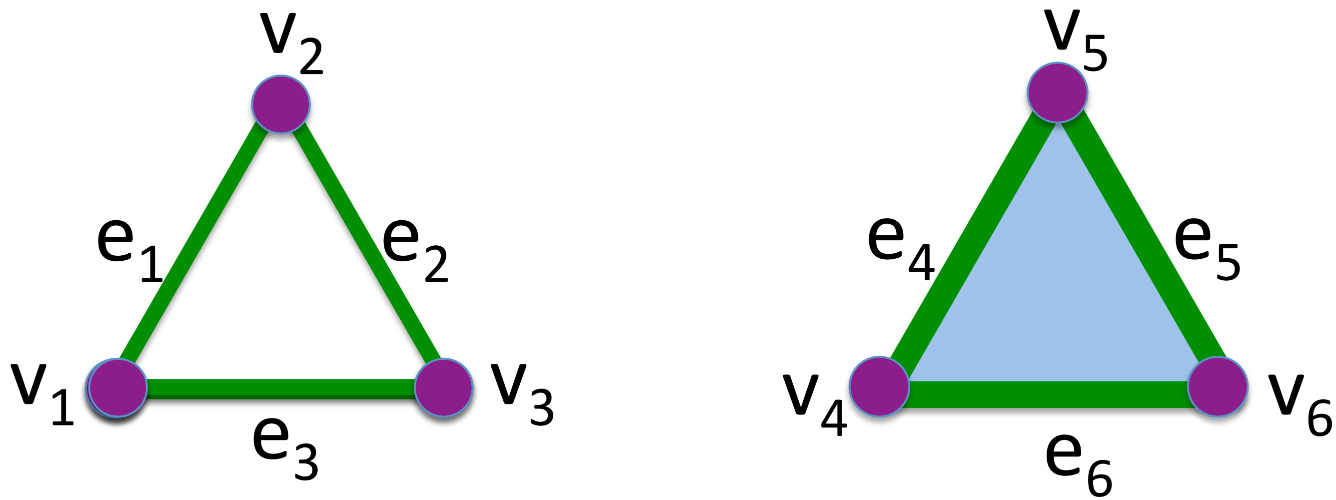


Topology and Data. Gunnar Carlsson

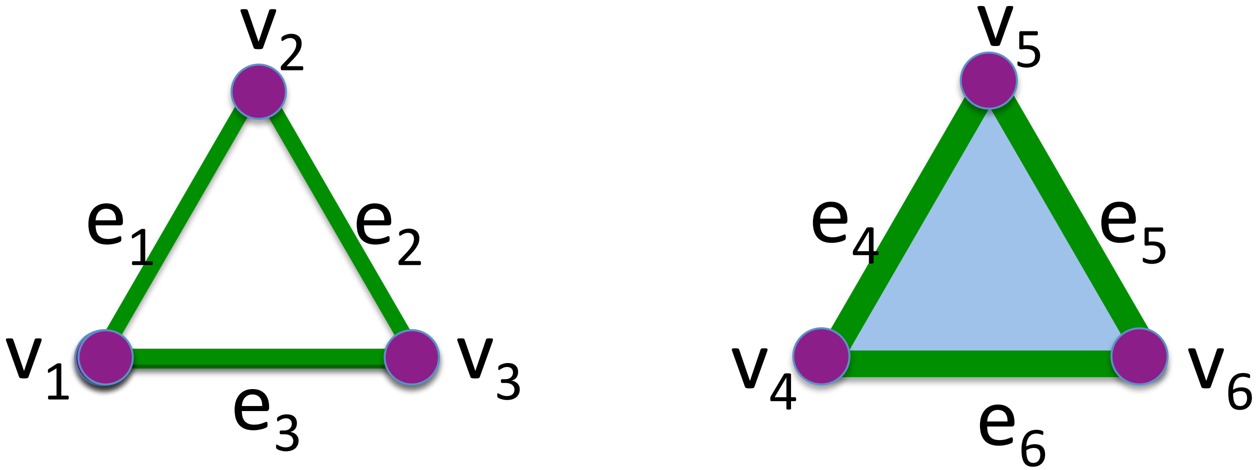
www.ams.org/journals/bull/2009-46-02/S0273-0979-09-01249-X



$$
\begin{array}{c c}
 & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}
$$



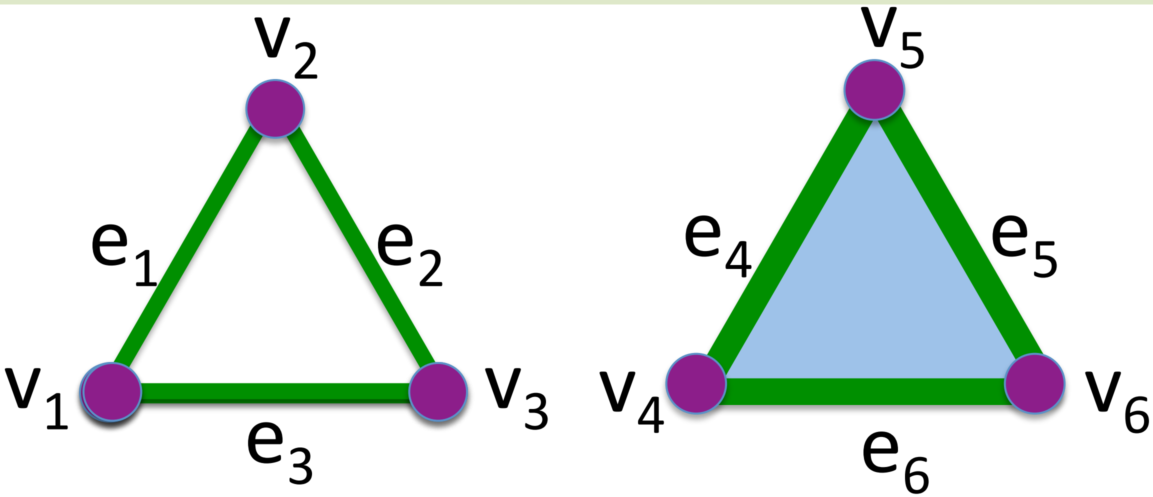
$$
\begin{array}{c@{\quad}c}
 & \begin{array}{cccccc} e_1 & e_2 & e_1+e_2+e_3 & e_4 & e_5 & e_4+e_5+e_6 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$



$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_1+e_2+e_3 & e_4 & e_5 & e_4+e_5+e_6 \\
\hline
v_1 & 1 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 1 & 0 & 0 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 1 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 1 & 1 & 0 \\
v_6 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

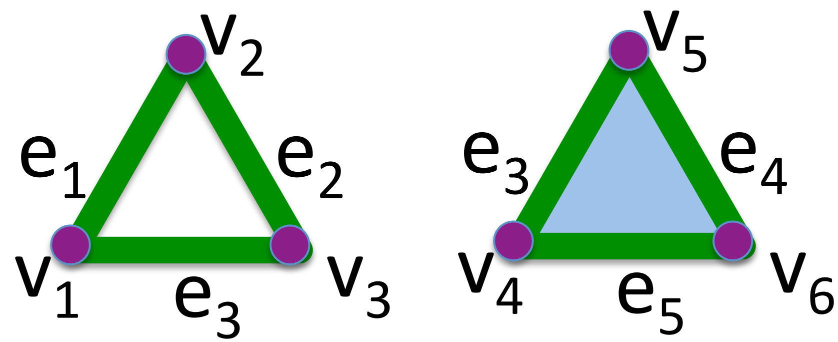
Z_1 = kernel of ∂_1 = null space of M_1

$= \langle e_1 + e_2 + e_3,\ e_4 + e_5 + e_6 \rangle$



$$
\begin{array}{c c}
 & \{v_4, v_5, v_6\} \\
\begin{array}{c} \{v_1, v_2\} \\ \{v_2, v_3\} \\ \{v_1, v_3\} \\ \{v_4, v_5\} \\ \{v_5, v_6\} \\ \{v_4, v_6\} \end{array} & \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{array} \right)
\end{array}
$$

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$



$$H_1 = Z_1/B_1 = (\text{kernel of } \partial_1)/\ (\text{image of } \partial_2)$$

$$= \frac{\text{null space of } M_1}{\text{column space of } M_2}$$

$$= \frac{\langle e_1 + e_2 + e_3,\ e_4 + e_5 + e_6 \rangle}{\langle e_4 + e_5 + e_6 \rangle}$$

$$\text{Rank } H_1 = \text{Rank } Z_1 - \text{Rank } B_1 = 2 - 1 = 1$$