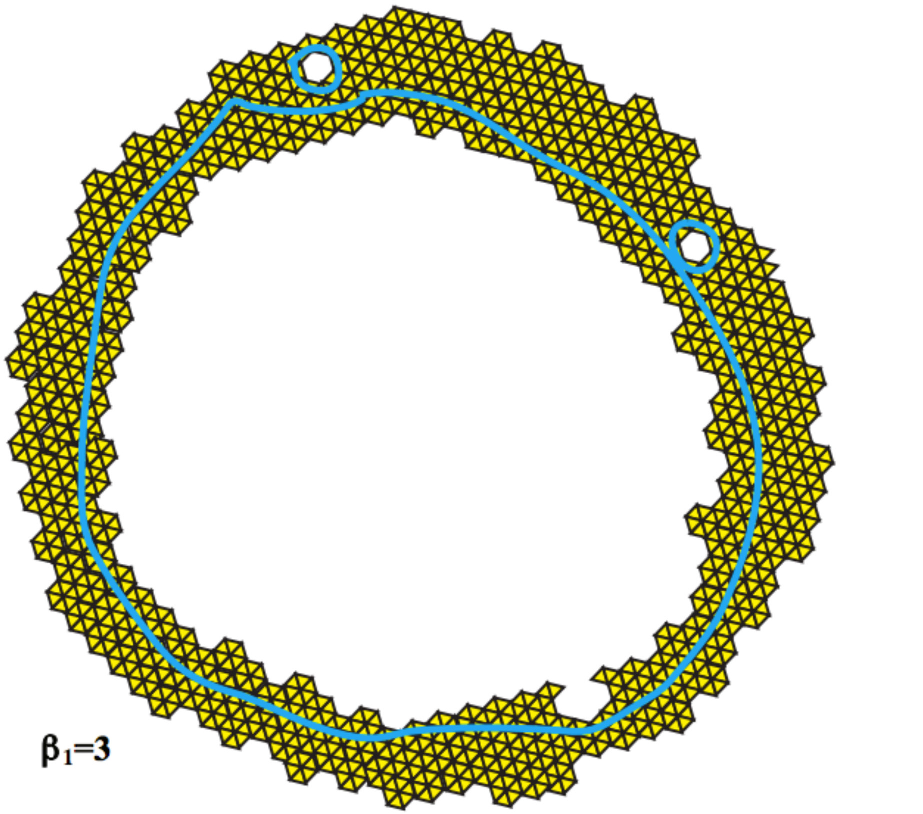
MATH:7450 (22M:305) Topics in Topology: Scientific and Engineering Applications of Algebraic Topology

Sept 11, 2013: Persistent homology.

Fall 2013 course offered through the
University of Iowa Division of Continuing Education

Isabel K. Darcy, Department of Mathematics
Applied Mathematical and Computational Sciences,
University of Iowa

http://www.math.uiowa.edu/~idarcy/AppliedTopology.html



Topology and Data. Gunnar Carlsson

www.ams.org/journals/bull/2009-46-02/S0273-0979-09-01249-X

Discrete Comput Geom 33:249–274 (2005)
DOI: 10.1007/s00454-004-1146-y

Discrete & Computational
Geometry

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y

Computing Persistent Homology*

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Discrete Comput Geom 28:511–533 (2002)
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Discrete & Computational
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Topological Persistence and Simplification*

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Stillwater, OK 74078, USA

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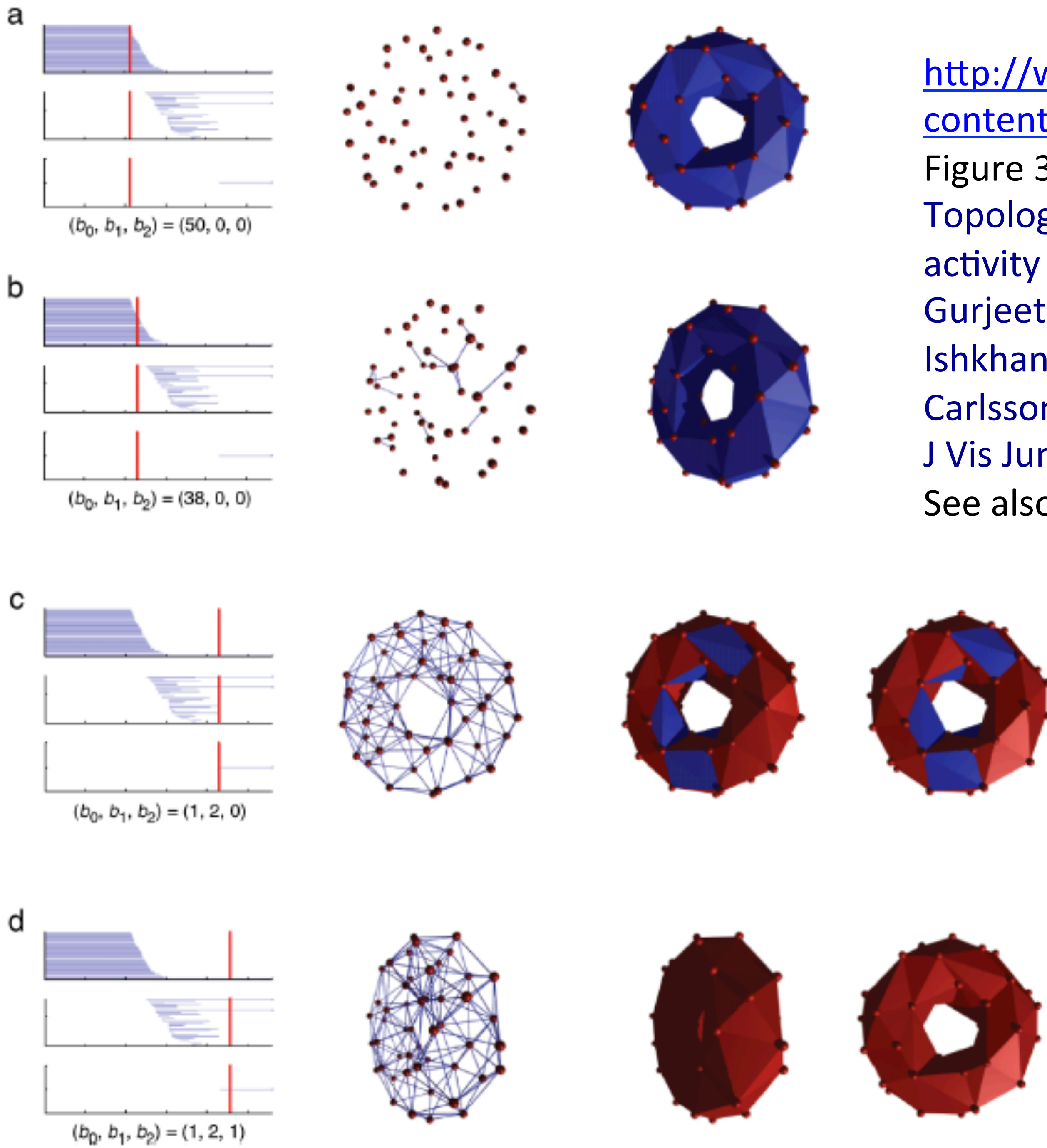
Urbana, IL 61801, USA



http://www.journalofvision.org/content/8/8/11.full

Figure 3 from

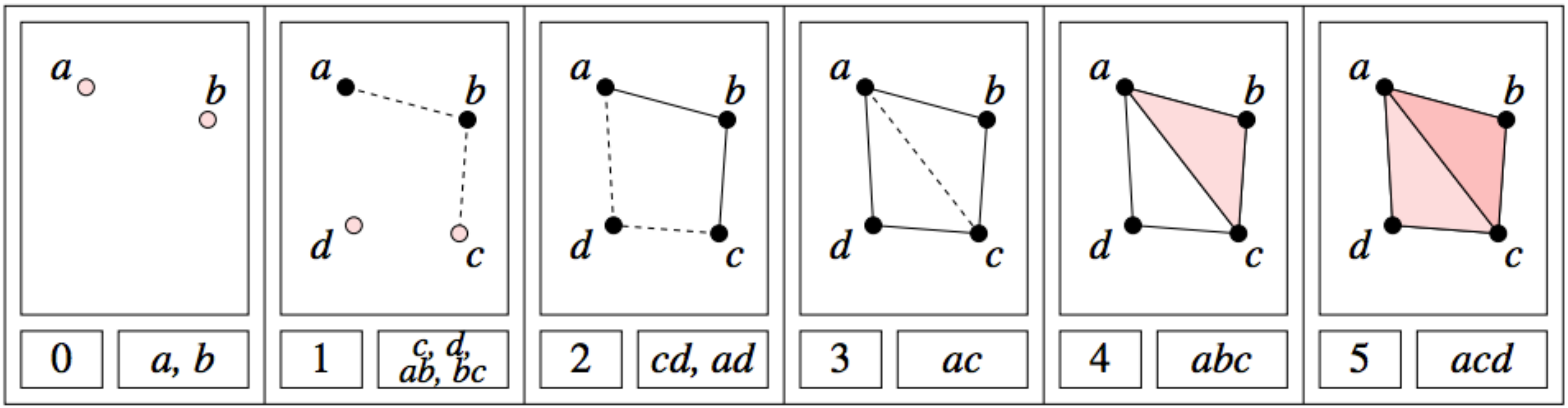
Topological analysis of population activity in visual cortex

Gurjeet Singh, Facundo Memoli, Tigran Ishkhanov, Guillermo Sapiro, Gunnar Carlsson, and Dario L. Ringach

J Vis June 30, 2008 8(8): 11

See also Figure 4 animation

A *filtered complex* is an increasing sequence of simplicial complexes: $C^0 \subset C^1 \subset C^2 \subset \ldots$



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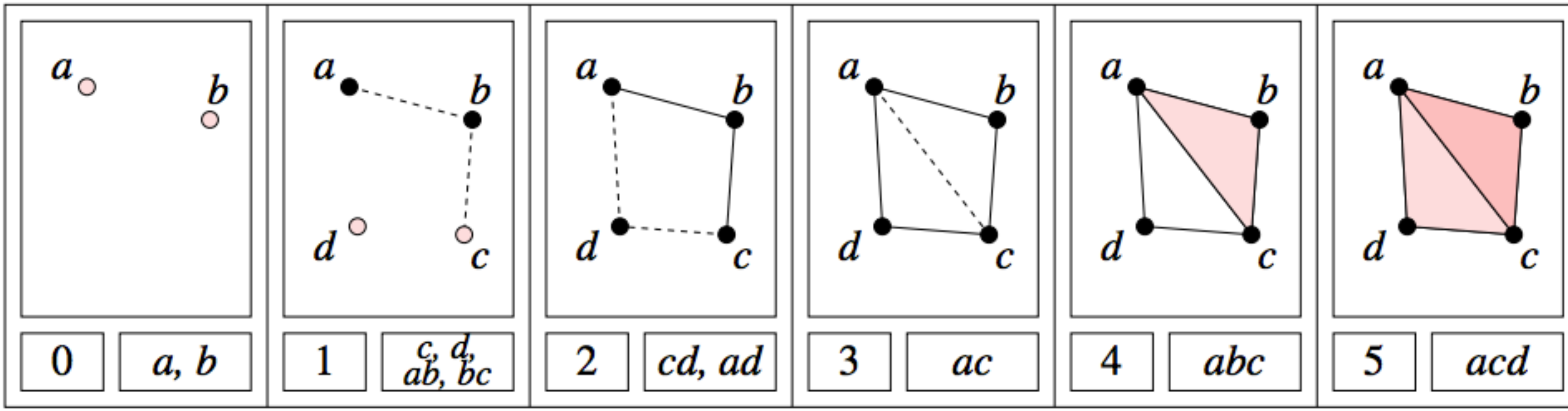
A *filtered complex* is an increasing sequence of simplicial complexes: $C^0 \subset C^1 \subset C^2 \subset \ldots$



a, b is in $C_0^0 \subset C_0^1 \subset C_0^2 \subset \ldots \subset C_0^5$

{a, b, c} is in $C_2^4 \subset C_2^5$

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$$\xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$H_0 = Z_0/B_0 =$ (kernel of ∂_0)/ (image of ∂_1)

$$= \frac{\text{null space of } M_0}{\text{column space of } M_1}$$

Rank H_0 = Rank Z_0 – Rank B_0

$$
\begin{array}{c}
\partial_3 \downarrow \\
C_2 \\
\partial_2 \downarrow \\
C_1 \\
\partial_1 \downarrow \\
C_0 \\
\partial_0 \downarrow \\
0
\end{array}
$$

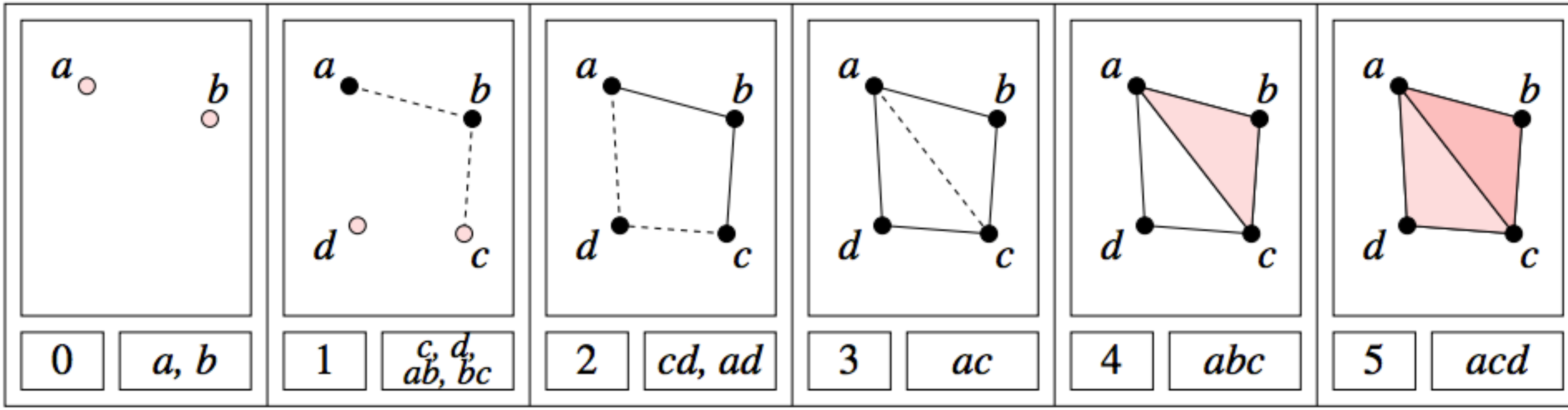
$H_0 = Z_0/B_0 = (\text{kernel of } \partial_0)/ (\text{image of } \partial_1)$

$$= \frac{\text{null space of } M_0}{\text{column space of } M_1}$$

$\text{Rank } H_0 = \text{Rank } Z_0 - \text{Rank } B_0$

$$
\begin{tikzcd}
{} \arrow[d, "\partial_3"'] & {} \arrow[d, "\partial_3"'] & {} \arrow[d, "\partial_3"'] & \\
C_2^0 \arrow[r, "f^0"] \arrow[d, "\partial_2"'] & C_2^1 \arrow[r, "f^1"] \arrow[d, "\partial_2"'] & C_2^2 \arrow[r, "f^2"] \arrow[d, "\partial_2"'] & \cdots \\
C_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & C_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & C_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
C_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"'] & C_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"'] & C_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"'] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}
$$

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y



$$
\begin{tikzcd}
C_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & C_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & C_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
C_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"'] & C_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"'] & C_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"'] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}
$$

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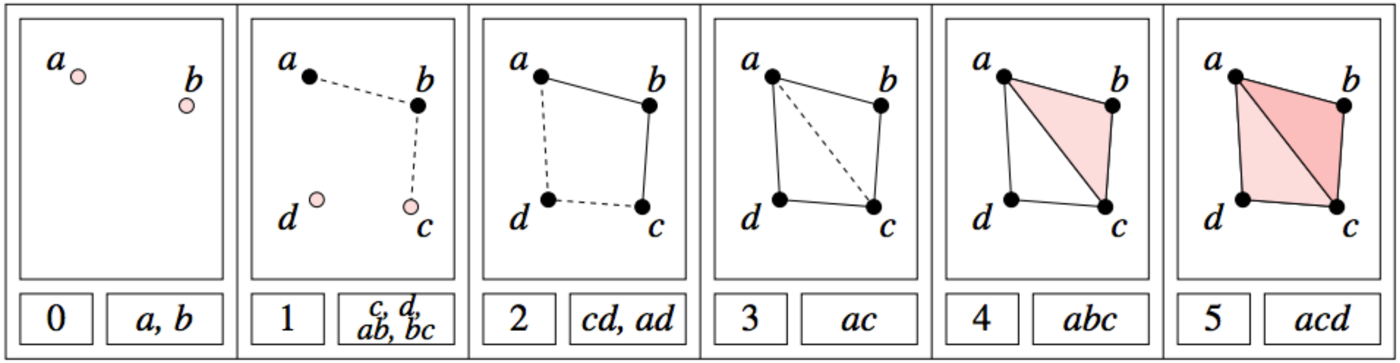
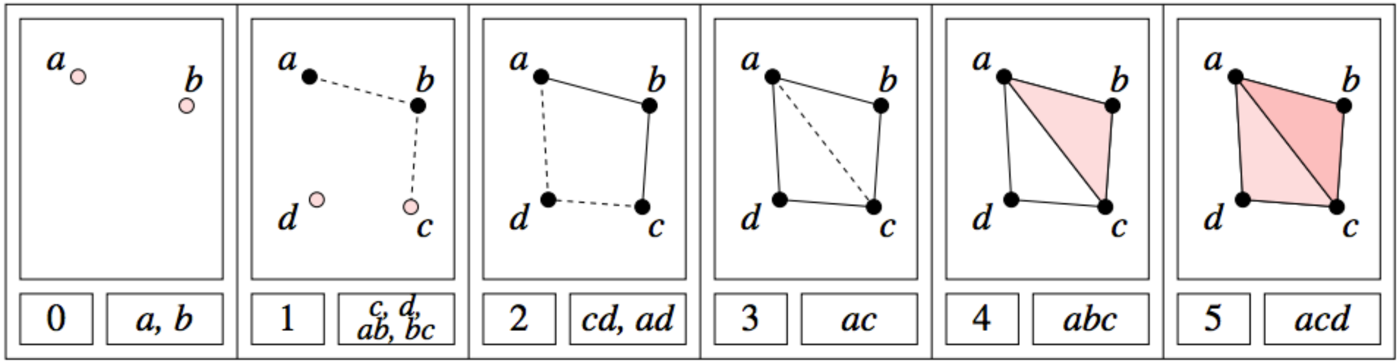


$$
\begin{tikzcd}
C_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & C_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & C_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
C_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"'] & C_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"'] & C_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"'] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}
$$

$H_0 = Z_0/B_0 =$ (kernel of ∂_0)/ (image of ∂_1)

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y



$$
\begin{tikzcd}
\mathrm{C}_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & \mathrm{C}_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & \mathrm{C}_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
\mathrm{C}_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"'] & \mathrm{C}_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"'] & \mathrm{C}_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"'] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}
$$

$$H_0^{i,0} = Z_0^i / B_0^i = (\text{kernel of } \partial_0)/\ (\text{image of } \partial_1)$$

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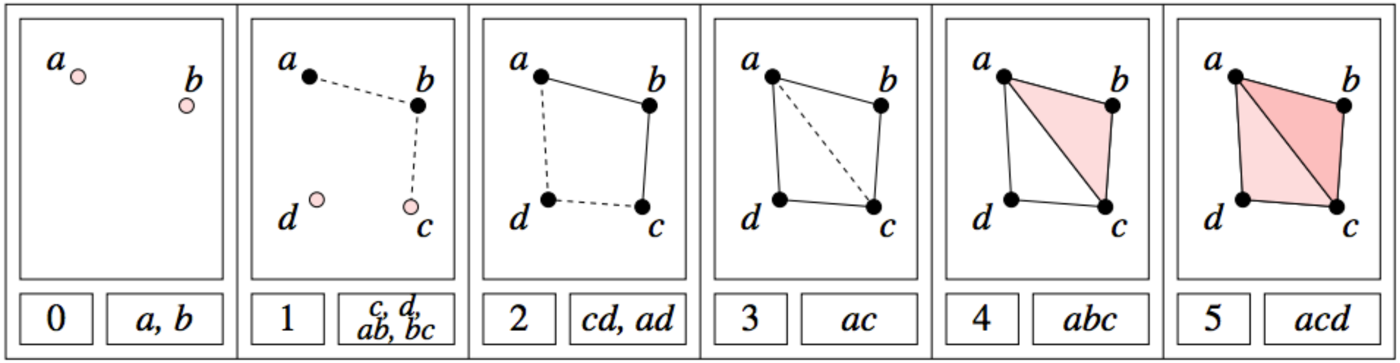

$$\begin{tikzcd}
C_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & C_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & C_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
C_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"'] & C_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"'] & C_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"'] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}$$

$$\boxed{H_0^{0,1} = Z_0^0 / (B_0^{0+1} \cap Z_0^0)}$$

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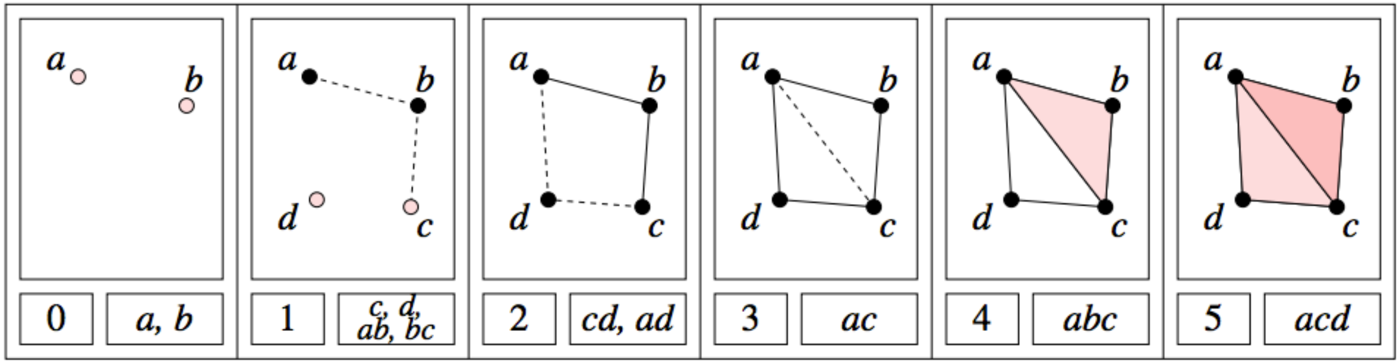
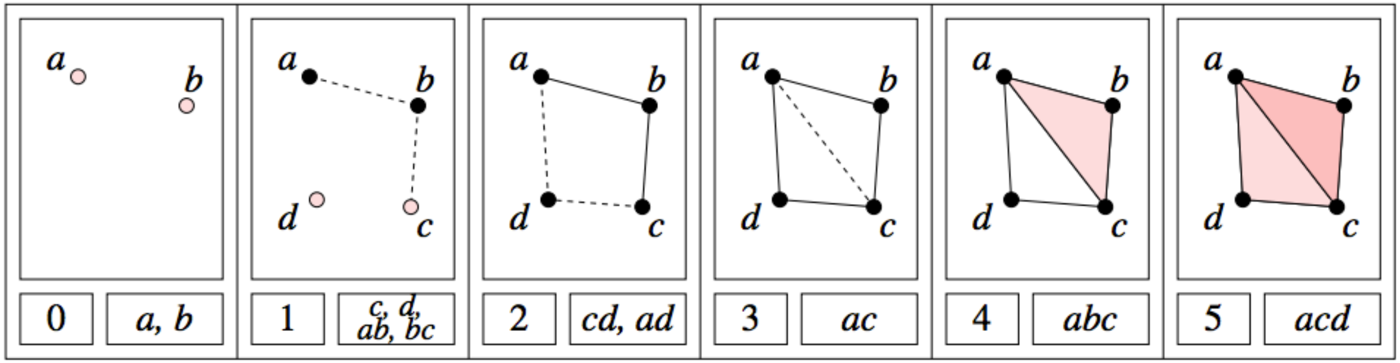


$$
\begin{tikzcd}
C_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & C_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & C_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
C_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"'] & C_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"'] & C_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"'] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}
$$

$$\boxed{H_0^{0,2} = Z_0^0 / (B_0^{0+2} \cap Z_0^0)}$$

http://link.springer.com/article/10.1007%2Fs00454-004-1146-y



$$
\begin{tikzcd}
C_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & C_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & C_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
C_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"'] & C_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"'] & C_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"'] & \cdots \\
0 & 0 & 0 &
\end{tikzcd}
$$

$$\boxed{H_0^{i,1} = Z_0^i / (B_0^{i+1} \cap Z_0^i)}$$

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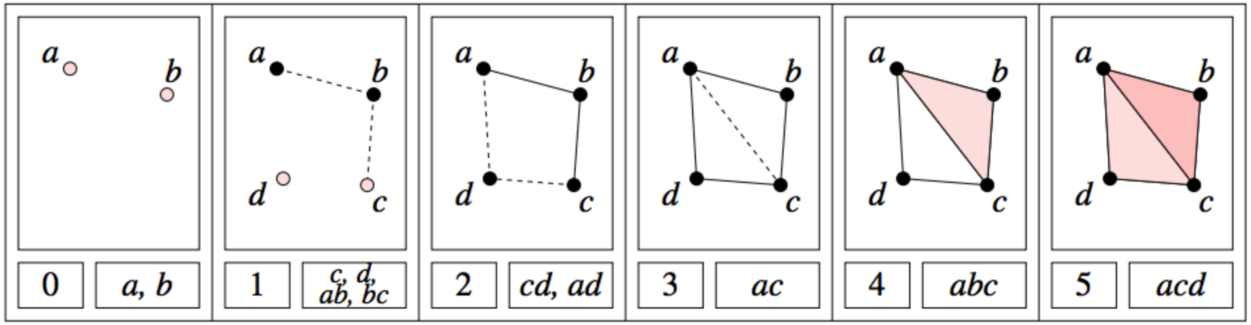

$$
\begin{tikzcd}
\mathsf{C}_2^0 \arrow[r, "f^0"] \arrow[d, "\partial_2"'] & \mathsf{C}_2^1 \arrow[r, "f^1"] \arrow[d, "\partial_2"'] & \mathsf{C}_2^2 \arrow[r, "f^2"] \arrow[d, "\partial_2"'] & \cdots \\
\mathsf{C}_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & \mathsf{C}_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & \mathsf{C}_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
\mathsf{C}_0^0 \arrow[r, "f^0"] & \mathsf{C}_0^1 \arrow[r, "f^1"] & \mathsf{C}_0^2 \arrow[r, "f^2"] & \cdots
\end{tikzcd}
$$

$$\boxed{H_0^{i,2} = Z_0^i / (B_0^{i+2} \cap Z_0^i)}$$

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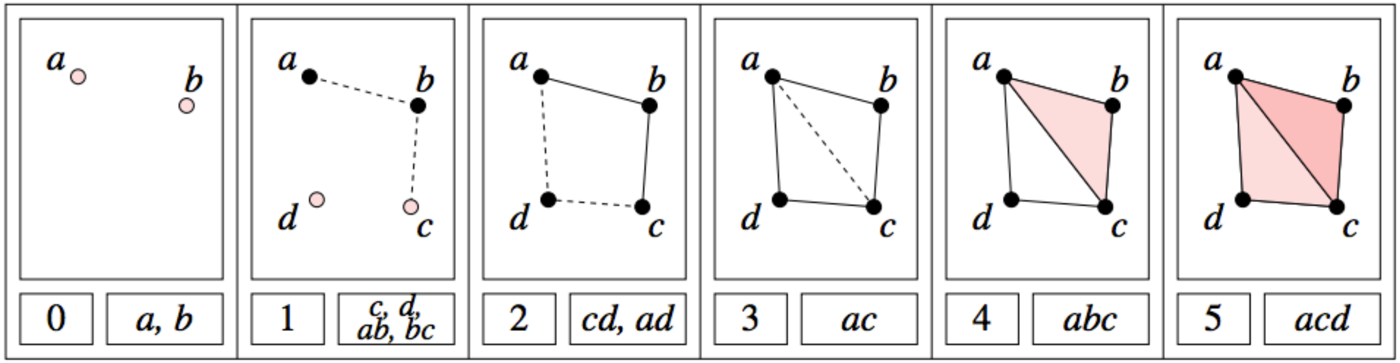

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\begin{tikzcd}
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\mathsf{C}_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"'] & \mathsf{C}_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"'] & \mathsf{C}_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"'] & \cdots \\
\mathsf{C}_0^0 \arrow[r, "f^0"] & \mathsf{C}_0^1 \arrow[r, "f^1"] & \mathsf{C}_0^2 \arrow[r, "f^2"] & \cdots
\end{tikzcd}
$$

p-persistent k^{th} homology group:

$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i)$$

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$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

$H_1 = Z_1/B_1 = $ (kernel of ∂_1)/ (image of ∂_2)

$$= \frac{\text{null space of } M_1}{\text{column space of } M_2}$$

$$= \frac{\langle e_1 + e_2 + e_3,\ e_4 + e_5 + e_6 \rangle}{\langle e_4 + e_5 + e_6 \rangle}$$

Rank H_1 = Rank Z_1 – Rank B_1 = 2 – 1 = 1

$C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1}$.

Let σ be a k-simplex

σ is a *cycle* if σ is in Z_k = kernel of ∂_k

= null space of M_k

I.e., σ is a *cycle* if $\partial(\sigma) = 0$

σ is a ***boundary*** if σ is in B_k = image of ∂_{k+1}

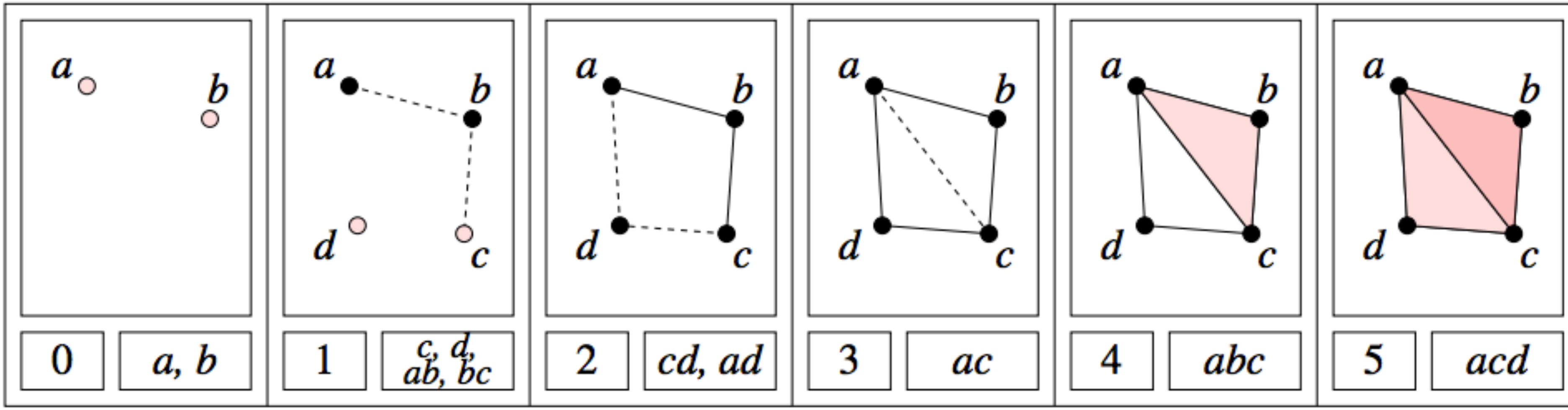
= column space of M_{k+1}

I.e., σ is a ***boundary*** if there exits τ in C_{k+1}

such that $\partial(\tau) = \sigma$

Topological Persistence and Simplification, 2002

Figure from: Computing Persistent Homology, 2005



Let σ be a k-simplex

σ is a *positive simplex* if it creates a cycle when it enters.

σ is a *negative simplex* if it destroys a cycle when it enters.

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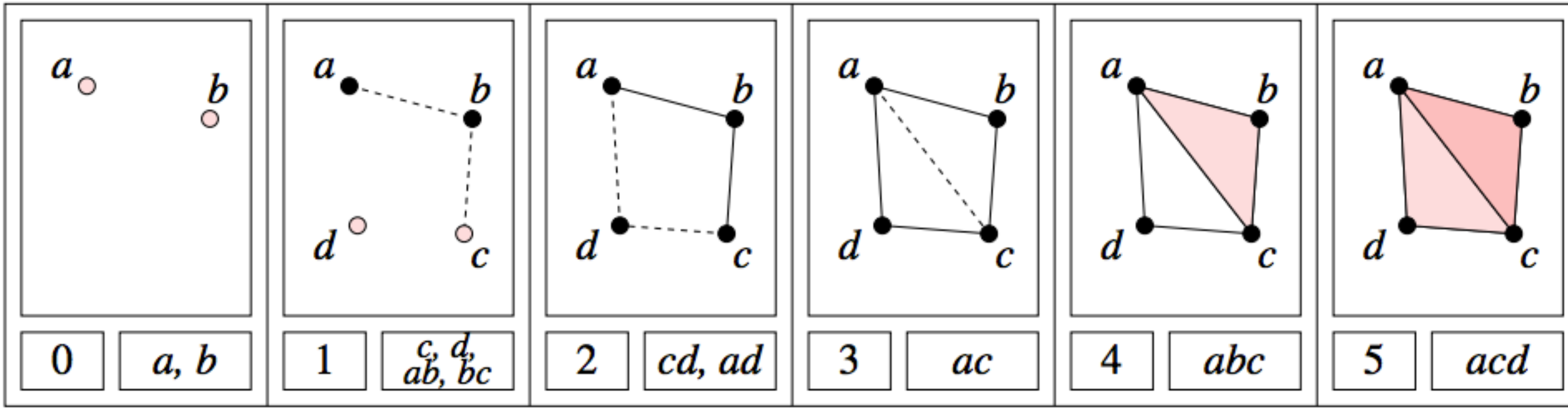
Computing Persistent Homology by Afra Zomorodian, Gunnar Carlsson



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a	b	c	d	ab	bc	cd	ad	ac	abc	acd
0	1	2	3	4	5	6	7	8	9	10

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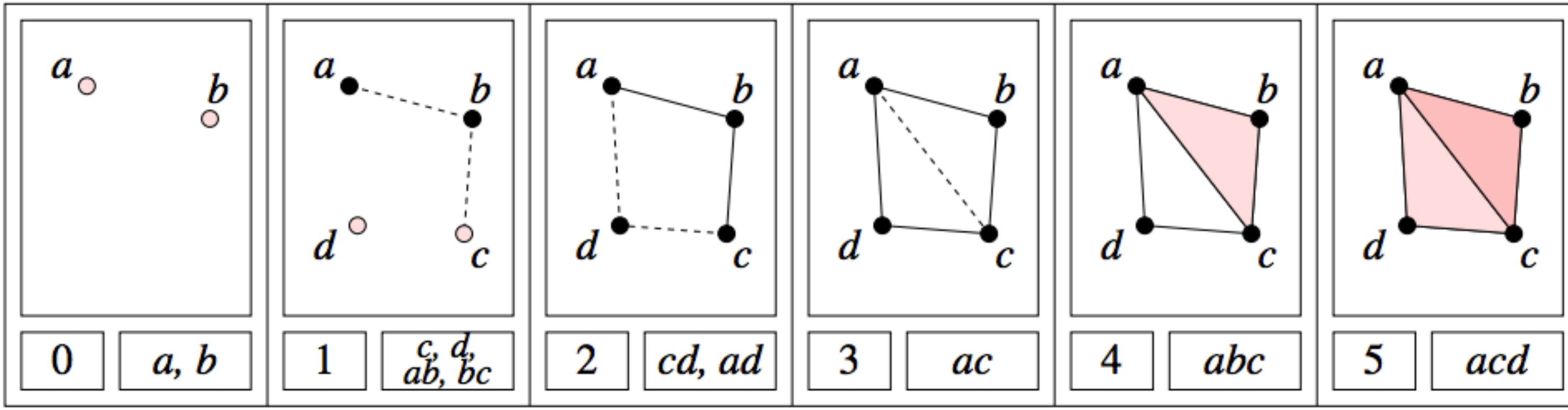
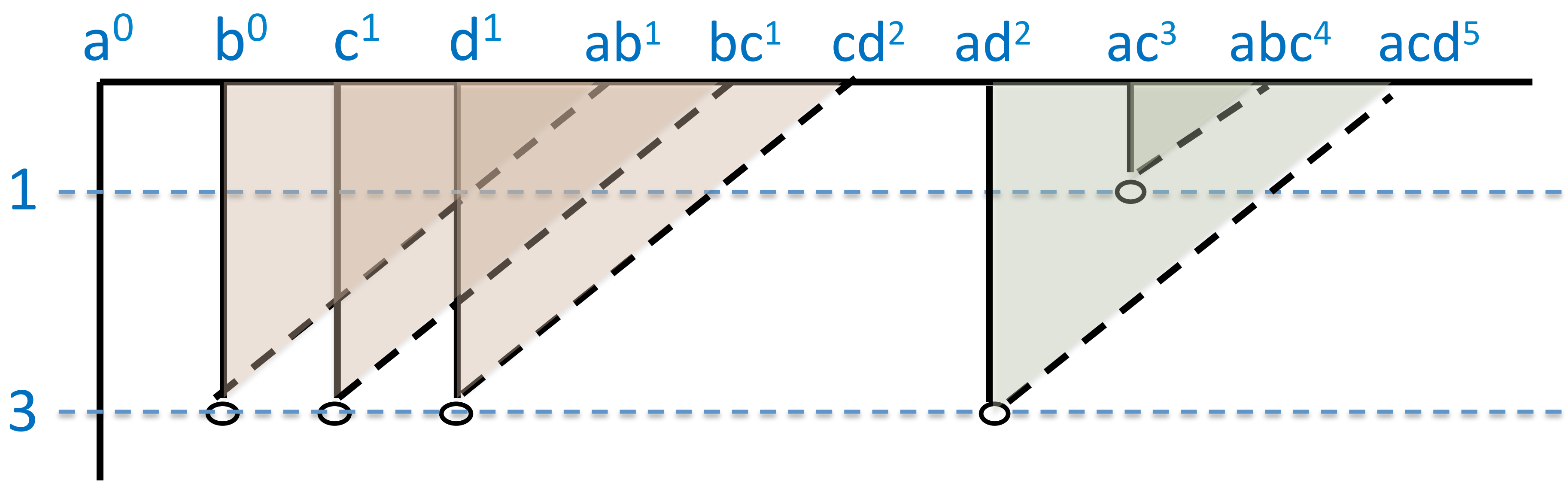
Computing Persistent Homology by Afra Zomorodian, Gunnar Carlsson

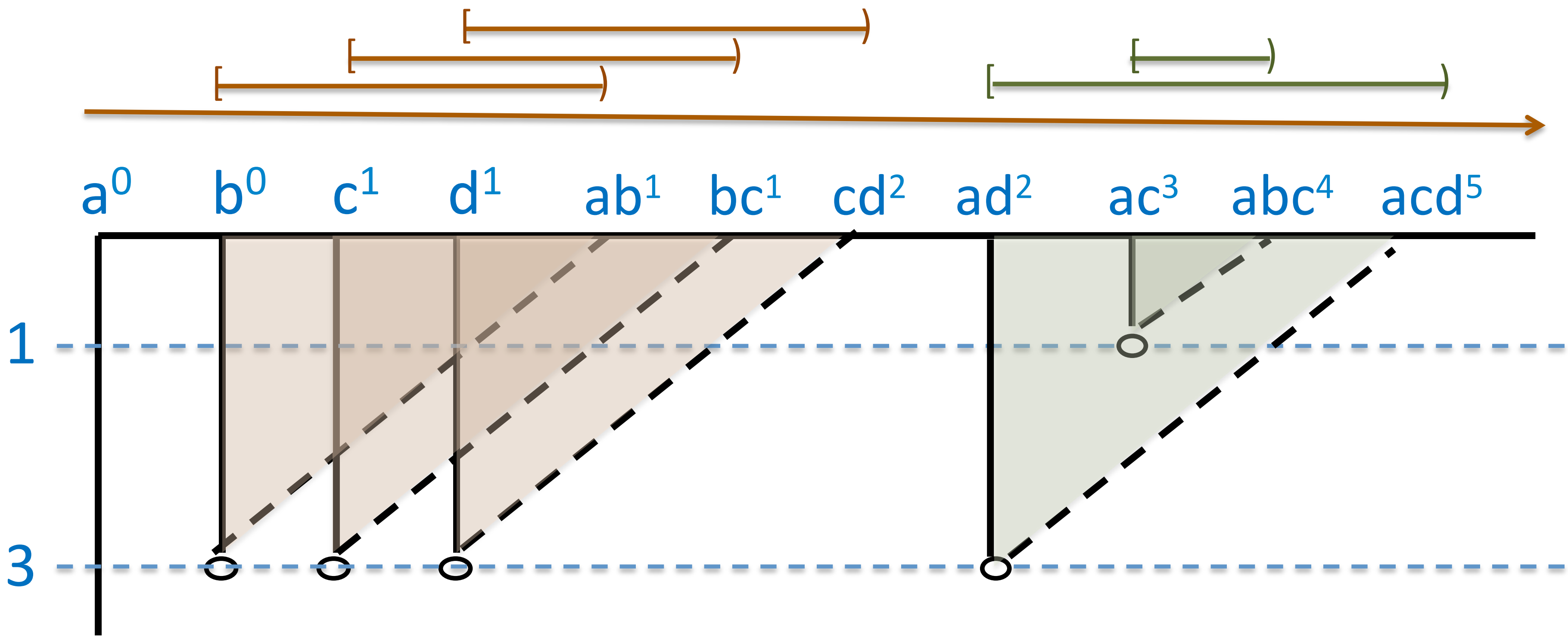


0	0	1	1	1	1	2	2	3	4	5
+	+	+	+	-	-	-	+	+	-	-
a	b	c	d	ab	bc	cd	ad	ac	abc	acd
0	1	2	3	4	5	6	7	8	9	10
	4	5	6				10	9		
	a,b	b,c	c,d				ad	ac		

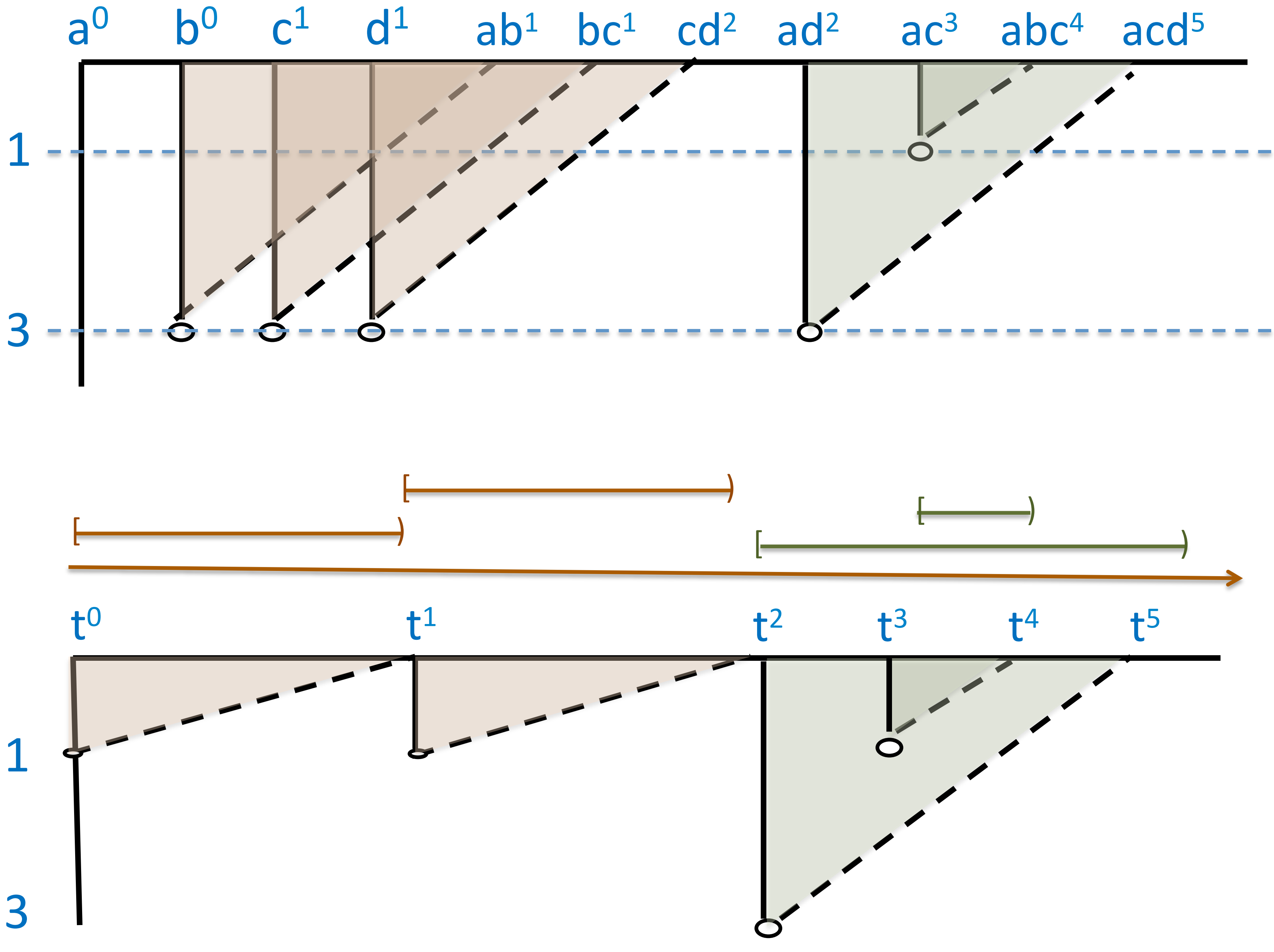
http://link.springer.com/article/10.1007%2Fs00454-004-1146-y

0	0	1	1	1	1	2	2	3	4	5
+	+	+	+	-	-	-	+	+	-	-
a	b	c	d	ab	bc	cd	ad	ac	abc	acd
0	1	2	3	4	5	6	7	8	9	10
	4	5	6				10	9		
	a,b	b,c	c,d				ad	ac		





a0
b0
c1
d1
ab1
bc1
cd2
ad2
ac3
abc4
acd5
1
3



a0
b0
c1
d1
ab1
bc1
cd2
ad2
ac3
abc4
acd5
1
3
t0
t1
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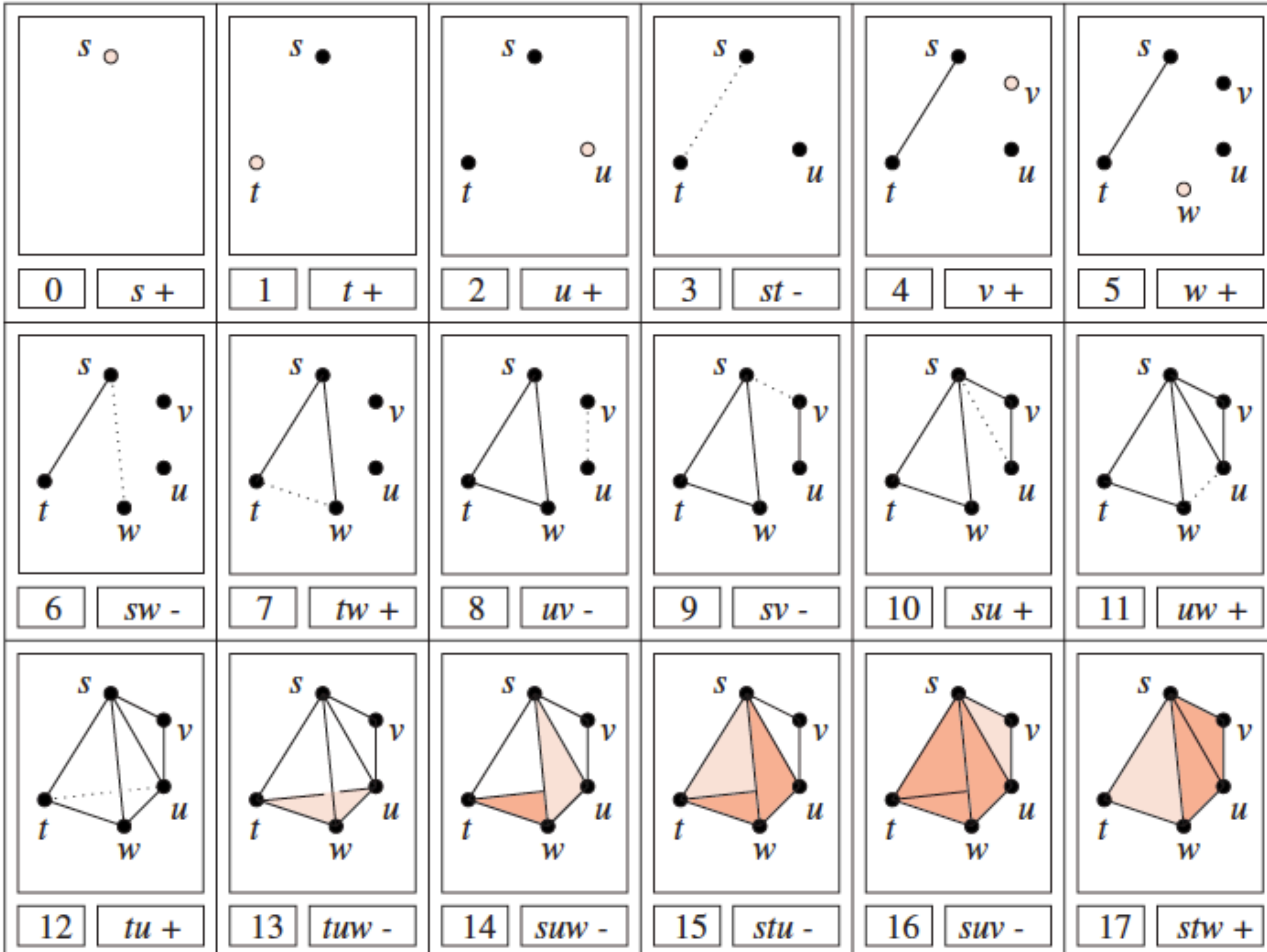
t4
t5
1
3



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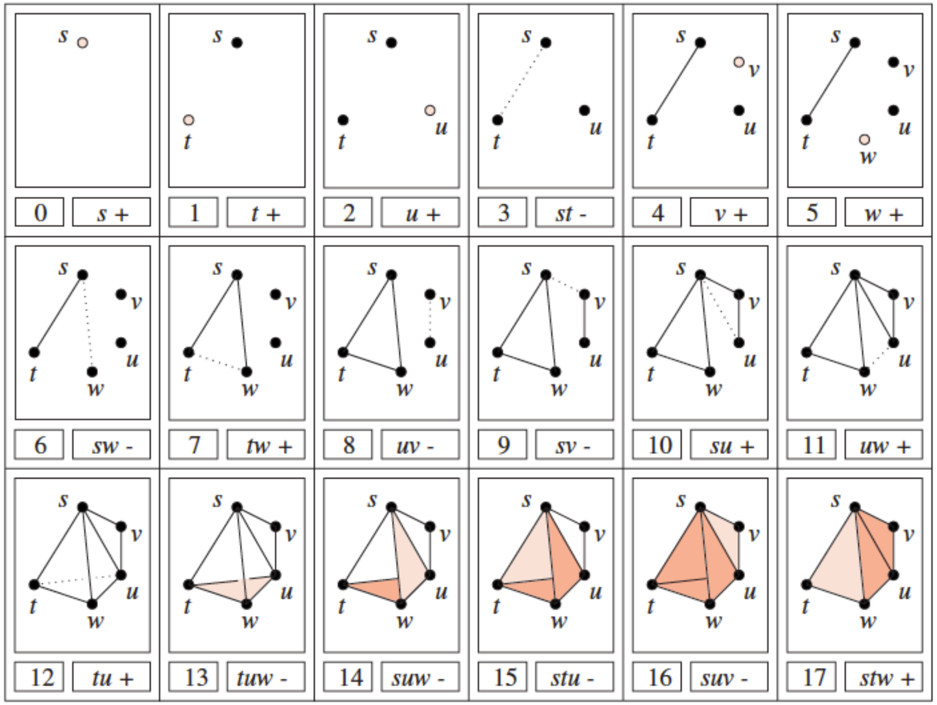

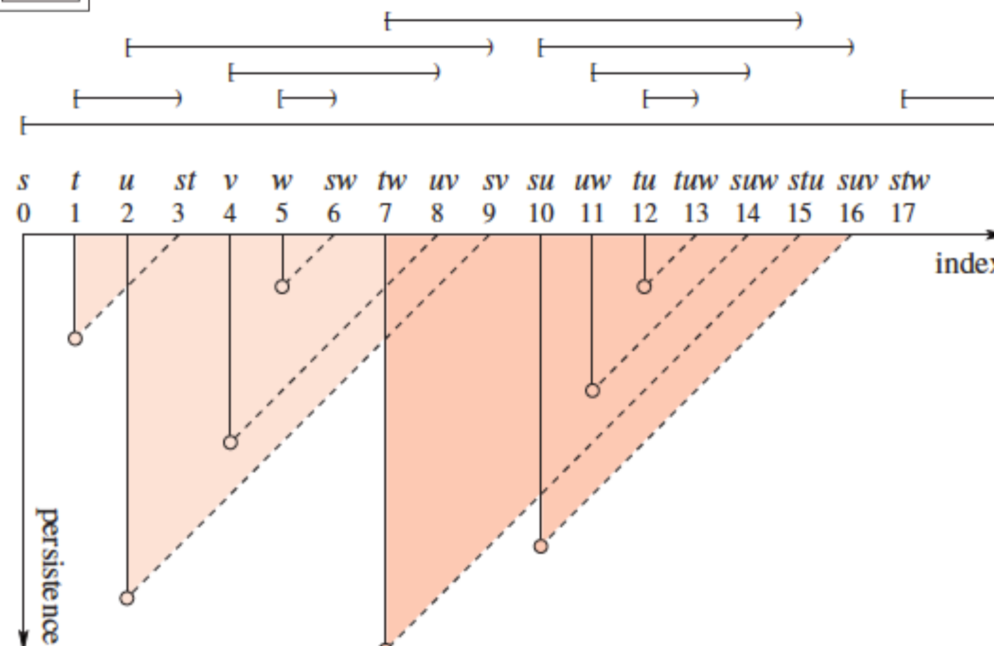

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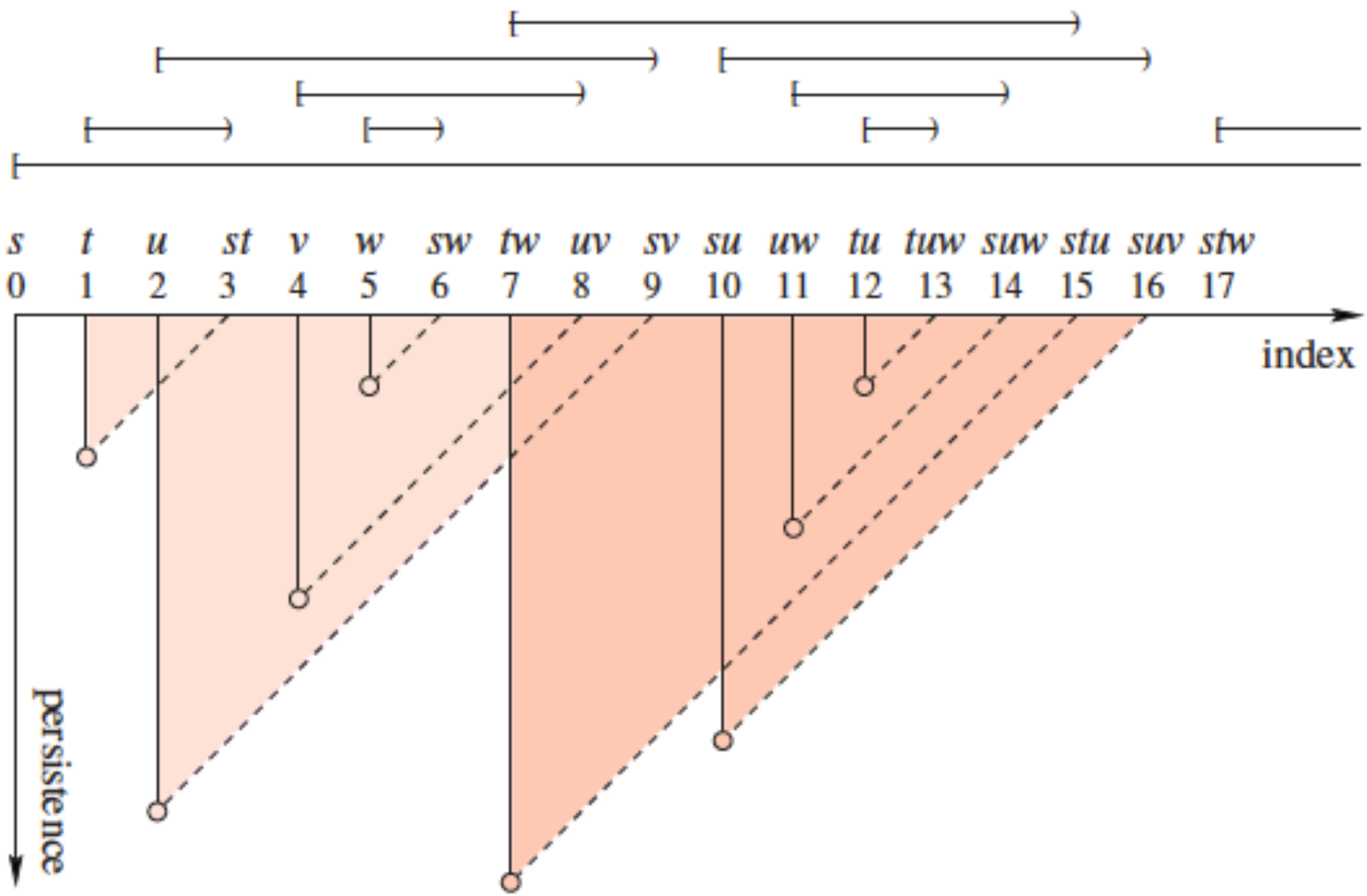

Topological Persistence and Simplification:link.springer.com/article/10.1007/s00454-002-2885-2