MATH 115: Introduction to Real Analysis
Midterm I, Fall 2014

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Rules of the exam

- You have 50 minutes to complete this exam.
- Show your work! – any correct answer without an explanation will get you only ten percent of the total points on that problem.
- Please read the questions carefully; some ask for more than one thing.
- When applicable, BOX the answer.
- Do not forget to write your name.

Good luck!
PROBLEM 1: (25 points) Define each of the terms listed below:

1. Countable set
2. An accumulation point of a set $E \subseteq \mathbb{R}$
3. The outer measure $\mu^*(E)$ of a set $E \subseteq \mathbb{R}$
4. A $F_\sigma$-set
5. Open cover of a set $E \subseteq \mathbb{R}$

See the book!
PROBLEM 2: (25 points) State and prove the Bolzano-Weierstrass Theorem. (You are allowed to use without proof the Nested Set Theorem).

See theorem 16 on page 21 in the book.
PROBLEM 3: (20 points) Solve at your choice ONE of the following problems:

1. Show that a sequence of real numbers \( a_n \) is convergent if and only if
\[
\liminf_{n \to \infty} \{a_n\} = \limsup_{n \to \infty} \{a_n\} = x \in \mathbb{R}.
\]

2. If \( E_1, E_2, \ldots, E_k \subseteq \mathbb{R} \) then show that
\[
\overline{E_1 \cup E_2 \cup \cdots \cup E_k} = \overline{E_1} \cup \overline{E_2} \cup \cdots \cup \overline{E_k}.
\]

2) Using mathematical induction it suffices to show the statement for \( k = 2 \)

First notice that
\[
E_1 \subseteq E_1 \cup E_2 \implies \overline{E_1} \subseteq \overline{E_1 \cup E_2}, \quad \text{and} \quad \overline{E_2} \subseteq \overline{E_1 \cup E_2},
\]

For the reversed implication, let \( x \in (\overline{E_1 \cup E_2})^c = \mathbb{R} \setminus (\overline{E_1 \cup E_2}) \). Thus there exists \( \varepsilon > 0 \) open such that \( \forall x \in E_i \), \( \forall x \in E_j \), \( \varepsilon = \phi \). Hence \( \phi = (\bigcap \overline{E_i}) \cup (\bigcap \overline{E_j}) = \bigcap (E_i \cup E_j)^c \), which further implies that \( x \in \mathbb{R} \setminus (\overline{E_1 \cup E_2}) = (\overline{E_1 \cup E_2})^c \).

In conclusion we have proved that \( (\overline{E_1 \cup E_2})^c \subseteq \overline{E_1 \cup E_2}^c \) which gives that \( \overline{E_1 \cup E_2} \subseteq \overline{E_1 \cup E_2} \).

1) \( \Leftarrow \)

Notice that if \( b_n = \inf_{k \geq n} \{a_k\} > 0 \),
\[
x = \liminf_{n \to \infty} \{a_n\} = \lim_{n \to \infty} \left( \inf_{k \geq n} \{a_k\} \right) = \lim_{n \to \infty} b_n \quad \text{(1)}
\]

Notice that if \( c_n = \sup_{k \geq n} \{a_k\} \), \( c_n \geq 0 \) and
\[
x = \limsup_{n \to \infty} \{a_n\} = \lim_{n \to \infty} \left( \sup_{k \geq n} \{a_k\} \right) = \lim_{n \to \infty} c_n \quad \text{(2)}
\]

We also have that
\[
b_n = \inf_{k \geq n} \{a_k\} \leq a_n \leq \sup_{k \geq n} \{a_k\} = c_n \quad \text{for all } n \in \mathbb{N}
\]

and hence
\[
b_n - x \leq a_n - x \leq c_n - x \quad \text{for all } n \in \mathbb{N}
\]

So by squeeze theorem \( a_n - x \to 0 \) and hence \( a_n \to x \) as \( n \to \infty \).
Assume that \( \lim_{n \to \infty} a_n = x \in \mathbb{R} \).

Fix \( \varepsilon > 0 \). Then for \( n \leq n_{\varepsilon} \),

\[
  |a_n - x| \leq \frac{\varepsilon}{3} \quad \forall n \geq n_{\varepsilon}
\]

and

\[
  -\frac{\varepsilon}{3} \leq x \leq \frac{\varepsilon}{3} \quad \forall n \geq n_{\varepsilon}
\]

and

\[
  -\frac{2\varepsilon}{3} \leq a_n - x \leq \frac{2\varepsilon}{3} \quad \forall n \geq n_{\varepsilon}
\]

Hence

\[
  \lim_{l \to \infty} a_l = x
\]

But as in the previous implication,

\[
  x = \lim_{l \to \infty} a_l = \lim_{l \to \infty} \left( \sup_{1 \leq n \leq l} |a_n| \right) = \limsup_{l \to \infty} |a_l|
\]

The equality \( x = \limsup_{l \to \infty} a_l \) can be shown in a similar manner.
PROBLEM 4: (10 points) Solve at your choice ONE of the following problems:

1. Show that a non-empty set $E$ is closed and bounded if and only if every continuous real-valued function on $E$ takes a minimum value.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function. Prove that the system

$$
\begin{align*}
x &= f(y) \\
y &= f(z) \\
z &= f(x)
\end{align*}
$$

has unique solution.

\[ \begin{array}{l}
\Rightarrow " \Rightarrow " \text{ follows directly from EVT} \\
\Leftarrow \text{ Fix } x \in \mathbb{R}. \text{ Consider the map } f : E \to \mathbb{R} \text{ defined by } f(y) = -1 \cdot x - y \quad \forall y \in E. \text{ Using the triangle inequality, we have that } \\
|f(y) - f(z)| = |-1 \cdot x - y + 1 \cdot x - z| \leq |x - z| \text{ for all } y, z \in E. \\
\text{Thus } f \text{ is Lipschitz and hence continuous. By assumption, } f \text{ takes a minimum value, so } \exists m \in \mathbb{R} \text{ such that } f(y) \geq m \quad \forall y \in E. \\
\text{Thus further implies that } -1 \cdot x - y \geq m \quad \forall y \in E \text{ and hence, using the reverse triangle inequality, we get that } \\
|y| - |x| \leq |x - y| \leq |x - z| \quad \forall y \in E. \\
\text{ hence } |y| \leq |x| + m \quad \forall y \in E, \text{ which in particular shows that } E \text{ is bounded.}
\end{array} \]

To show that $E = \overline{E}$, assume that $x \in \overline{E}$ and consider the function $g : E \to \mathbb{R}_+ \text{ defined by } g(y) = |y - x| \quad \forall y \in E$. As before, $g$ is Lipschitz and hence continuous. So by assumption, $g$ takes a minimum, so

\[
\exists m \in \mathbb{R}_+ \text{ such that } g(y) \geq m = g(y_0) \quad \forall y \in E. \tag{1}
\]

Since $x \in \overline{E}$, $\exists (y_n)_{n \geq 1} \subseteq E$ such that $y_n \to x \text{ as } n \to \infty$. \textit{(=>)} $|y_n - x| \to 0 \text{ as } n \to \infty$. But this together with (1) shows that

$m = 0 \Rightarrow 0 = g(y_0) = |y_0 - x| \Rightarrow x = y_0 \in E$. In particular, $\overline{x_E} = E$. Since $x \in \overline{E}$ was arbitrary, we get that $\overline{E} \subseteq E$ and hence $\overline{E} = E$. So $E$ is closed.
2) Since $f$ is decreasing, then we have that
\[ \lim_{x \to -\infty} f(x) - x = \infty \]
\[ \lim_{x \to \infty} f(x) - x = -\infty \]
So by IVT \( \exists x_0 \in \mathbb{R} \) s.t. \( f(x_0) - x_0 = 0 \) i.e. \( f(x_0) = x_0 \)
However the function $f$ cannot have another fixed point $x<y$ thus if $x = f(x) \geq f(y) = y$ which is impossible.
Notice that \((x_0, x_0, x_0)\) is clearly a solution to the system.
Let's briefly argue that there is no other solution.
Indeed if \((x, y, z)\) is a solution then \( f(x) = f(y) = f(z) = x \) and \( f(f(x)) = y = f(z) \) and since $f$ is continuous and decreasing it follows that \( x = y = z \). Again, continuous and decreasing so by above argument, it has a unique fixed point. And thus unique fixed point can only be \( x_0 \). Thus $x = y = z = x_0$ so the solution is unique!
PROBLEM 5: (20 points) Solve at your choice ONE of the following problems:

1. If \( E = [0, 1] \setminus Q \) then find \( \mu^*(E) \). Make sure you include all the arguments in your proof.

2. Let \( E \subset \mathbb{R} \) such that \( \mu^*(E) < \infty \). Show that there exists a sequence of open sets \( O_n \subset \mathbb{R} \) such that

   (a) \( E \subset O_n \) for all \( n \), and

   (b) \( \mu^*(O_n) \) converges to \( \mu^*(E) \) as \( n \) tends to \( \infty \).

1. First we show that \( \mu^*(Q) = 0 \). Since \( Q \) is countable let \( Q = \{ q_n : n \in \mathbb{N} \} \) be an enumeration. Fix \( \varepsilon > 0 \) and consider

\[
I_n = (r_n - \frac{\varepsilon}{2^{n+1}}, r_n + \frac{\varepsilon}{2^{n+1}}) \ni r_n \quad \forall n \in \mathbb{N}.
\]

We clearly have that

\[
Q = \bigcup_{n} I_n \subset \bigcup_{n} (r_n - \frac{\varepsilon}{2^{n+1}}, r_n + \frac{\varepsilon}{2^{n+1}}) = \bigcup_{n} I_n \quad \text{and by definition of outer measure we get}
\]

\[
0 \leq \mu^*(Q) = \inf_{\mathcal{U}} \sum_{n} l(I_n) \quad \text{where } \mathcal{U} \text{ is a open cover by bounded intervals}
\]

\[
\leq \sum_{n} l(I_n) = \sum_{n} \left( \frac{\varepsilon}{2^{n+1}} + r_n - (r_n - \frac{\varepsilon}{2^{n+1}}) \right)
\]

\[
= \sum_{n} \frac{\varepsilon}{2^n} = \varepsilon.
\]

In conclusion \( 0 \leq \mu^*(Q) \leq \varepsilon \), \( \forall \varepsilon > 0 \); Hence \( \mu^*(Q) = 0 \).

In particular \( Q \) is measurable and by excision property we get

\[
\mu^*(E) = \mu^*([0,1] \setminus Q) = \mu^*([0,1]) \setminus \mu^*(Q) = 1 - 0 = 1.
\]

2. Since \( \mu^*(E) = \inf \{ \sum l(I_n) : \bigcup_{n} I_n \supseteq E \} \), let \( E \) be open cover by

then \( \forall k \in \mathbb{N} \exists O_k = \bigcup_{n} I_{n}^{k} \supseteq E \) open set such that

\[
(1) \quad \mu^*(E) \leq \sum_{n} l(I_{n}^{k}) \leq \mu^*(E) + \frac{1}{k}
\]

Also notice that since \( E \subseteq O_k \), then by monotonicity of outer measure we have

\[
(2) \quad \mu^*(E) \leq \mu^*(O_k) \leq \sum_{n} l(I_{n}^{k}).
\]

Second inequality above in \( 2 \) follows from the def of \( \mu^*(O_k) \) using the fact \( \bigcup_{n} I_{n}^{k} \) is an
An open cover of $O_k$.

Altogether (1) and (2) show that

$$\mu^*(E) \leq \mu^*(O_k) \leq \mu^*(E) - \frac{1}{k} \quad \forall k \in \mathbb{N}$$

Since $\mu^*(E) < \infty$ this further implies that

$$0 \leq \mu^*(O_k) - \mu^*(E) \leq \frac{1}{k} \quad \forall k \in \mathbb{N}$$

Letting $k \to \infty$ we get that $\mu^*(O_k) \to \mu^*(E)$ as $k \to \infty$. 