MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 20, 2013: Persistent homology.

Fall 2013 course offered through the
University of Iowa Division of Continuing Education

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Applied Mathematical and Computational Sciences,
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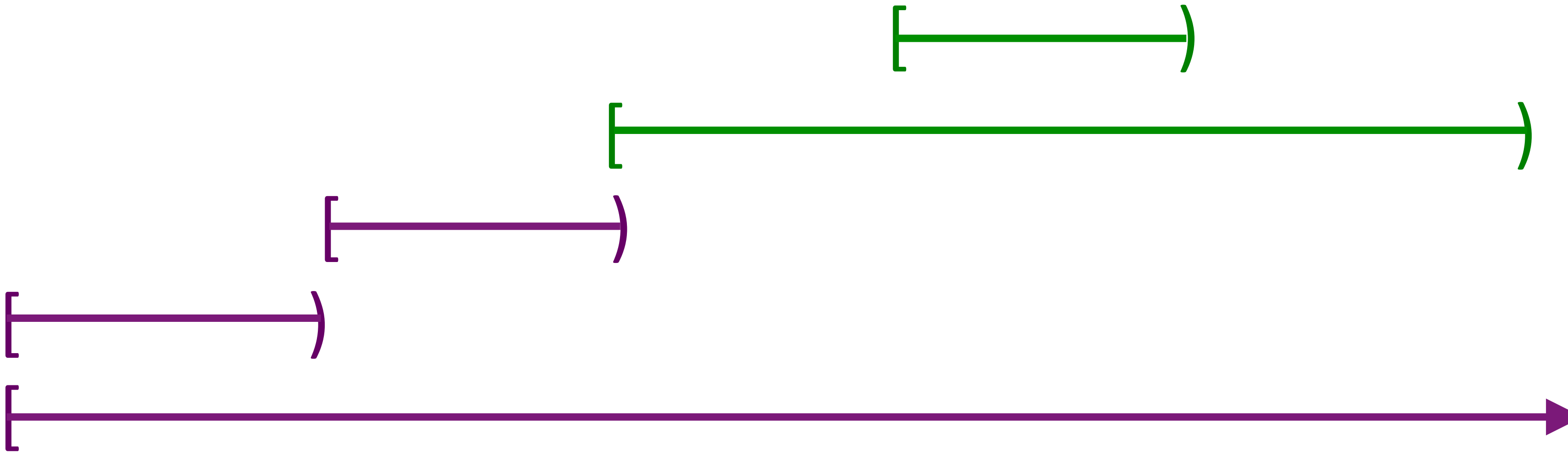
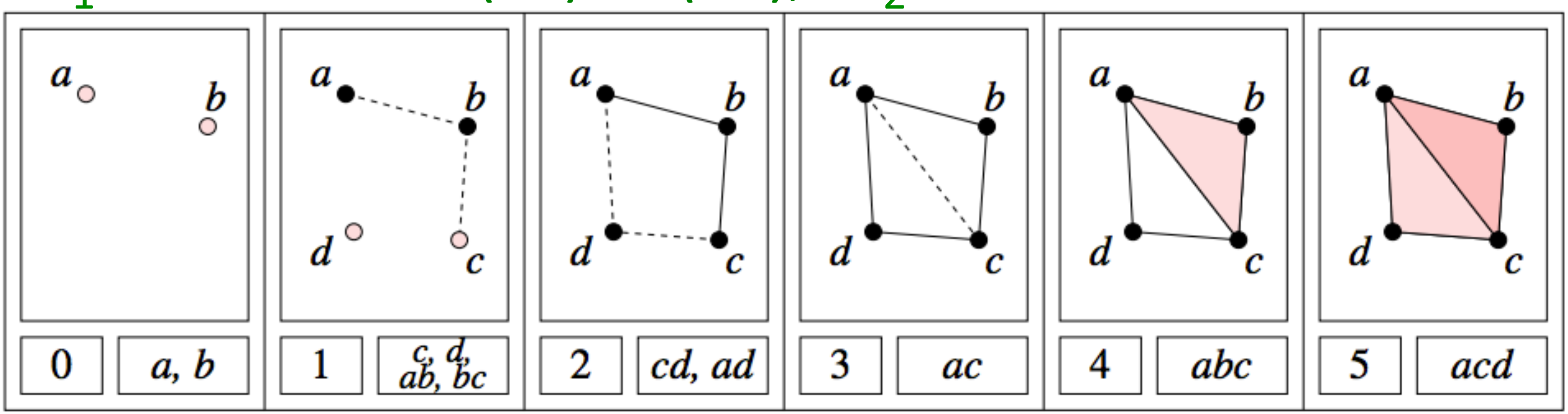
http://www.math.uiowa.edu/~idarcy/AppliedTopology.html

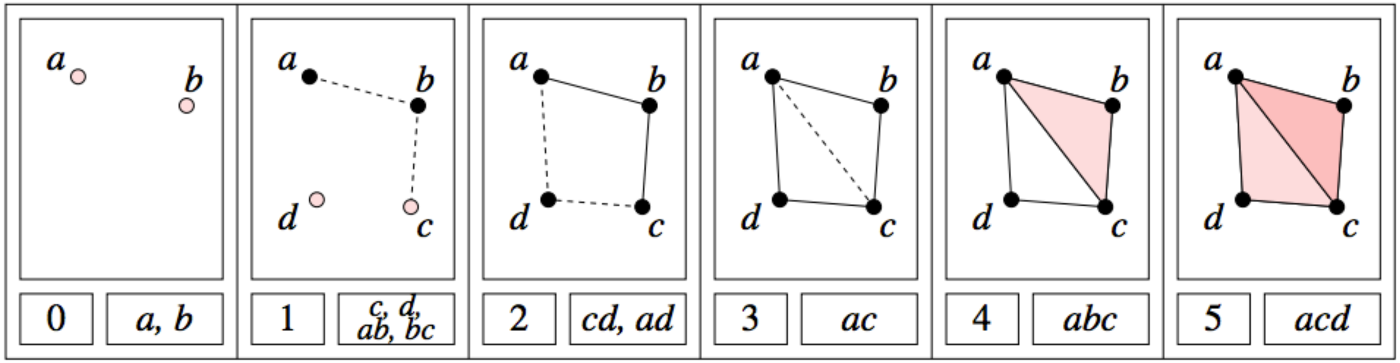
$$H_0 = \langle a, b, c, d \; : \; tc + td,\; tb + c,\; ta + tb \rangle$$

$$H_1 = \langle z_1, z_2 \; : \; t\, z_2,\; t^3 z_1 + t^2 z_2 \rangle$$



$$z_1 = ad + cd + t(bc) + t(ab), \quad z_2 = ac + t^2 bc + t^2 ab$$





$$
\begin{tikzcd}
\mathsf{C}_2^0 \arrow[r, "f^0"] \arrow[d, "\partial_2"'] & \mathsf{C}_2^1 \arrow[r, "f^1"] \arrow[d, "\partial_2"'] & \mathsf{C}_2^2 \arrow[r, "f^2"] \arrow[d, "\partial_2"'] & \cdots \\
\mathsf{C}_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & \mathsf{C}_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & \mathsf{C}_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
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\end{tikzcd}
$$

p-persistent k^{th} homology group:

$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}[abc, acd]$

$\partial_2 \downarrow$

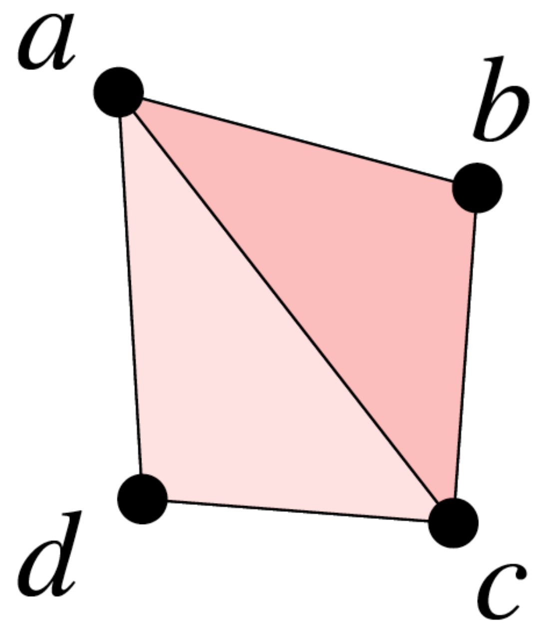
$C_1 = \mathbf{Z_2}[ab, bc, ac, ad, cd]$

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}[a, b, c, d]$

$\partial_0 \downarrow$

0



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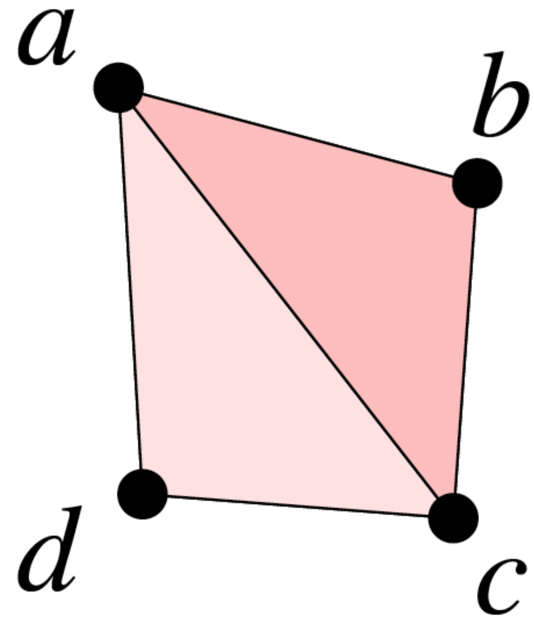
$$\partial_0 \downarrow$$

$$0$$



$$\boxed{\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots}$$

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}[abc, acd]$

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$\partial_1 \downarrow$

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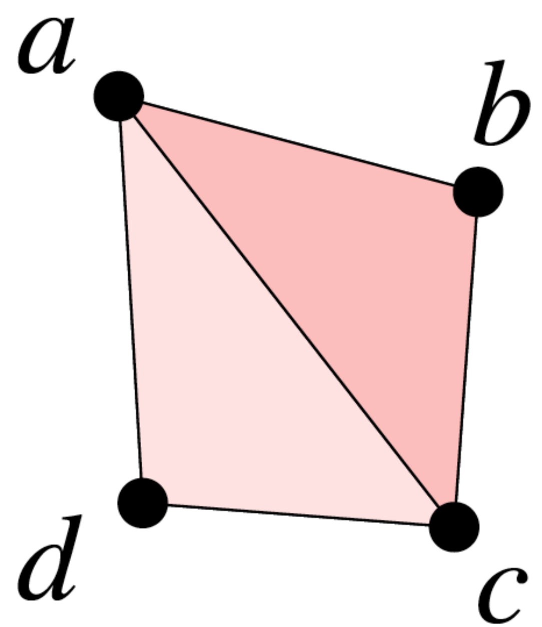
$\partial_0 \downarrow$

0



$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}[abc, acd]$

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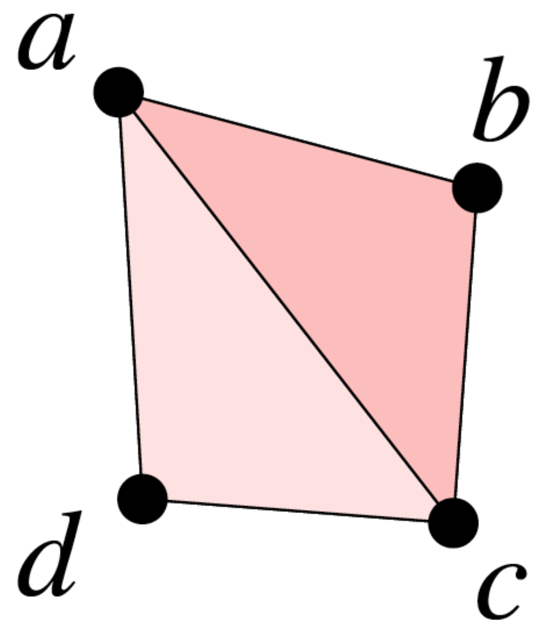
0



$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

$a + b + bc + abc + acd$

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}[abc, acd]$

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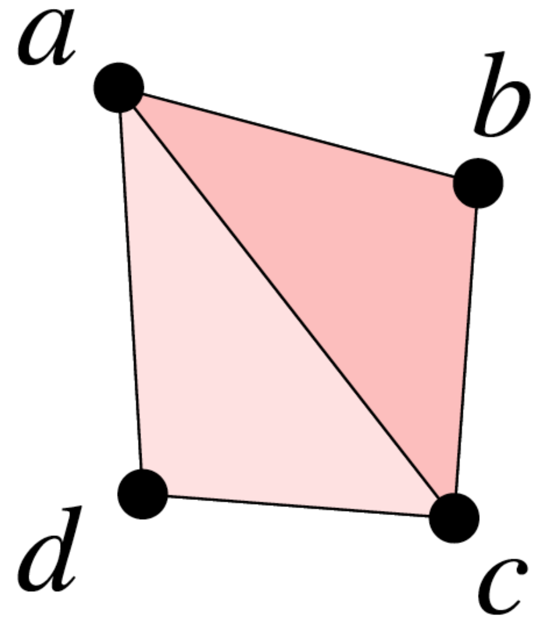
$C_0 = \mathbf{Z_2}[a, b, c, d]$

$\partial_0 \downarrow$

0



$\oplus C_k = C_0 \oplus C_1 \oplus C_2$

$a + b \oplus bc \oplus abc + acd$

$(a + b, \ bc, \ abc + acd)$

$$\partial_3 \downarrow$$

$$C_2 = \mathbf{Z_2}[abc, acd]$$

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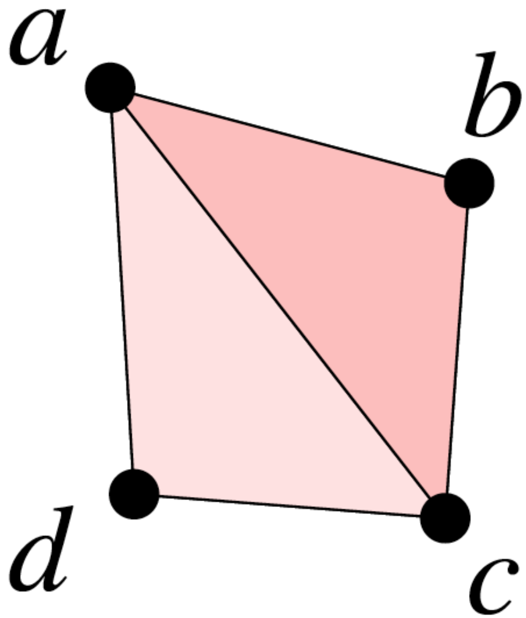
$$\partial_0 \downarrow$$

$$0$$



$$\oplus C_k = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus \ldots$$

$$\partial : \oplus C_k \rightarrow \oplus C_k$$

$$\partial(0, ab, 0) = (a + b, 0, 0)$$

$$\partial^2 = 0$$

$\partial_3 \downarrow$

$C_2 = \mathbf{Z_2}[abc, acd]$

$\partial_2 \downarrow$

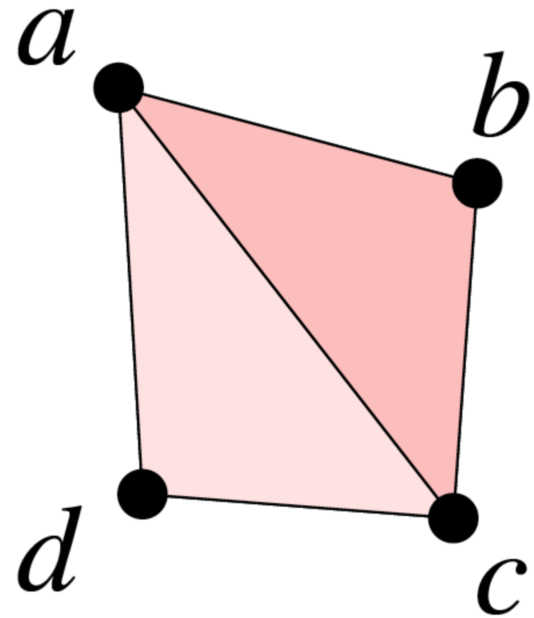
$C_1 = \mathbf{Z_2}[ab, bc, ac, ad, cd]$

$\partial_1 \downarrow$

$C_0 = \mathbf{Z_2}[a, b, c, d]$

$\partial_0 \downarrow$

0



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$\partial : \oplus C_k \rightarrow \oplus C_k$

$\partial(0, ab, 0) = (a + b, 0, 0)$

$\partial(0 \oplus ab \oplus 0) = a{+}b \oplus 0 \oplus 0$

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$C_2 = \mathbf{Z_2}[abc, acd]$

$\partial_2 \downarrow$

$C_1 = \mathbf{Z_2}[ab, bc, ac, ad, cd]$

$\partial_1 \downarrow$

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$\partial_0 \downarrow$

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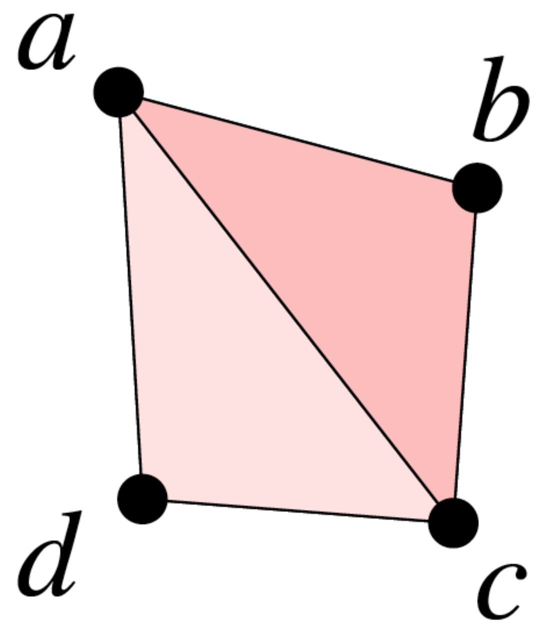

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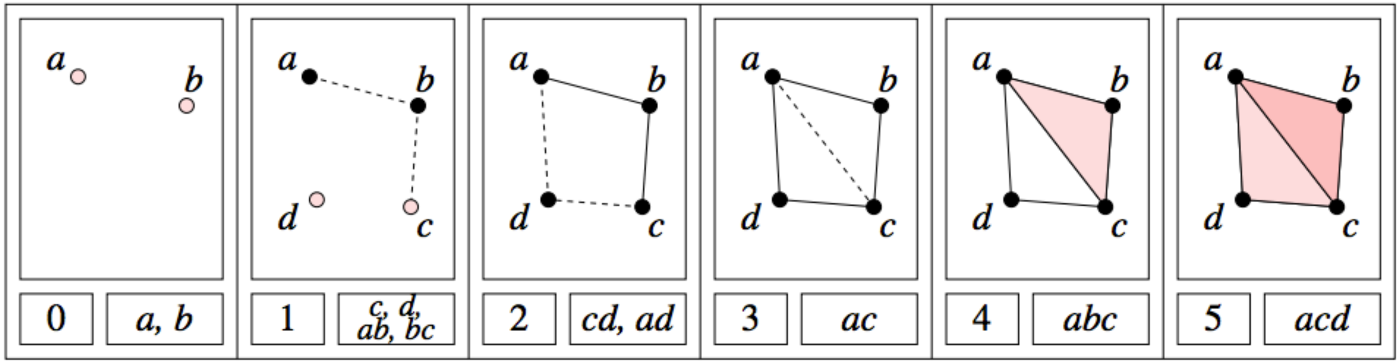

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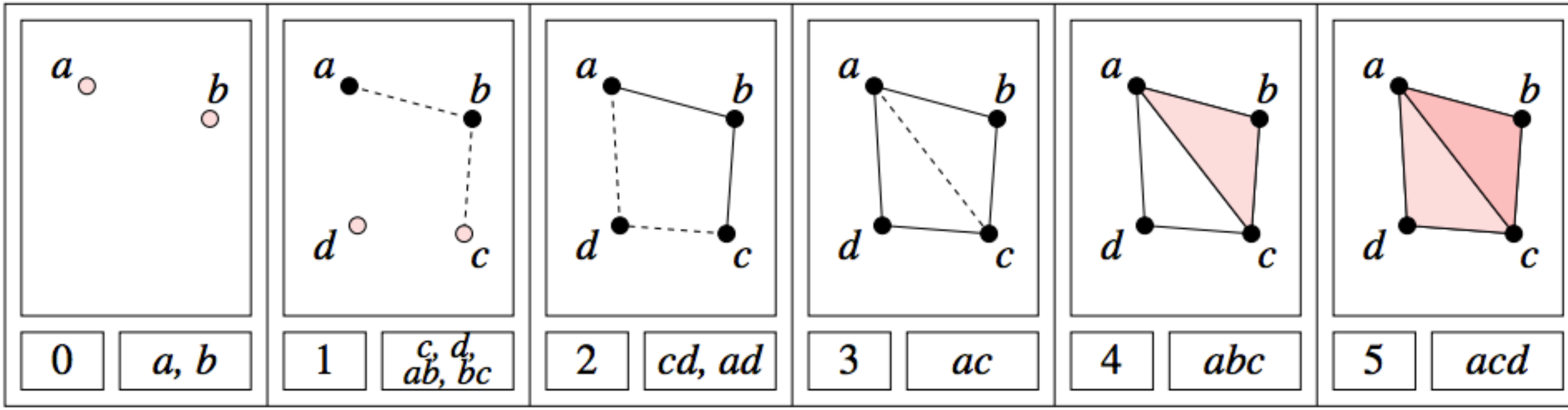
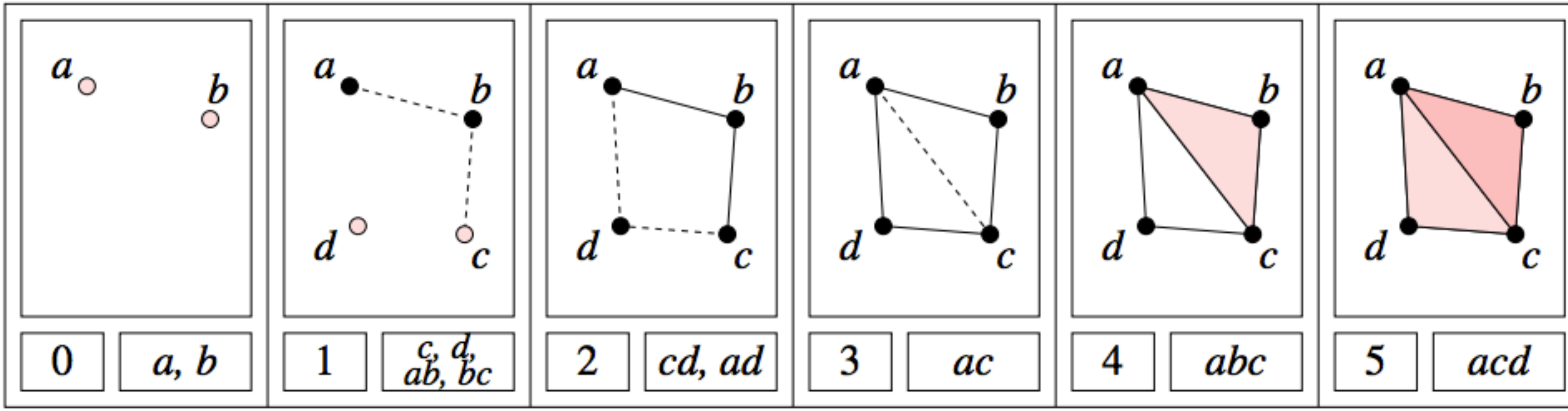
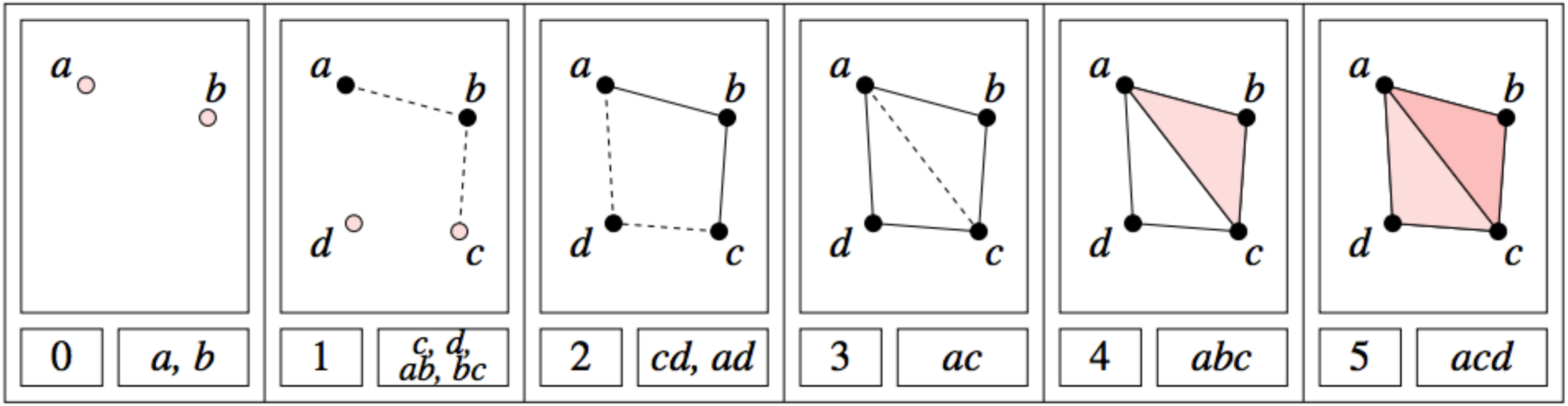


$$C_j^0 \to C_j^1 \to C_j^2 \to C_j^3 \to C_j^4 \to C_j^5$$



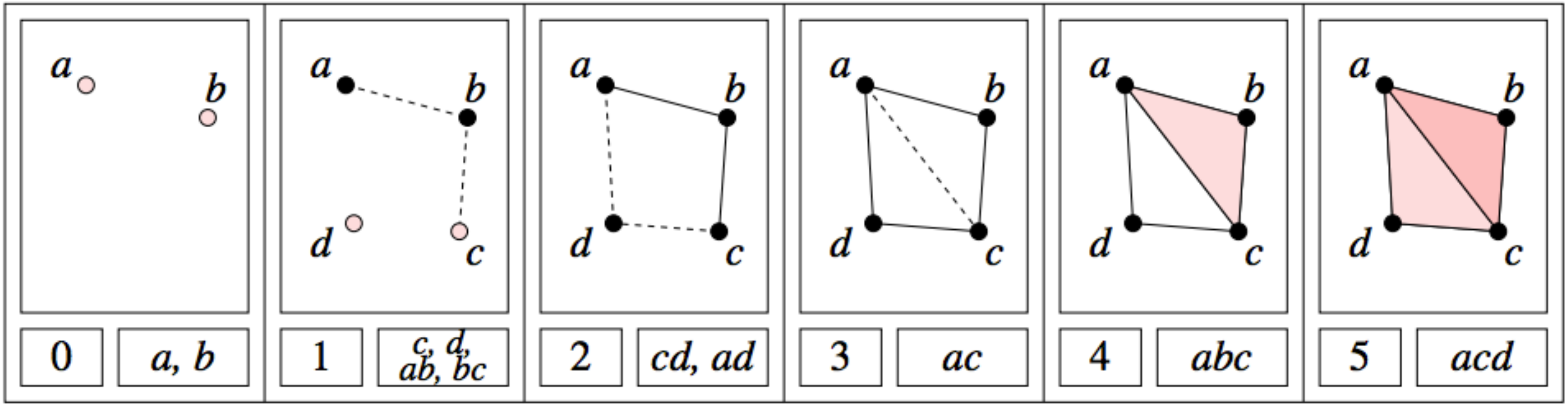
$$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5$$



$$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \ldots$$

$$\oplus C_0^i = \langle a, b\rangle \oplus \langle c, d, ta, tb\rangle \oplus \langle tc, td, t^2a, t^2b\rangle \oplus \ \ldots$$

$$(a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \ldots)$$



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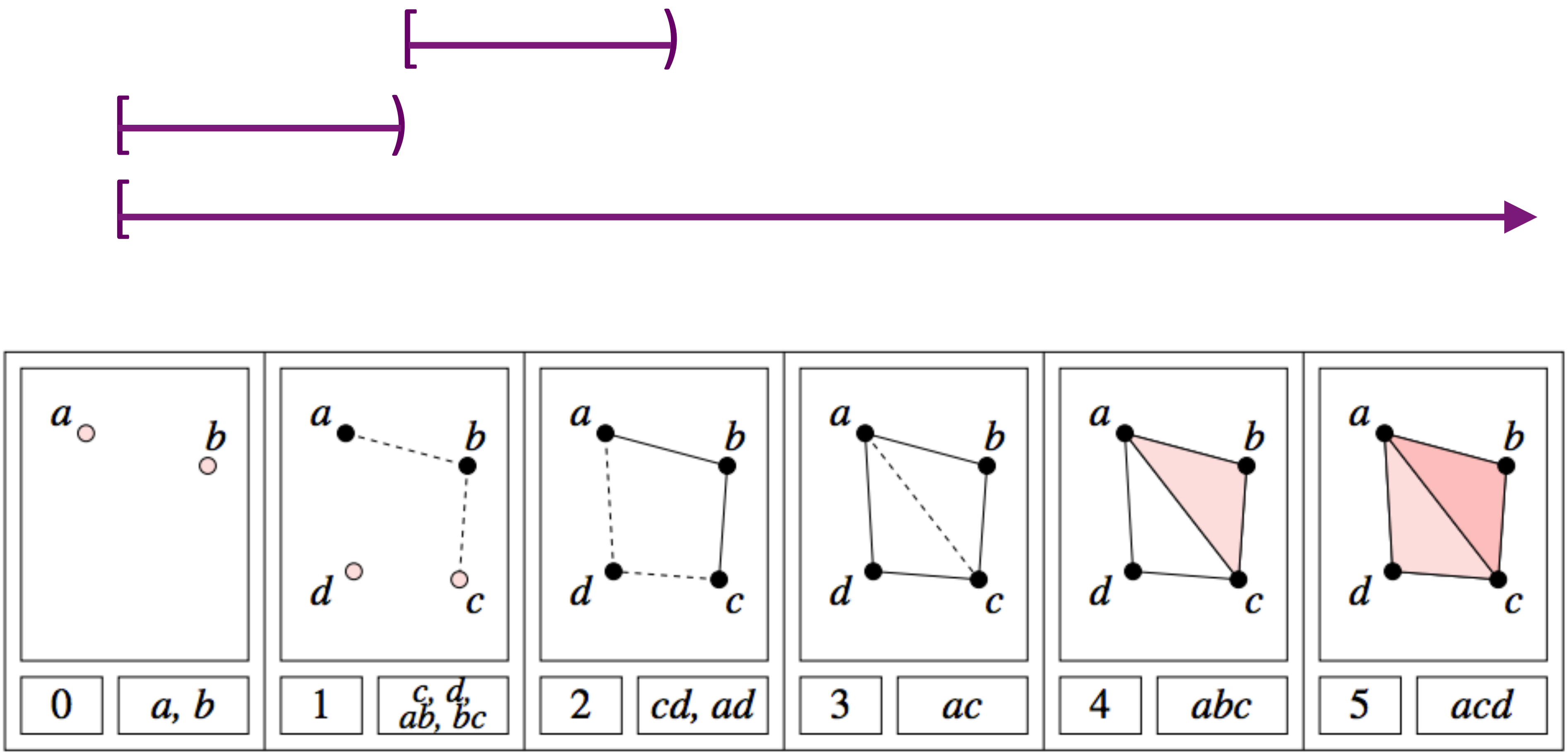
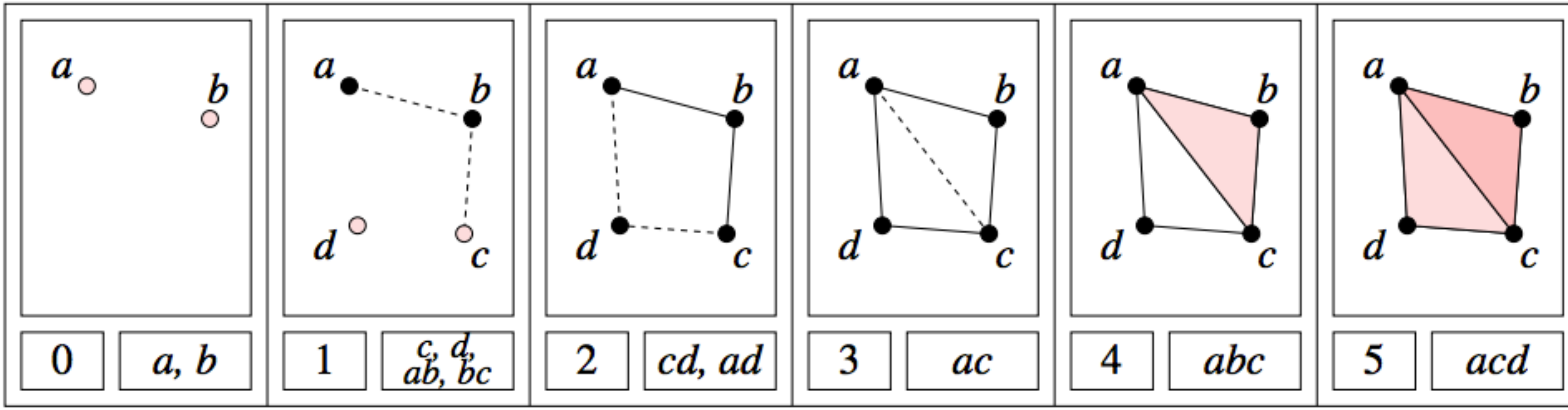
$$(a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \ldots)$$

$$t \bullet (a, c + ta, tc + t^2b, t^2c, t^4b, t^5b, t^6b, \ldots)$$

$$= (0, ta, tc + t^2a, t^2c + t^3b, t^3c, t^5b, t^6b, t^7b, \ldots)$$

$$H_0 = \langle a, b, c, d : tc + td,\ tb + c,\ ta + tb \rangle$$





$$C_0^0 \rightarrow C_0^1 \rightarrow C_0^2 \rightarrow C_0^3 \rightarrow C_0^4 \rightarrow C_0^5 \rightarrow \dots$$

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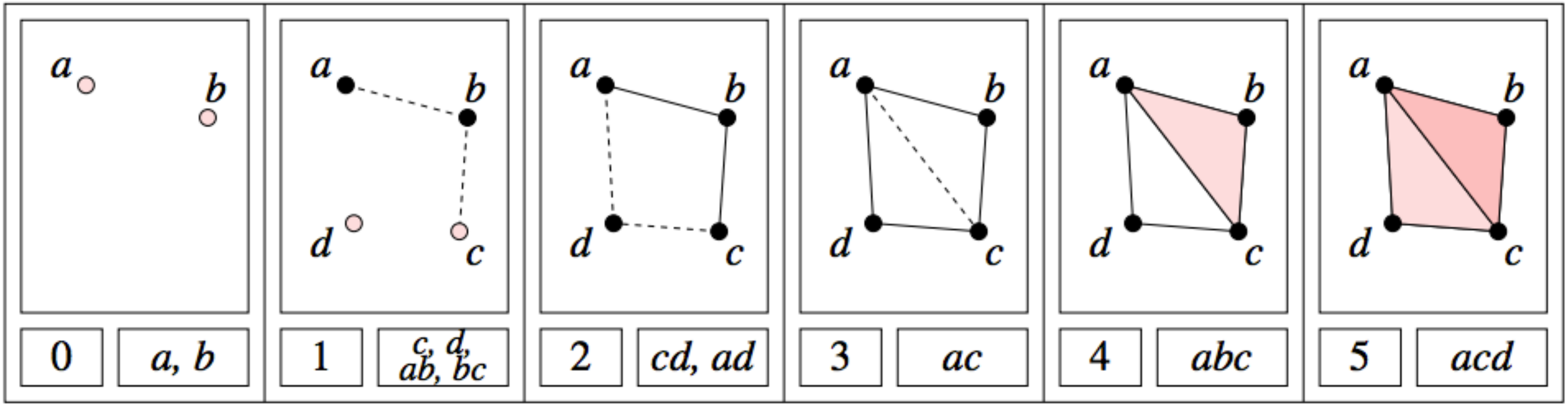
a: $\mathbf{Z_2}[t]$ b: $\mathbf{Z_2}[t]$ c: $\Sigma^1 \mathbf{Z_2}[t]$ d: $\Sigma^1 \mathbf{Z_2}[t]$



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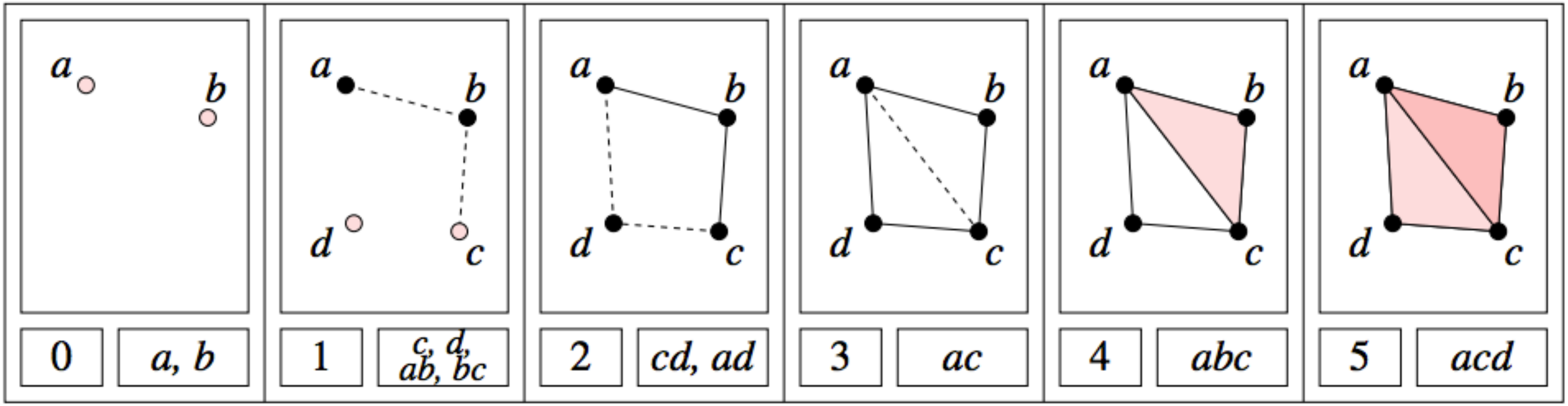
a: $\mathbf{Z_2}[t]$ b: $\mathbf{Z_2}[t]$ c: $\Sigma^1 \mathbf{Z_2}[t]$ d: $\Sigma^1 \mathbf{Z_2}[t]$

acd: $\Sigma^5 \mathbf{Z_2}[t]$



$$C_0^0 \to C_0^1 \to C_0^2 \to C_0^3 \to C_0^4 \to C_0^5 \to \ldots$$

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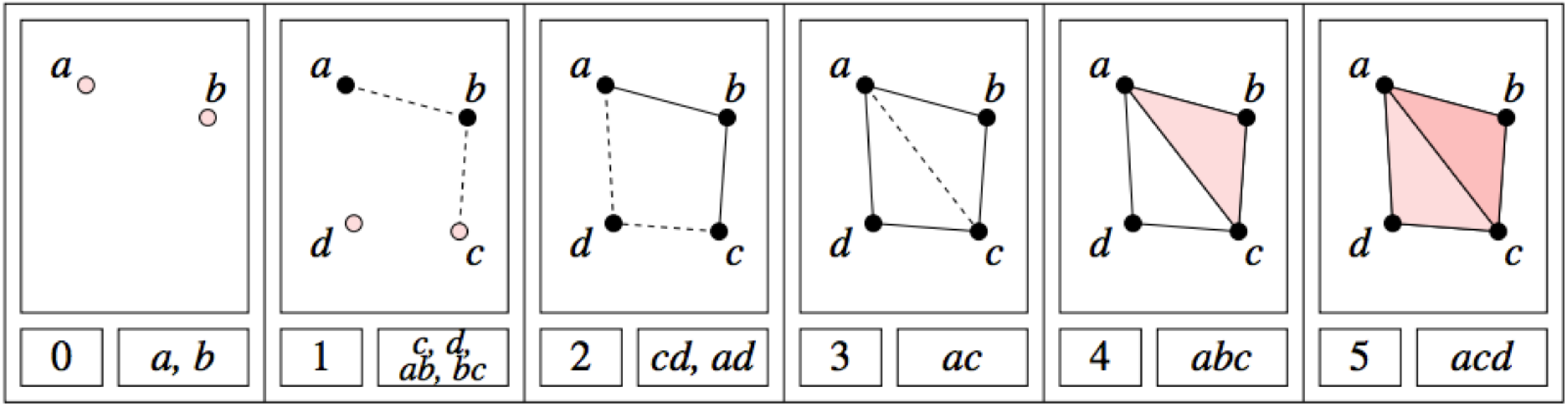
a: $\mathbf{Z_2}[t] = \{n_0 + n_1t + n_2t^2 + \ldots + n_kt^k : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$

$= \{ (n_0, n_1, n_2, \ldots, n_k, 0, 0, \ldots) : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$



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b: $\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \ldots + n_k t^k : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$

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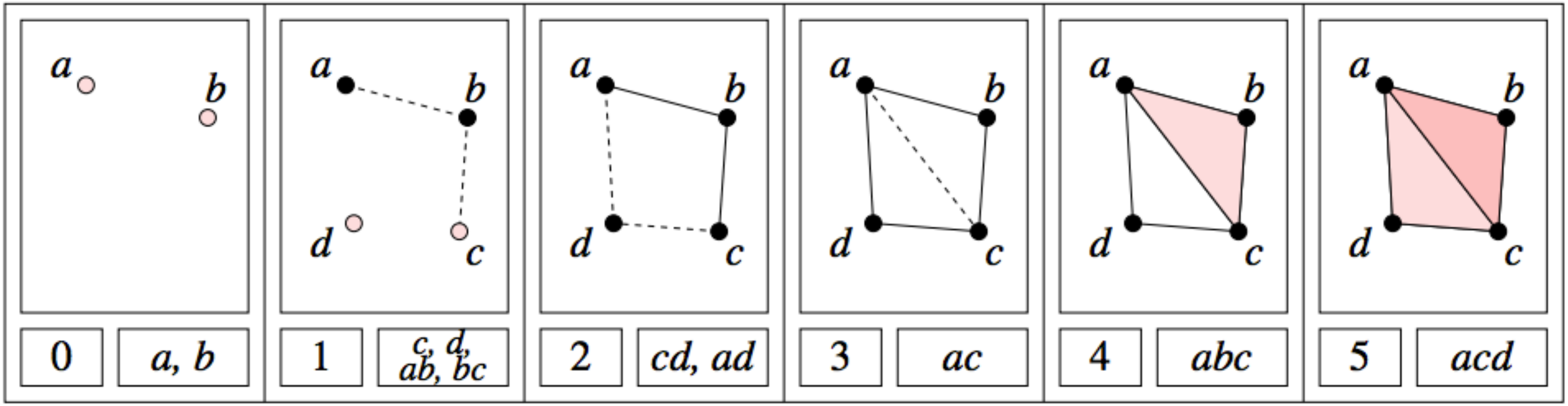

$$C_0^0 \to C_0^1 \to C_0^2 \to C_0^3 \to C_0^4 \to C_0^5 \to \dots$$

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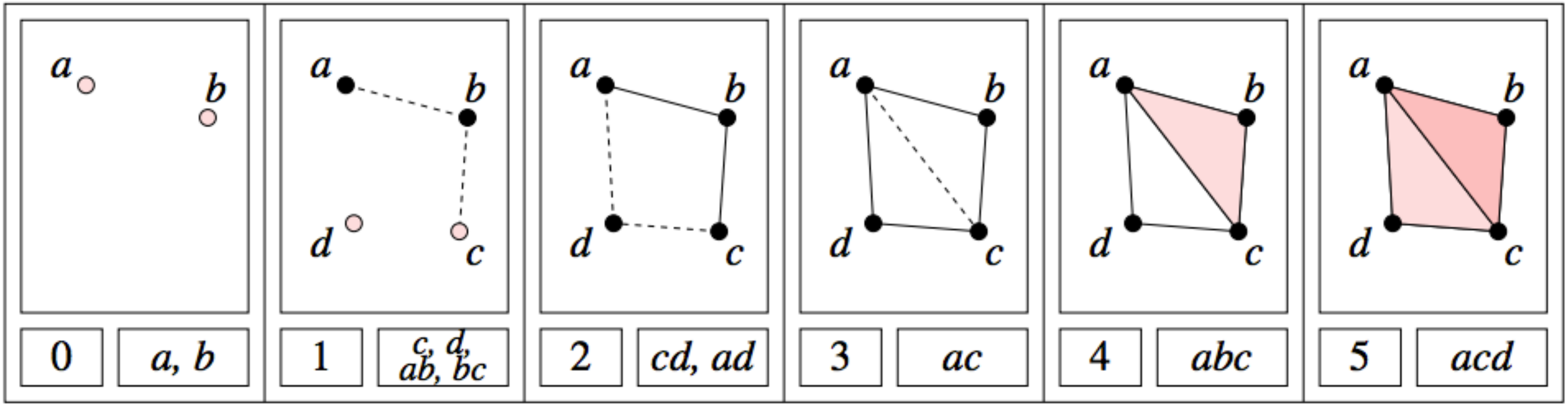
$$c: \quad \Sigma^1 \mathbf{Z_2}[t] = \{n_0 t + n_1 t^2 + \dots \;:\; n_i \text{ in } \mathbf{Z_2}\}$$

$$= \{ (0, n_0, n_1, \dots) \;:\; n_i \text{ in } \mathbf{Z_2}\}$$



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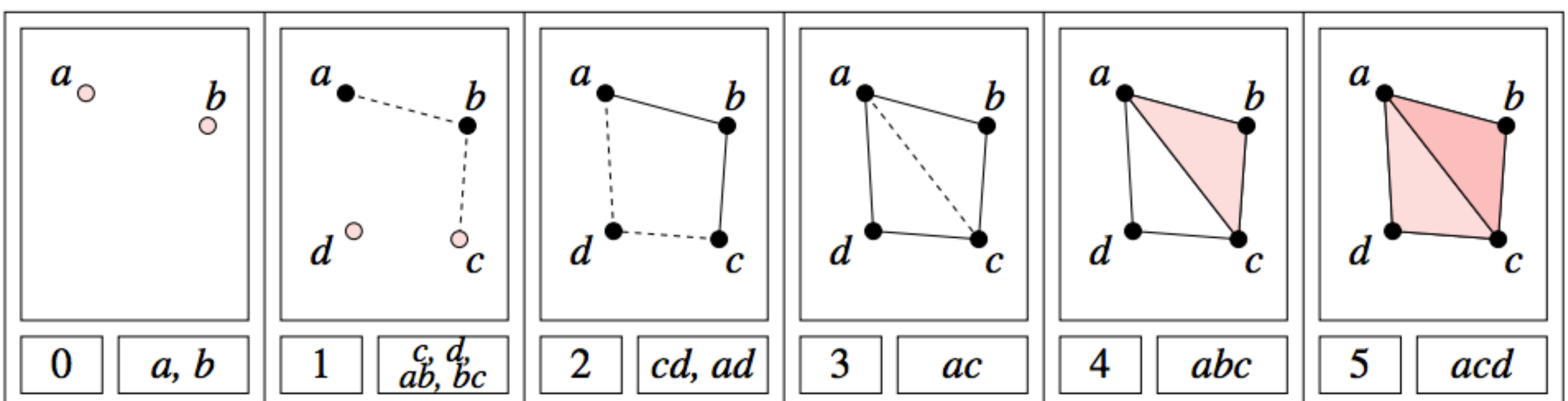
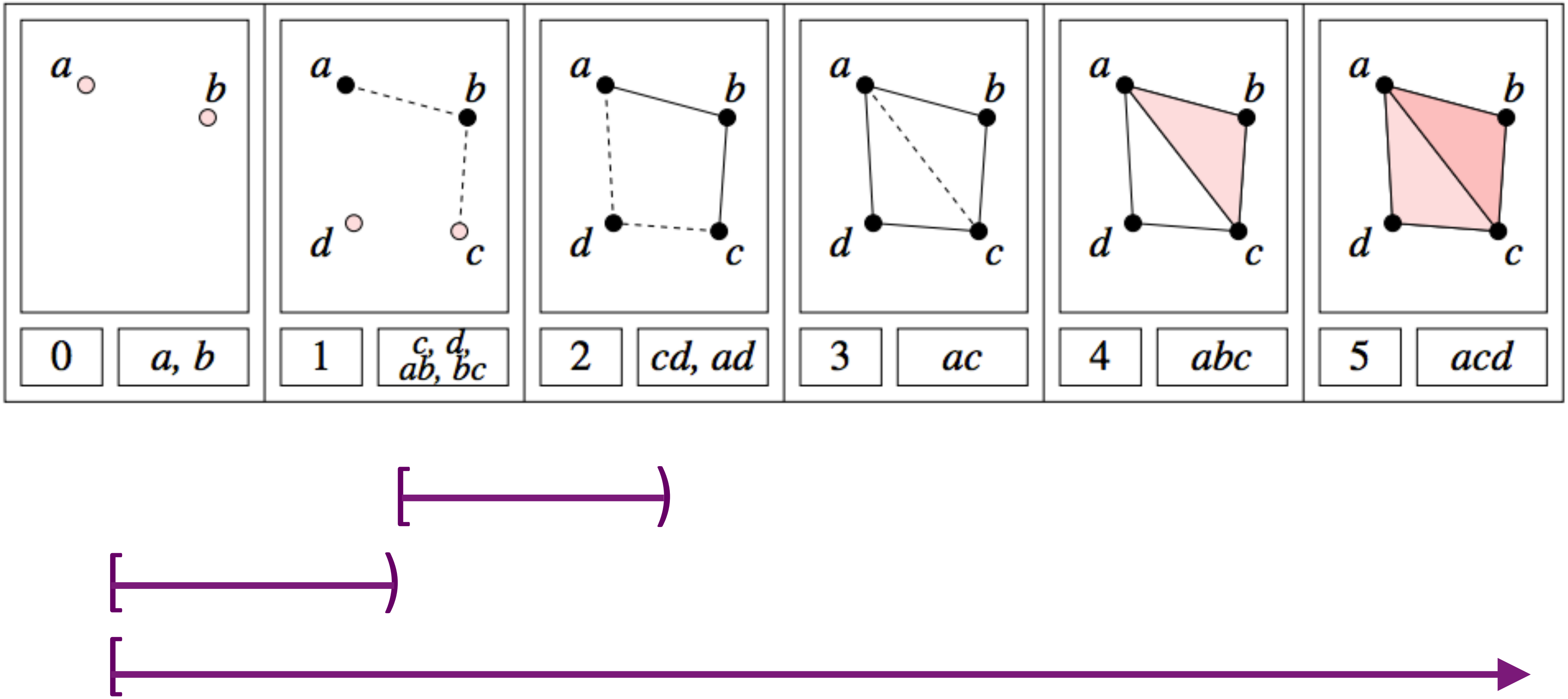
$$H_0 = \langle a, b, c, d : tc + td,\ tb + c,\ ta + tb \rangle$$

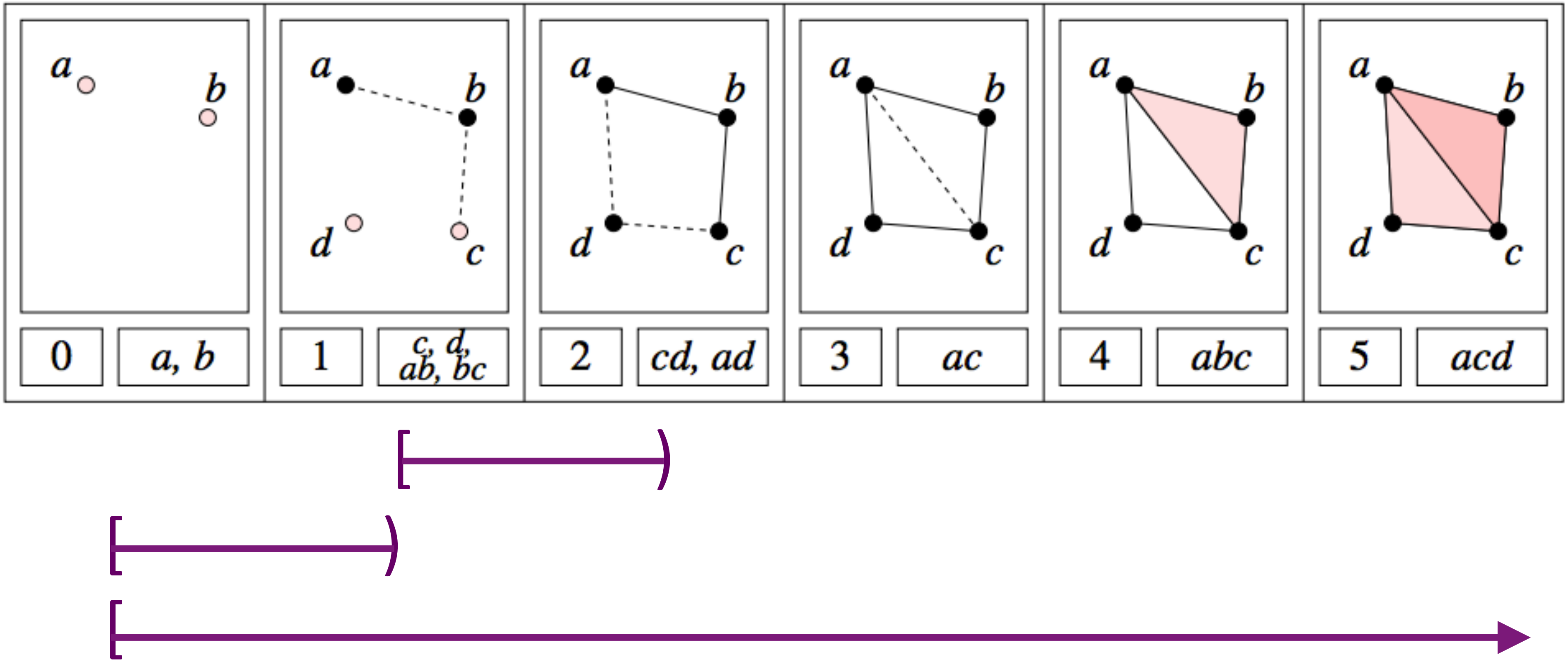




$$H_0 = \langle a, b, c, d : tc + td,\ tb + c,\ ta + tb \rangle$$

a: $\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \ldots + n_k t^k : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$

$$= \{ (n_0, n_1, n_2, \ldots, n_k, 0, 0, \ldots) : n_i \text{ in } \mathbf{Z_2}, k \text{ in } \mathbf{Z_+}\}$$

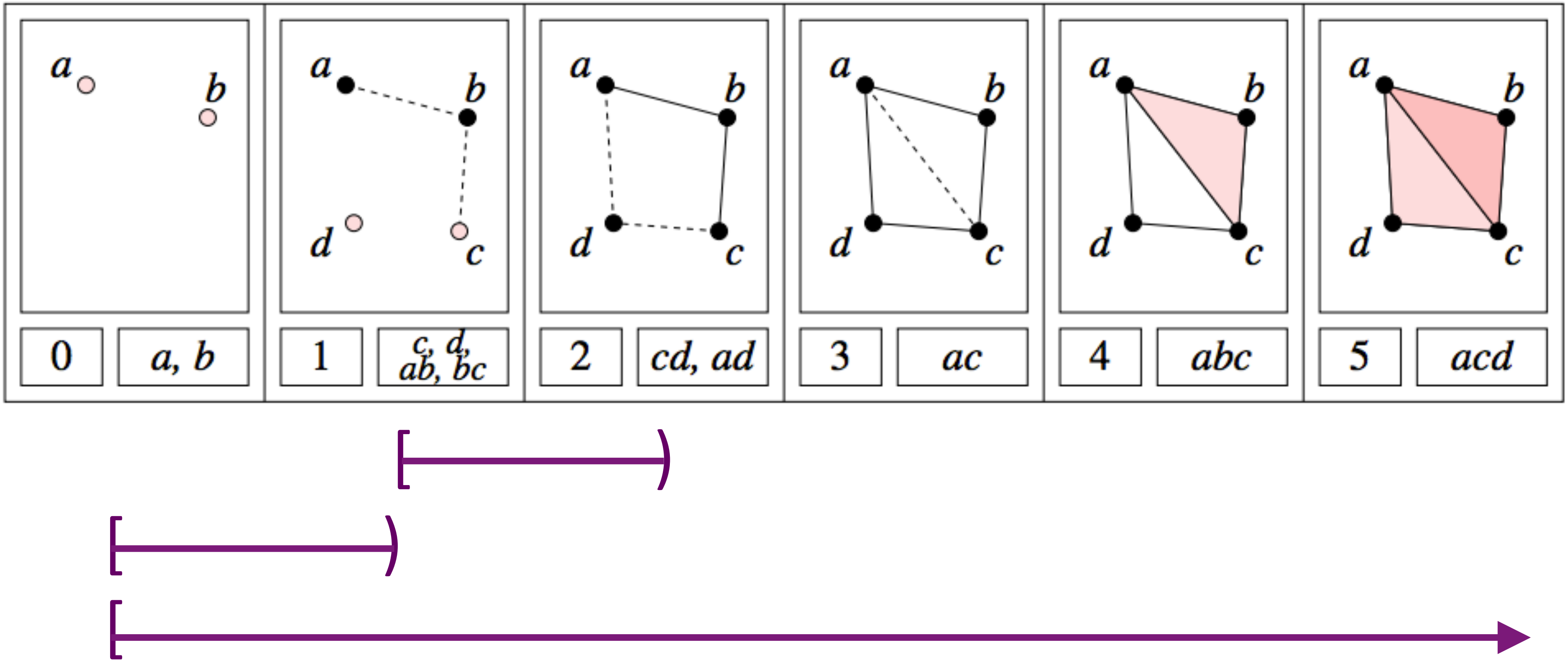


$$H_0 = \langle a, b, c, d \ : \ tc + td,\ tb + c,\ ta + tb \rangle$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \dots \ : \ n_i \text{ in } \mathbf{Z_2}\}$$

b: $\mathbf{Z_2}[t]/t = \{n_0 \ : \ n_0 \text{ in } \mathbf{Z_2}\} = \mathbf{Z_2}$

$$= \{ (n_0, 0, 0, 0, \dots) \ : \ n_i \text{ in } \mathbf{Z_2}\}$$

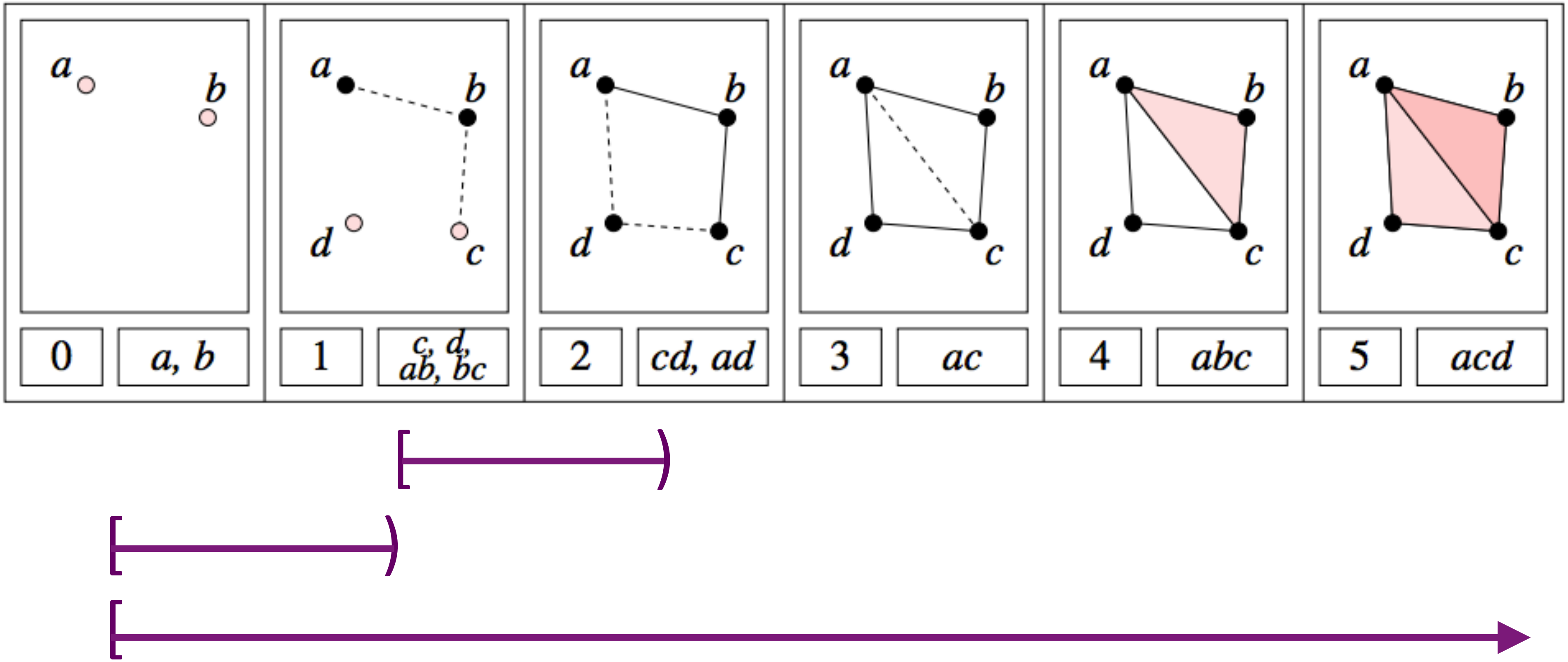


$$H_0 = < a, b, c, d \;:\; tc + td,\; tb + c,\; ta + tb>$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \ldots \;:\; n_i \text{ in } \mathbf{Z_2}\}$$

$$d: \;\Sigma^1 \mathbf{Z_2}[t]/t = \Sigma^1 \{n_0 \;:\; n_0 \text{ in } \mathbf{Z_2}\}$$

$$= \{ (0, n_0, 0, 0, 0, \ldots) : n_0 \text{ in } \mathbf{Z_2}\}$$



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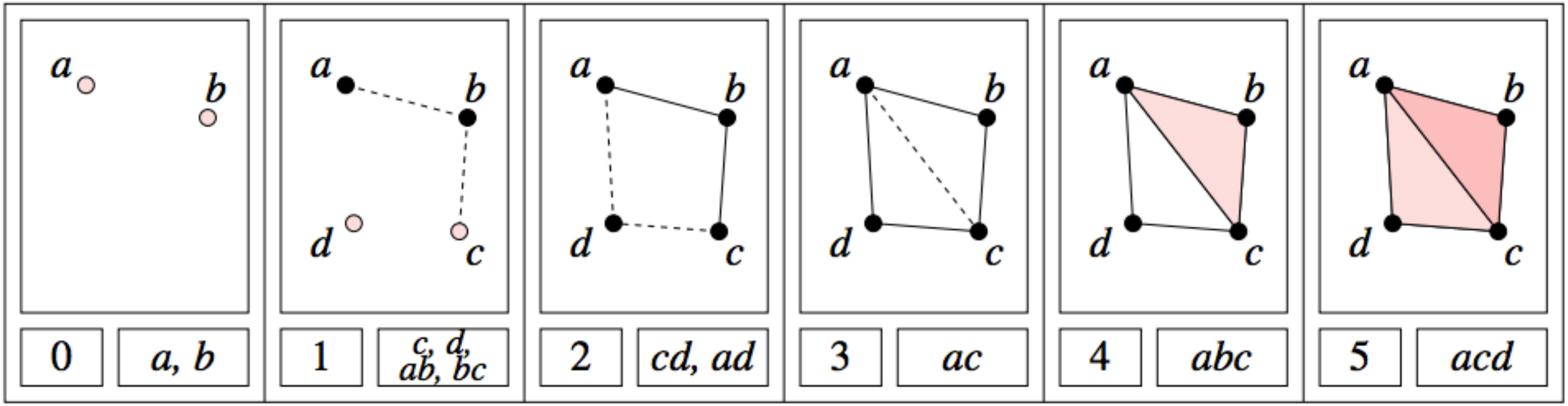
$$c:\ \Sigma^1 \mathbf{Z_2}[t]/1 = \{ (0, 0, 0, \ldots) \}$$



$$C^0_0 \rightarrow C^1_0 \rightarrow C^2_0 \rightarrow C^3_0 \rightarrow C^4_0 \rightarrow C^5_0 \rightarrow \ldots$$

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a: $\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + \ldots\ n_j t^j : n_i \text{ in } \mathbf{Z_2}\}$

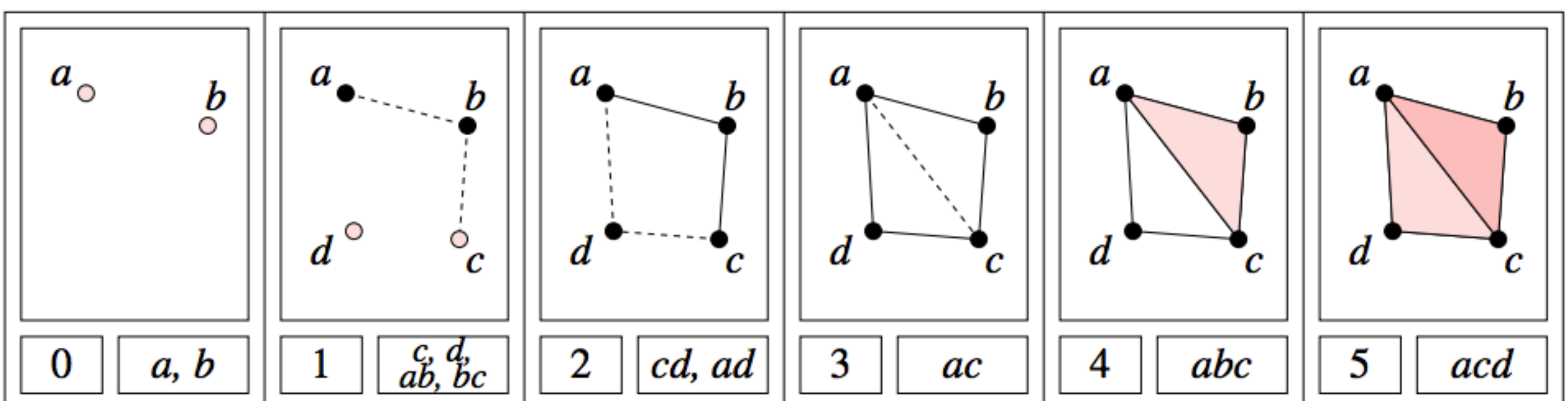
b: $\mathbf{Z_2}[t]/t = \{n_0 : n_0 \text{ in } \mathbf{Z_2}\} = \mathbf{Z_2}$

c: $\Sigma^1\, \mathbf{Z_2}[t]/1$ = empty set

d: $\Sigma^1\, \mathbf{Z_2}[t]/t = \{n_0 : n_0 \text{ in } \mathbf{Z_2}\} = \{(0, n_0, 0, 0, 0, \ldots)\}$

$$H_0 = \langle a, b, c, d : tc + td,\ tb + c,\ ta + tb \rangle$$

$$= \mathbf{Z_2}[t] \oplus \mathbf{Z_2}[t]/t \oplus (\Sigma^1 \mathbf{Z_2}[t])/t$$



$$H_0 = \langle a, b, c, d : tc + td,\ tb + c,\ ta + tb \rangle$$

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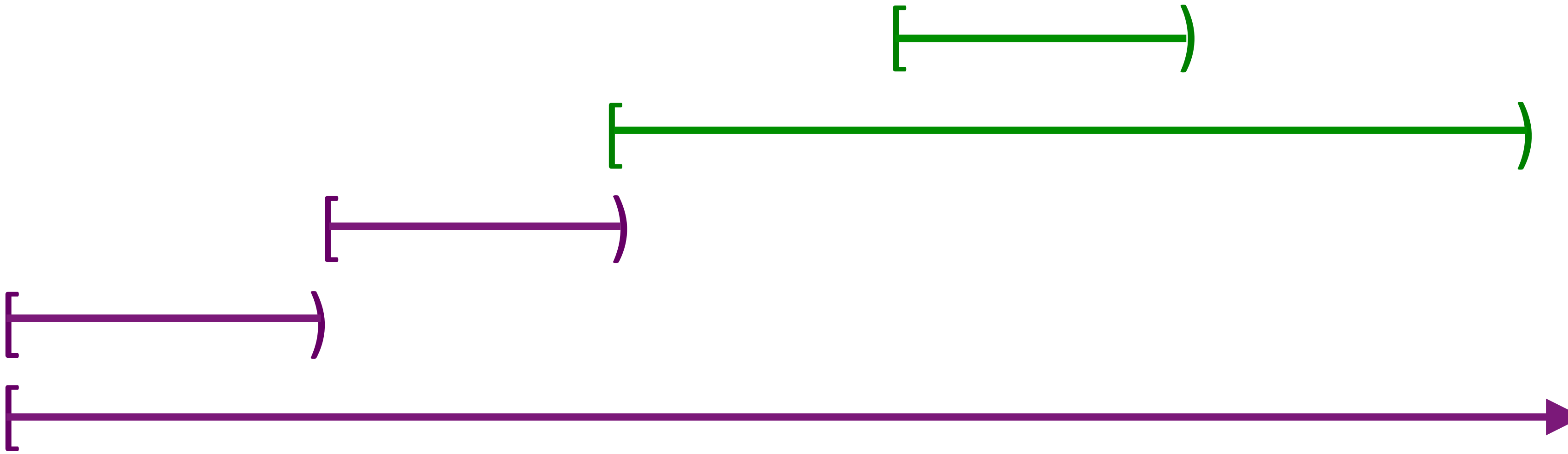
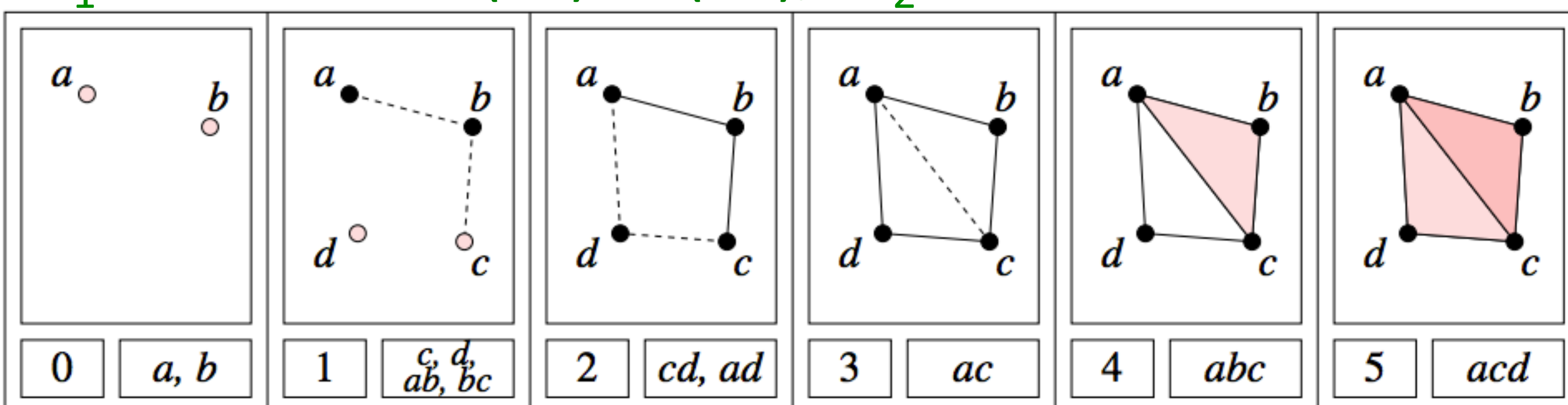
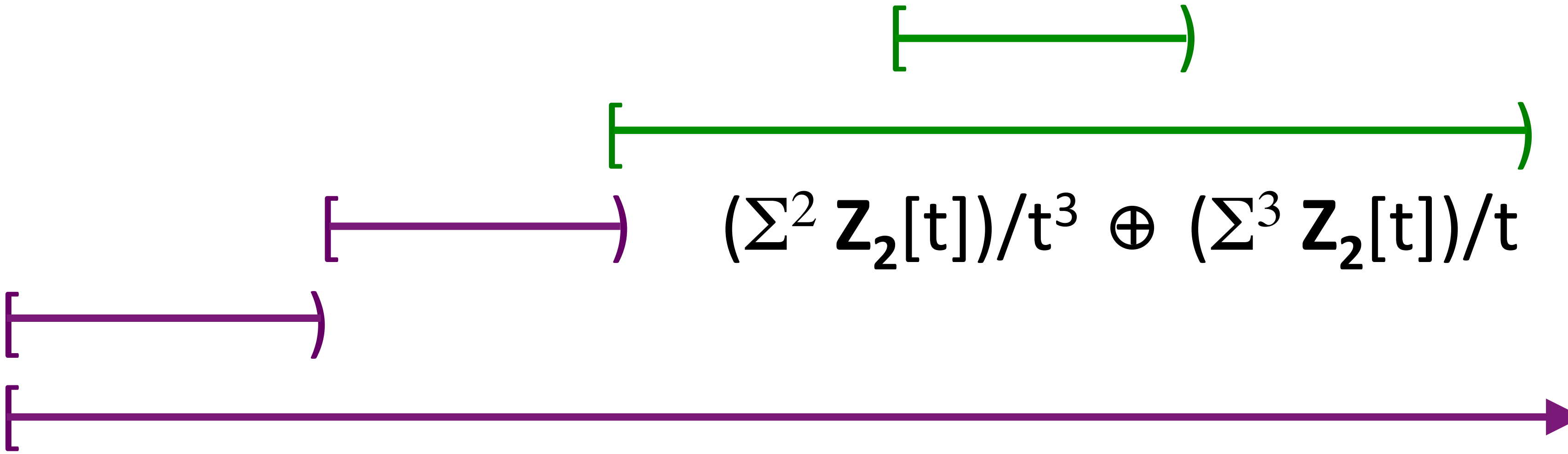


$$z_1 = ad + cd + t(bc) + t(ab), \quad z_2 = ac + t^2bc + t^2ab$$



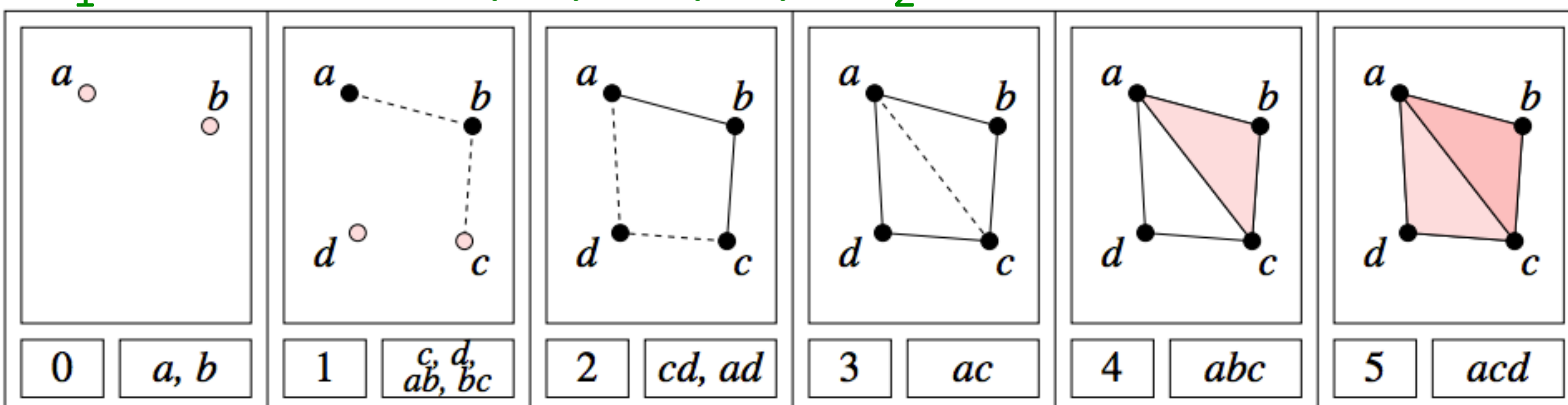
$$H_0 = < a, b, c, d \;:\; tc + td,\; tb + c,\; ta + tb >$$

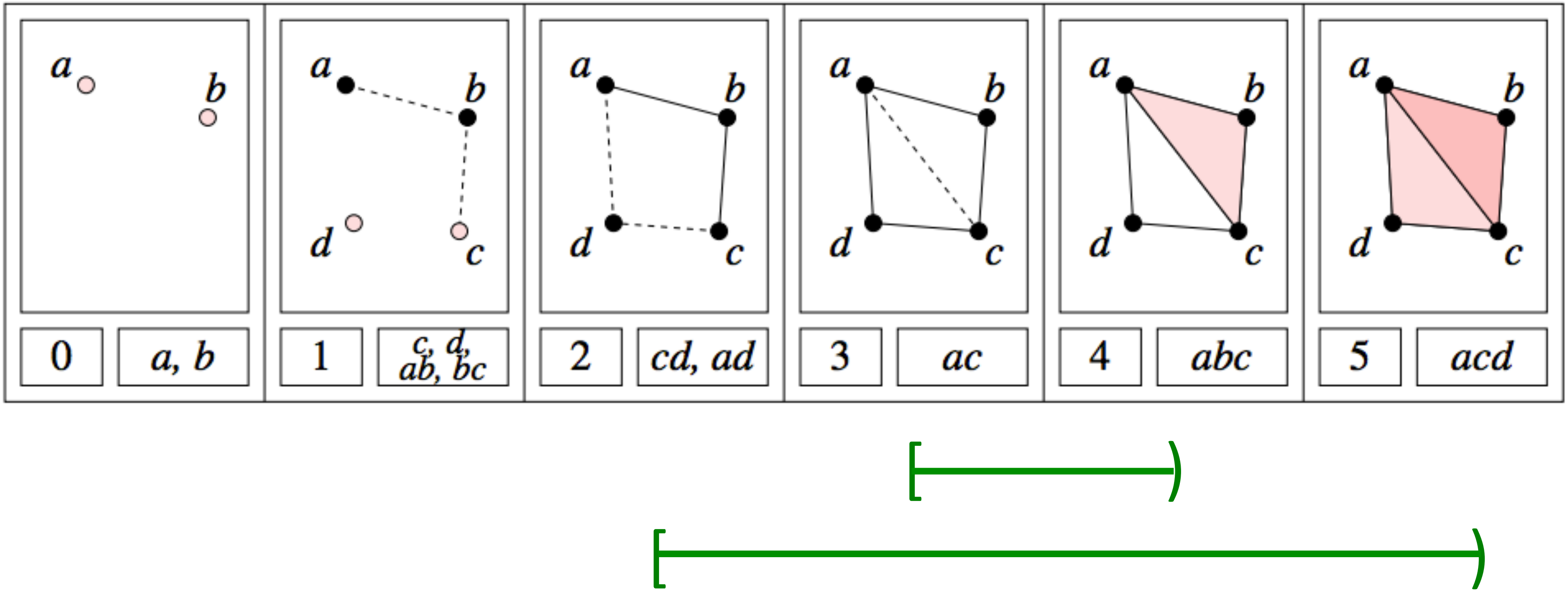
$$H_1 = < z_1, z_2 \;:\; t\, z_2,\; t^3 z_1 + t^2 z_2 >$$



$$(\Sigma^2\, \mathbf{Z_2}[t])/t^3 \oplus (\Sigma^3\, \mathbf{Z_2}[t])/t$$

$$z_1 = ad + cd + t(bc) + t(ab), \quad z_2 = ac + t^2 bc + t^2 ab$$



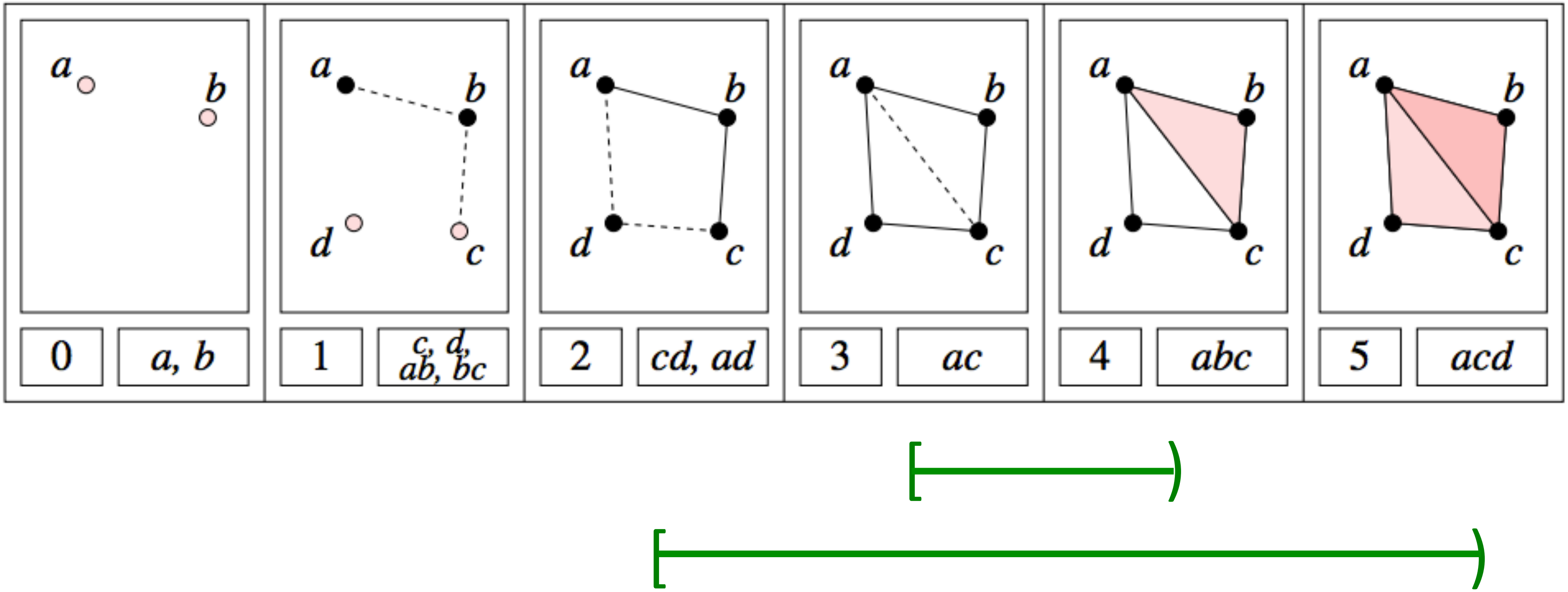


$$H_1 = < z_1, z_2 \; : \; t\, z_2, \; t^3 z_1 + t^2 z_2 >$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + n_2 t^3 + n_2 t^4 + \dots \; : \; n_i \text{ in } \mathbf{Z_2}\}$$

$$z_1: \quad (\Sigma^2\, \mathbf{Z_2}[t])/t^3$$

$$= \{ (0, 0, n_0, n_1, n_2, 0, 0, 0, \dots) \; : \; n_i \text{ in } \mathbf{Z_2}\}$$



$$H_1 = \langle z_1, z_2 : t z_2,\ t^3 z_1 + t^2 z_2 \rangle$$

$$\mathbf{Z_2}[t] = \{n_0 + n_1 t + n_2 t^2 + n_2 t^3 + n_2 t^4 + \dots : n_i \text{ in } \mathbf{Z_2}\}$$

$$z_2: \quad (\Sigma^3\, \mathbf{Z_2}[t])/t$$

$$= \{ (0, 0, 0, n_0, 0, 0, 0, \dots) : n_i \text{ in } \mathbf{Z_2}\}$$

In general when calculating homology over the field **F**

$$H_k = \left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} \mathbf{F}[t] \right) \oplus \left(\bigoplus_{i=1}^{n} \Sigma^{\gamma_j} \mathbf{F}[t]/(t^{k_j}) \right)$$

Lecture 9: Visualizing Data via Homology

Friday June 19, 2009

Coffee	
"Visualizing data via homology" image statistics data, range patches, neuroscience	**Gunnar Carlsson** (Stanford University)

http://www.ima.umn.edu/videos/?id=856

http://ima.umn.edu/2008-2009/ND6.15-26.09/activities/Carlsson-Gunnar/imafive-handout4up.pdf

Topological Methods for Large and Complex Data Sets

IMA Workshop on Machine Learning, Minneapolis

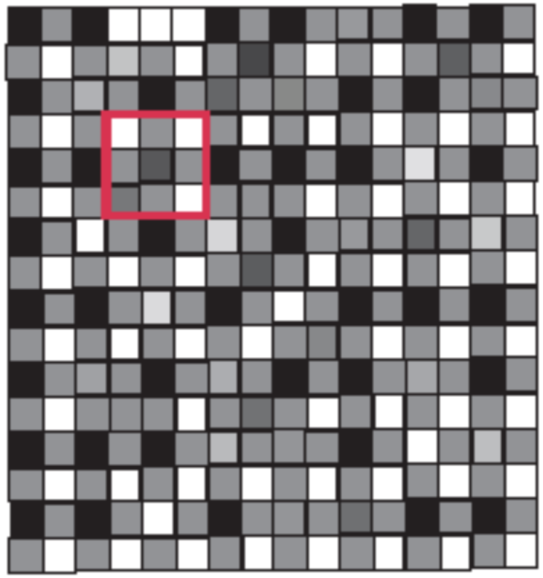
Application to Natural Image Statistics
With V. de Silva, T. Ishkanov, A. Zomorodian

Gunnar Carlsson, Stanford University

March 26, 2012

http://www.ima.umn.edu/videos/?id=1846

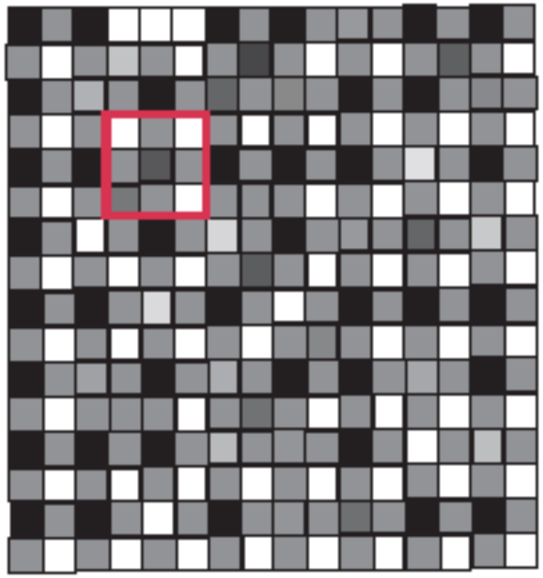
http://www.ima.umn.edu/2011-2012/W3.26-30.12/activities/Carlsson-Gunnar/imamachinefinal.pdf



An image taken by black and white digital camera can be viewed as a vector, with one coordinate for each pixel

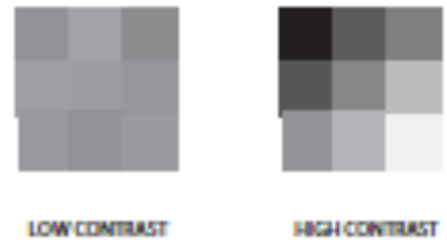
Each pixel has a “gray scale” value, can be thought of as a real number (in reality, takes one of 255 values)

Typical camera uses tens of thousands of pixels, so images lie in a very high dimensional space, call it pixel space, P



Observations:

1. Each patch gives a vector in $\mathbb{R}^9$
2. Most patches will be nearly constant, or *low contrast*, because of the presence of regions of solid shading in most images



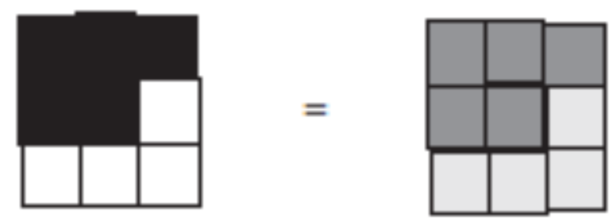
3. Low contrast will dominate statistics, not interesting

Lee-Mumford-Pedersen [LMP] study only high contrast patches.

Collection: 4.5 x 10^6 high contrast patches from a collection of images obtained by van Hateren and van der Schaaf

- Normalize mean intensity by subtracting mean from each pixel value to obtain patches with mean intensity $= 0$
- Puts data on an 8-dimensional hyperplane, $\cong \mathbb{R}^8$
- Means that we will consider as equivalent patches which can be obtained from each other by turning the intensity knob

- Normalize contrast by dividing by the D-norm, so obtain patches with D-norm $= 1$
- Means that data now lies on a 7-D ellipsoid, $\cong S^7$



Analysis

First Observation: The points fill out S^7 in the sense that every point in S^7 is "close" to a point in $\mathcal{M}$

Initially disappointing, since it means that nothing special can be said about the actual patches different from patches chosen at random

However, density of points varies a great deal from region to region

How to analyze?

Codensity

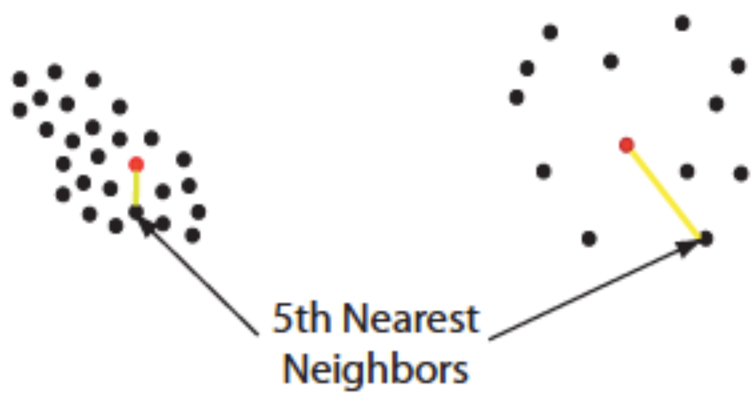
For integer $k > 0$, and PCD $\mathbb{X}$

$$\delta_k(x) = d(x, x')$$

$$x' = \text{ any } k\text{-th nearest neighbor to } x \in \mathbb{X}$$

$$\delta_k(x) \text{ large} \Longrightarrow x \text{ is sparse}$$

$$\delta_k(x) \text{ small} \Longrightarrow x \text{ is dense}$$



$\mathcal{M}[k, T]$ is $T\%$ densest points as measured by δ_k

What is the persistent homology of these $\mathcal{M}[k, T]$?

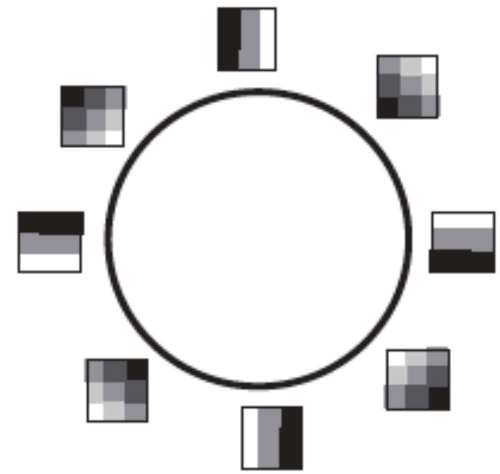
Primary Circle

5×10^4 points, $k = 300$, $T = 25$

One-dimensional barcode, suggests $\beta_1 = 1$

Is the set clustered around a circle?

Primary Circle



PRIMARY CIRCLE

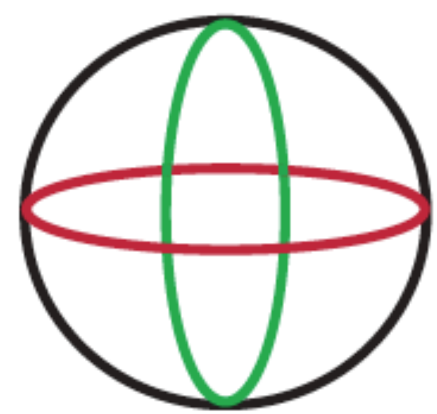
Three Circle Model

5×10^4 points, $k = 15$, $T = 25$

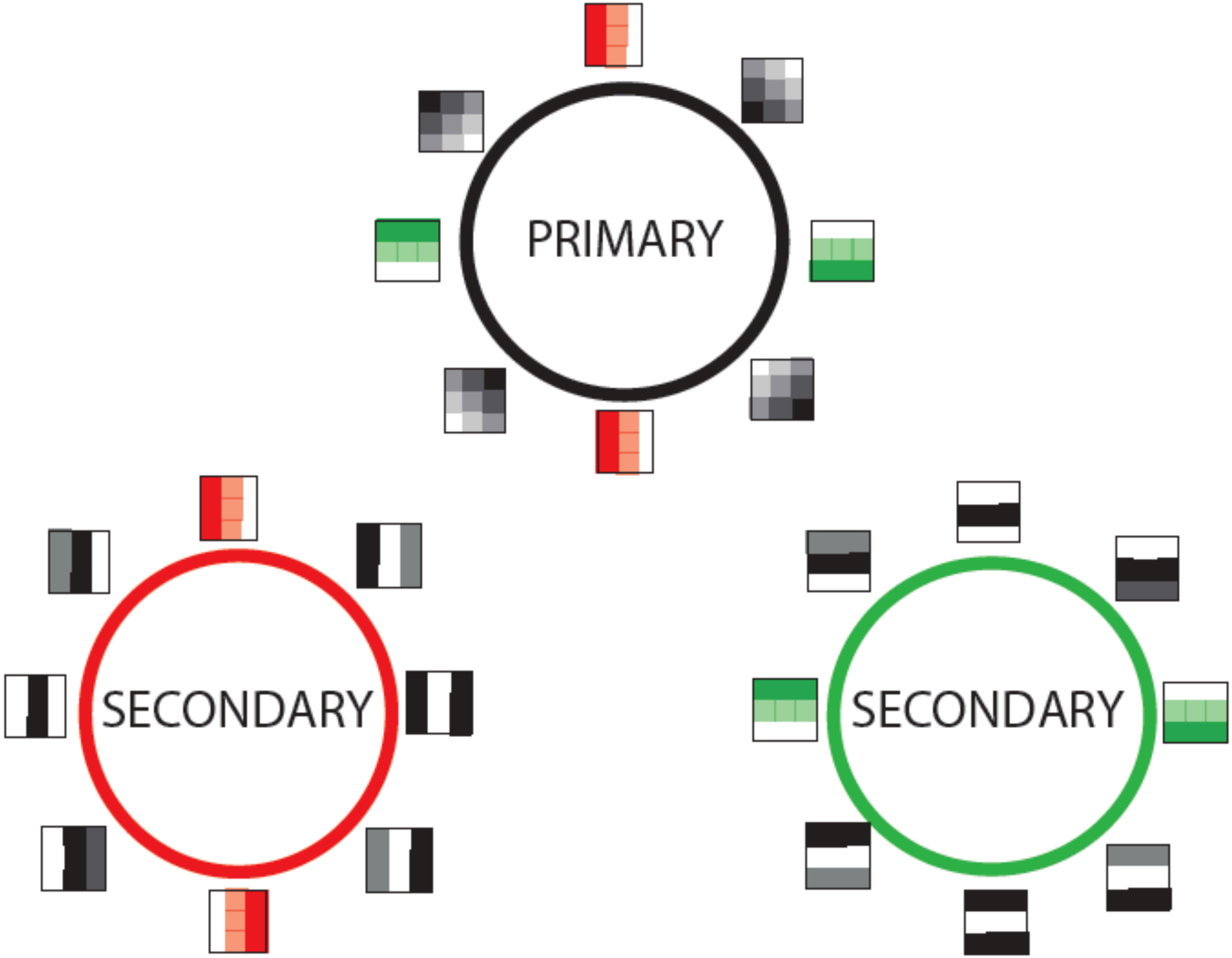
One-dimensional barcode, suggests $\beta_1 = 5$

What's the explanation for this?

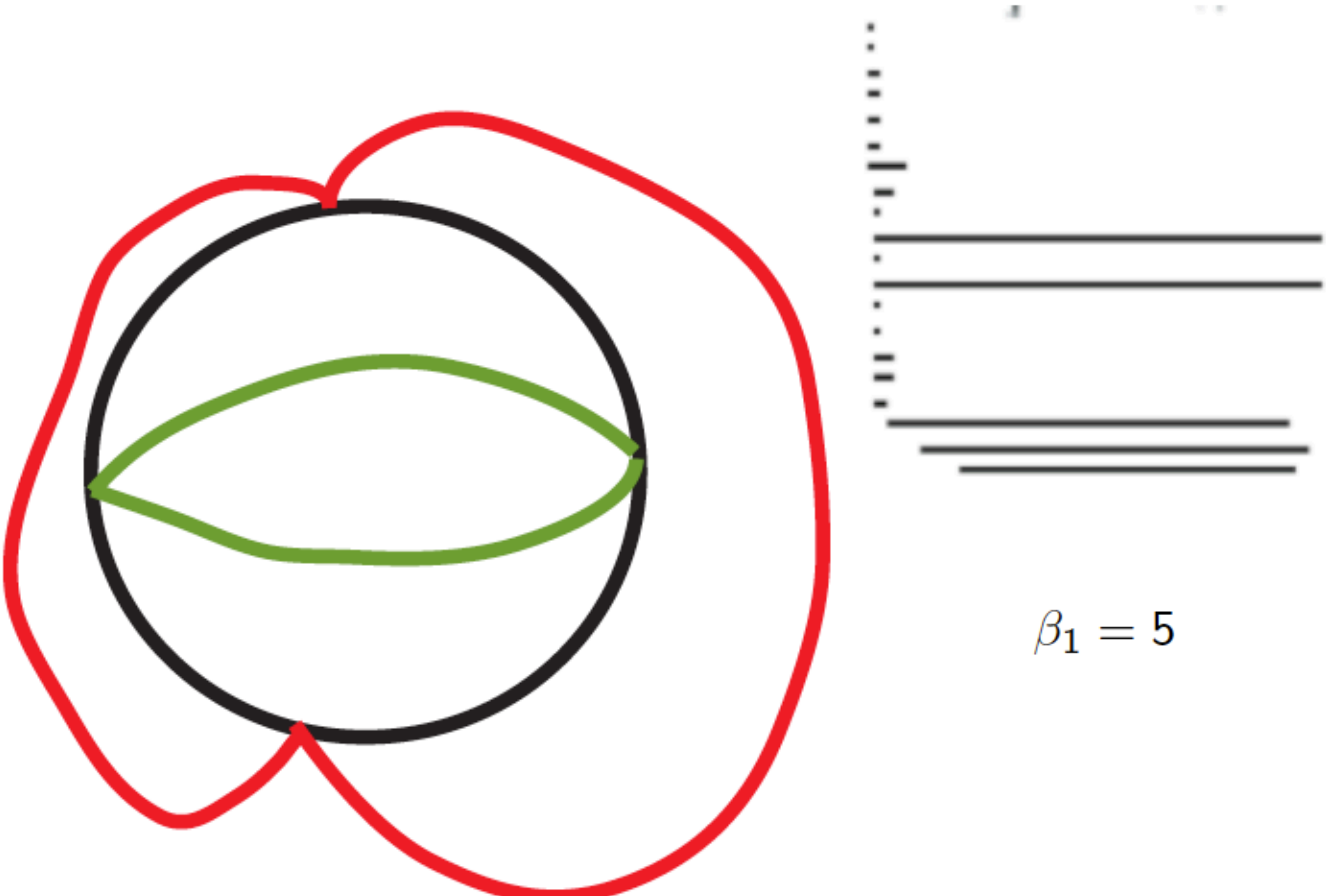
Three Circle Model



THREE CIRCLE MODEL


PRIMARY
SECONDARY
SECONDARY

5×10^4 points, $k = 15$, $T = 25$



$\beta_1 = 5$

4.5×10^6 points, $k = 100$, $T = 10$