CROSSING CHANGES AND MINIMAL DIAGRAMS

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The unknotting number of a knot is the minimal number of crossing changes needed to convert the knot into the unknot where the minimum is taken over all possible diagrams of the knot. For example, the minimal diagram of the knot 10_8 requires 3 crossing changes in order to change it to the unknot. Thus, by looking only at the minimal diagram of 10_8, it is clear that $u(10_8) \leq 3$ (figure 1). Nakanishi [5] and Bleiler [2] proved that $u(10_8) = 2$. They found a non-minimal diagram of the knot 10_8 in which two crossing changes suffice to obtain the unknot (figure 2). Lower bounds on unknotting number can be found by looking at how knot invariants are affected by crossing changes. Nakanishi [5] and Bleiler [2] used signature to prove that $u(10_8) = 2$.

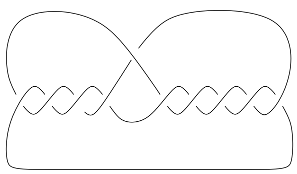

FIGURE 1. Minimal diagram of knot 10_8

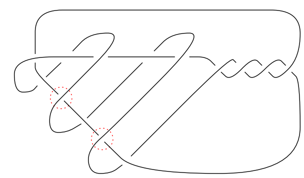

FIGURE 2. A non-minimal diagram of knot 10_8

Bernhard [1] noticed that one can determine that $u(10_8) = 2$ by using a sequence of crossing changes within minimal diagrams and ambient isotopies as shown in figure. A crossing change within the minimal diagram of 10_8 results in the knot 6_2. We then use an ambient isotopy to change this non-minimal diagram of 6_2 to a minimal diagram. The unknot can then be obtained by changing one crossing within the minimal diagram of 6_2. Bernhard hypothesized that the unknotting number of a knot could be determined by only looking at minimal diagrams by using ambient isotopies between crossing changes.

If we remove the condition that only minimal diagrams are used, the unknotting number of a knot can be found by using a sequence of crossing changes (in both minimal and non-minimal diagrams) and ambient isotopies. In the original definition of unknotting number we were restricted to doing all the crossing changes in a single diagram and then minimizing over all diagrams. It is well known that alternating between crossing changes and ambient isotopies is an equivalent method for finding the unknotting number. To see this we can transform a knot into the unknot by a sequence of crossing changes and ambient isotopies, but whenever we do a crossing change we tie an imaginary string between the segments of the knot that pass through each other. Once we obtain the unknot, we shorten all the imaginary strings and project onto 2-dimensions. Changing all the crossings with the imaginary strings results in a projection of the original knot in which all the crossing changes needed to transform the knot to the unknot can be seen.

A generalization of the unknotting number is the crossing change distance between knots, $d(K_1, K_2) =$ the minimum number of crossing changes needed to change K_1 into K_2. In [3], it was shown that non-minimal diagrams are required in order to calculate $d(K_1, K_2)$. For example, $d_2(5_1, 5_2) = 1$, but the crossing change cannot be seen in either the minimal diagram of 5_1 or 5_2 (see figure 3). However, Kohn [4] proved that only minimal diagrams are needed to determine if a 4-plat knot has unknotting number 1. He similarly proved that only minimal diagrams are needed to determine if a 4-plat link has crossing change distance one to the unlink of 2-components.

When non-minimal diagrams are needed is also a question posed by the biologist Andrzej Stasiak. He studies a class of proteins called topoisomerases. These proteins will perform crossing changes on knotted circular DNA molecules. The crossing change distance can be used to determine the minimum number of times a topoisomerase performs a crossing change in order to change one knotted DNA configuration into a different knotted DNA configuration. Thus, Dr. Stasiak wished to know if the crossing changes could always be seen within minimal diagrams or if their were any non-obvious crossing changes.

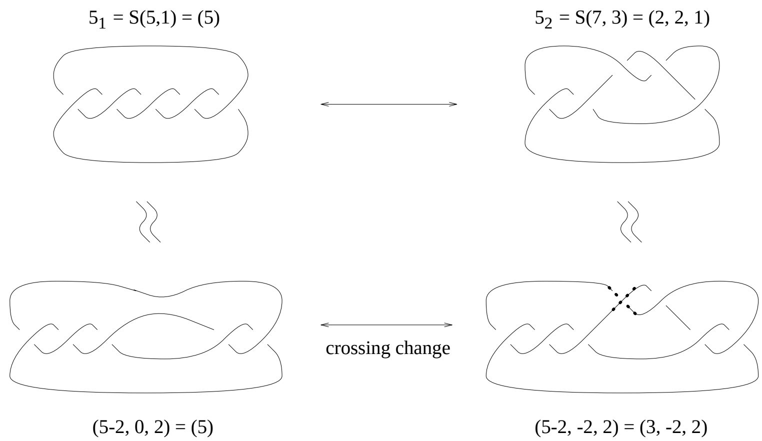

FIGURE 3. $d_2(5_1, 5_2) = 1$

In Theorem 1 we will determine exactly when a non-minimal diagram is needed to change one knot/link into another knot/link when both knots/links belong to the 4-plat family. Four-plats (also called 2-bridge or rational) are knots or 2-component links of the form shown in figure 4. Note that $5_1, 5_2, 10_8$ and all prime knots with less than 8 crossings are 4-plats. The 4-plat $< c_1, ..., c_n >$ can also be denoted by $S(a, b)$ where $\frac{a}{b} = c_1 + \frac{1}{c_2 + \dots \frac{1}{c_n}}$. Two 4-plats, $S(a_1, b_1)$ and $S(a_2, b_2)$, $a_1 a_2 \geq 0$, are equivalent if and only if $a_1 = a_2$ and $b_1 {b_2}^{\pm 1} = 1 \bmod a_1$.

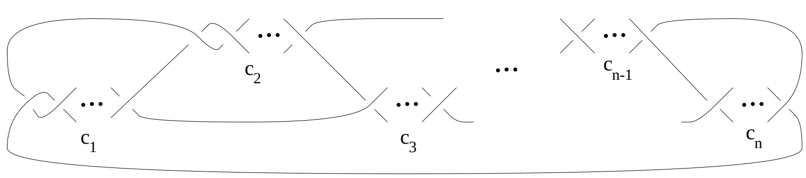

FIGURE 4. 4-plat $< c_1, ..., c_n >$

The following seven lemmas will be used to prove theorem 1.

1.) [3] If $S(u, v) \; = < c_1, ..., c_n >$ and $d(S(a, b), S(u, v)) \; = \; 1$, then $S(a, b) =< c_n, ..., c_1 + a_1, ...a_k, \pm 2, -a_k, ..., -a_1 >$, where k is odd and one of the following holds:

i.) $a_1 \geq 0$ and $a_i > 0$ for all $i > 1$.

ii.) $a_1 \leq 0$ and $a_i < 0$ for all $i > 1$.

iii.) $k = 3$ and $(a_1, a_2, a_3) = (0, 1, -1)$.

(can also take $|c_1| \geq 2$ or can take n to be even or odd.)

2.) [] A reduced alternating diagram is minimal. Hence $< c_1, ..., c_n >$ is minimal if $c_1 \neq 0$, $c_n \neq 0$ and $c_i \geq 0$ for all i or $c_i \leq 0$ for all i.
3.) $< c_1, ..., c_n, \pm 1 >=< c_1, ..., c_n \pm 1 >$
4.) $< c_1, ..., c_{i-1}, 0, c_{i+1}, ..., c_n >=< c_1, ..., c_{i-1} + c_{i+1}, ..., c_n >$
5.) $< c_1, ..., c_{n-1}, c_n, 0 >=< c_1, ..., c_{n-1} >$
6.) $< c_1, ..., c_n > = < c_n, ..., c_1 >$?
7.) [Kohn] $< c_1, ..., c_{i-1}, c_i, c_{i+1}, ..., c_n >=< c_1, ..., c_{i-1} - 1, 1, -c_i - 1, -c_{i+1}, ..., -c_n >$

The proof consists of taking $S(u, v) =< c_n, ..., c_1 + a_1, ...a_k, \pm 2, -a_k, ..., -a_1 >$ and changing this projection into a minimal projection of the form in lemma 2 by using the operations in lemmas 3 – 7. Unfortunately many cases and subcases result. Hence a program was written to check all the cases and output the results in latex. A portion of the output is included below. The entire proof and/or program is available at www.math.uiowa.edu/∼idarcy.

Let $cr(K)$ denote the minimal crossing number of K.

Theorem. *Suppose $d(S(a, b), S(u, v)) = 1$ and $cr(S(a, b) \geq cr(S(u, v))$, then if a non-minimal diagram is needed to see the crossing change, $cr(S(a, b)) = cr(S(u, v))$. In this case if $S(u, v) = < c_1, ..., c_n >$, then $S(a, b)$ has one of the following forms:*
$S(a, b) =< c_n, ..., c_1 + 0, 2, 0 >= S(u, v)$
etc.

Proof:
From the continued fraction expansion of $\frac{u}{v}$, we see that $S(u, v)$ has a minimal diagram of the form $< c_1, ..., c_n >$ where $c_i \geq 0$ for all i or $c_i \leq 0$ for all i. By lemmas 3 - 6, we can take $|c_i| > 0$ for all i and $|c_n| > 1$ if $n > 1$.

Case 1: Suppose $S(u, v) =< c_1, ..., c_n >$ where $c_i > 0$ for all i and $c_n > 1$ if $n > 1$ (can also take ($c_1 > 1$ if needed).

By lemma 1, $S(a, b) =< c_n, ..., c_1 + a_1, ...a_k, \pm 2, -a_k, ..., -a_1 >$, where k is odd and one of the following holds:

i.) $a_1 \geq 0$ and $a_i > 0$ for all $i > 1$.
ii.) $a_1 \leq 0$ and $a_i < 0$ for all $i > 1$.
iii.) $k = 3$ and $(a_1, a_2, a_3) = (0, 1, -1)$.

Case 1i: Suppose $a_1 \geq 0$ and $a_i > 0$ for all $i > 1$

Case 1iA: S(a, b) = $< c_n, ..., c_1 + a_1, ..., a_k, 2, -a_k, ..., -a_1 >$
S(a, b) = $< c_n, ..., c_1 + a_1, ..., a_k, 2, -a_k, ..., -a_1 > = < c_n, ..., c_1 + a_1, ..., a_k, 2 - 1, 1, a_k - 1, ..., a_1 >$.
Note $< c_n, ..., c_1 + a_1, ..., a_k, 2 - 1, 1, a_k - 1, ..., a_1 >$ is a minimal diagram with more crossings than $S(u, v)$ if $k > 1$ or $a_k > 1$.
Note also that changing $2 - 1 = 1$ to -1 (i.e. changing a crossing) in this diagram results in $S(u, v)$:
$< c_n, ..., c_1 + a_1, ..., a_k, -1, 1, a_k - 1, ..., a_1 > = < c_n, ..., c_1 + a_1, ..., a_k - 1, 1, 0, -1, -a_k + 1, ..., -a_1 > =< c_n, ..., c_1 + a_1, ..., a_k - 1, 0, -a_k + 1, ..., -a_1 > =< c_n, ..., c_1 + 0 > =< c_n, ..., c_1 >= S(u, v)$

Case 1iAa) $k = 1, a_k = 1$.
$S(a, b) =< c_n, ..., c_1 + a_1, ..., a_k, 2, -a_k, ..., -a_1 > = < c_n, ..., c_1 + 1, 2, -1 > = < c_n, ..., c_1 + 1, 2 - 1, 1, 0 > = < c_n, ..., c_1 + 1, 2 - 1 >$ is a minimal diagram with more crossings than $S(u, v)$.

Note also that changing $2-1=1$ to -1 (i.e. changing a crossing) in this diagram results in $S(u,v)$:

$< c_n, ..., c_1+1, -1 > = < c_n, ..., c_1, 1, 0 > = < c_n, ..., c_1 >= S(u,v)$

Case 1iAb) $k=1, a_k=0$.

$S(a,b) =< c_n, ..., c_1+a_1, ..., a_k, 2, -a_k, ..., -a_1 > =< c_n, ..., c_1+0, 2, 0 > =< c_n, ..., c_1 >= S(u,v)$. Hence $S(a,b)$ and $S(u,v)$ have the same number of crossings.

References

[1] James A. Bernhard. Unknotting numbers and minimal knot diagrams. *J. Knot Theory Ramifications*, 3(1):1–5, 1994.

[2] Steven A. Bleiler. A note on unknotting number. *Math. Proc. Cambridge Philos. Soc.*, 96(3):469–471, 1984.

[3] Isabel Dazey Darcy and De Witt Sumners. A strand passage metric for topoisomerase action. In *KNOTS '96 (Tokyo)*, pages 267–278. World Sci. Publ., River Edge, NJ, 1997.

[4] Peter Kohn. Two-bridge links with unlinking number one. *Proc. Amer. Math. Soc.*, 113(4):1135–1147, 1991.

[5] Yasutaka Nakanishi. A note on unknotting number. *Math. Sem. Notes Kobe Univ.*, 9(1):99–108, 1981.