Higher Order Noncommutative Functions

Leonard C. Stevenson

Drexel University

NonCommutative Analysis, June 2016

The Noncommutative Space

Let

- $\mathcal{R}$ be a commutative ring with identity,
- $\mathcal{M}$ be an $\mathcal{R}$-module, and
- $\mathcal{M}^{n\times n}$ be the module of all $n \times n$ matrices with entries from $\mathcal{M}$.

Define the noncommutative space over $\mathcal{M}$ to be

$$\mathcal{M}_{nc} := \coprod_{n=1}^{\infty} \mathcal{M}^{n\times n}$$

Matrix Operations

The following operations on matrices over $\mathcal{M}$ and $\mathcal{R}$ can be defined:

1. Sum: For $X, Y \in \mathcal{M}^{n\times n}$,

$$X + Y := [x_{ij} + y_{ij}]_{i,j=1,\ldots,n} \in \mathcal{M}^{n\times n}$$

2. Direct Sum: For $X \in \mathcal{M}^{n\times n}$ and $Y \in \mathcal{M}^{m\times m}$

$$X \bigoplus Y := \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in M^{(m+n)\times(m+n)}$$

3. Ring Actions: For $X \in \mathcal{M}^{p\times q}$, $T \in \mathcal{R}^{r\times p}$ and $S \in \mathcal{R}^{q\times b}$,

$$TX := \left[\sum_{k=1}^{p} t_{ik} x_{kj}\right]_{i=1,\ldots r}^{j=1,\ldots q} \qquad XS = \left[\sum_{k=1}^{q} x_{ik} s_{kj}\right]_{i=1,\ldots p}^{j=1,\ldots b}$$

Matrix Operations

4. Kronecker Product: For $S \in \mathcal{R}^{p\times q}$ and $T \in \mathcal{R}^{n\times m}$, we define $S \otimes T = [s_{ij}T]_{i=1,\dots,p}^{j=1,\dots,q} \in \mathcal{R}^{np\times mq}$.
5. Generalized Matrix Product: For $\mathcal{R}$-modules $\mathcal{N}_1, \mathcal{N}_2$, $Z^1 \in \mathcal{N}_1^{n_0\times n_1}$, $Z^2 \in \mathcal{N}_2^{n_1\times n_2}$, integers s_1, s_2 such that $n_1 = s_1 m_1$ and $n_2 = s_2 m_2$ and the tensor product $\mathcal{N}_1^{s_0\times s_1} \otimes \mathcal{N}_2^{s_1\times s_2}$,

$$Z^1_{s_0,s_2} \odot_{s_1} Z^2 := \left[\left(Z^1_{s_0,s_2} \odot_{s_1} Z^2\right)_{\alpha_0,\alpha_2}\right]_{\alpha_0=1,\dots,m_0}^{\alpha_2=1,\dots,m_2}$$

Where,

$$\left(Z^1_{s_0,s_2} \odot_{s_1} Z^2\right)_{\alpha_0,\alpha_2} = \sum_{\alpha_1=1}^{m_1} Z^1_{\alpha_0,\alpha_1} \otimes Z^2_{\alpha_1,\alpha_2}$$

Noncommutative Sets

For $\Omega \subseteq \mathcal{M}_{nc}$

- $\Omega_n := \Omega \cap \mathcal{M}^{n \times n}$.
- Ω is a *noncommutative set* (nc set) if

$$X \in \Omega_n, Y \in \Omega_m \implies X \oplus Y \in \Omega_{n+m}$$

- Ω is *right admissible* if

$$X \in \Omega_n, Y \in \Omega_m, Z \in \mathcal{M}^{n \times m} \implies \exists r \in \mathrm{Gl}(1, \mathcal{R}) \text{ s.t. } \begin{bmatrix} X & rZ \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$$

The Similarity Envelope

Define,

$$\tilde{\Omega} := \{SXS^{-1} \mid X \in \Omega_n, S \in \mathrm{Gl}(n, \mathcal{R}), n \in \mathbb{N}\}$$

to be the similarity envelope of Ω.

Proposition

If $\Omega \subseteq \mathcal{M}_{nc}$ is a right admissible nc set, then so is its similarity envelope $\tilde{\Omega}$. Moreover, for any $\tilde{X} \in \tilde{\Omega}_n$, $\tilde{Y} \in \tilde{\Omega}_m$ and $Z \in \mathcal{M}^{n \times m}$, one has

$$\begin{bmatrix} \tilde{X} & Z \\ 0 & \tilde{Y} \end{bmatrix} \in \tilde{\Omega}_{n+m}$$

Definition of Noncommutative Function

A function $f : \Omega \to \mathcal{N}_{nc}$ s.t. $f(\Omega_n) \subseteq \mathcal{N}^{n \times n}$ for $n = 1, 2, \dots$ is called a noncommutative function if

- f respects direct sums:

$$X \in \Omega_n, Y \in \Omega_m \implies f(X \bigoplus Y) = f(X) \oplus f(Y)$$

- f respects similarities:

$$X \in \Omega_n, S \in \mathrm{Gl}(n, \mathcal{R}) \text{ s.t. } SXS^{-1} \in \Omega_n \implies f(SXS^{-1}) = Sf(X)S^{-1}$$

Examples of Noncommutative Functions

Consider the matrix polynomial $f(X) = X^2$. In this case,

$$f\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = \begin{bmatrix} X^2 & 0 \\ 0 & Y^2 \end{bmatrix} = \begin{bmatrix} f(X) & 0 \\ 0 & f(Y) \end{bmatrix}$$

$$f(SXS^{-1}) = SX(S^{-1}S)XS^{-1} = SX^2S^{-1} = Sf(X)S^{-1}$$

Examples of Noncommutative Functions

1. All polynomials and rational expressions in d matrices over $\mathcal{R}$.
2. Formal power series of matrices over $\mathcal{R}$.
3. Let $l : \mathcal{M} \to \mathcal{N}$ be a linear mapping. Define $L : \mathcal{M}^{n\times n} \to \mathcal{N}^{n\times n}$ by

$$L([x_{ij}]_{i,j=1,\cdots,n}) = [l(x_{ij})]_{i,j=1,\cdots,n}$$

Then, $L : \mathcal{M}_{nc} \to \mathcal{N}_{nc}$ is a noncommutative function.

The Difference-Differential Operator

Let f be a nc function on a nc set Ω. For any $X \in \Omega_n$, $Y \in \Omega_m$ and any $Z \in \mathcal{M}^{n\times m}$ such that $\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$, define $\Delta_R f(X,Y)(Z)$ by

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} f(X) & \Delta_R f(X,Y)(Z) \\ 0 & f(Y) \end{bmatrix}$$

Proposition

Take any nc function f on a right admissible, nc set Ω. Then, $\Delta_R f(X,Y)(Z)$, can be extended to a function linear in Z on the $\mathcal{R}$-module $\mathcal{M}^{n\times m}$.

Difference-Differential Operator Examples

1. If $f(X) = X^2$, then

$$f\left(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\right) = \begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix} = \begin{bmatrix} X^2 & XZ + ZY \\ 0 & Y^2 \end{bmatrix}$$

Thus, $\Delta_R f(X, Y)(Z) = XZ + ZY$.

Difference-Differential Operator Examples

2. If f is a polynomial of the form

$$\sum_{i=1}^{n} a_i X^i,$$

then

$$\Delta_R f(X,Y)(Z) = \sum_{i=1}^{n} a_i X^{i-1} Z Y^{n-i}.$$

3. For the extension of the linear function defined above

$$\Delta_R L(X,Y)(Z) = L(Z).$$

Difference Formula

Theorem

Let $f : \Omega \to \mathcal{N}_{nc}$ be an nc function where Ω is a right admissible nc set. Then, for all $n, m \in \mathbb{N}$, all $X \in \Omega_n$, $Y \in \Omega_m$ and $S \in \mathcal{R}^{n \times m}$ we have

$$Sf(Y) - f(X)S = \Delta_R f(X, Y)(SY - XS)$$

and, in the special case that $n = m$ and $S = I_n$, we get,

$$\Delta_R f(Y, X)(Y - X) = f(Y) - f(X) = \Delta_R f(X, Y)(Y - X)$$

Difference Formula

For our function $f(X) = X^2$, the difference formula looks like,

$$\begin{aligned} Sf(Y) - f(X)S = SY^2 - X^2S &= XSY - X^2S + SY^2 - XSY \\ &= X(SY - XS) + (SY - XS)Y = \Delta_R f(X,Y)(SY - XS) \end{aligned}$$

Or in the case that $S = I$ and X and Y have the same size,

$$\begin{aligned} f(Y) - f(X) = Y^2 - X^2 &= XY - X^2 + Y^2 - XY \\ &= X(Y - X) + (Y - X)Y = \Delta_R f(X,Y)(Y - X) \end{aligned}$$

Properties of the Difference-Differential Operator

The Difference-Differential Operator has the following properties with respect to direct sums,

$$\Delta_R f(X' \oplus X'', Y)\left(\begin{bmatrix} Z' \\ Z'' \end{bmatrix}\right) = \begin{bmatrix} \Delta_R f(X', Y)(Z') \\ \Delta_R f(X'', Y)(Z'') \end{bmatrix}$$

$$\Delta_R f(X, Y' \oplus Y'')\left(\begin{bmatrix} Z' & Z'' \end{bmatrix}\right) = \begin{bmatrix} \Delta_R f(X, Y')(Z') & \Delta_R f(X, Y'')(Z'') \end{bmatrix}$$

Properties of the Difference-Differential Operator

For our function $\Delta_R f(X,Y)(Z) = XZ + ZY$,

$$\begin{aligned}\Delta_R f(X' \oplus X'', Y)\left(\begin{bmatrix} Z' \\ Z'' \end{bmatrix}\right) &= \begin{bmatrix} X' & 0 \\ 0 & X'' \end{bmatrix}\begin{bmatrix} Z' \\ Z'' \end{bmatrix} + \begin{bmatrix} Z' \\ Z'' \end{bmatrix} Y \\ &= \begin{bmatrix} X'Z' \\ X''Z'' \end{bmatrix} + \begin{bmatrix} Z'Y \\ Z''Y \end{bmatrix} \\ &= \begin{bmatrix} X'Z' + Z'Y \\ X''Z'' + Z''Y \end{bmatrix} \\ &= \begin{bmatrix} \Delta_R f(X', Y)(Z') \\ \Delta_R f(X'', Y)(Z'') \end{bmatrix}\end{aligned}$$

Properties of the Difference-Differential Operator

and

$$\begin{aligned}
\Delta_R f(X, Y' \oplus Y'')\left(\begin{bmatrix} Z' & Z'' \end{bmatrix}\right) &= X\begin{bmatrix} Z' & Z'' \end{bmatrix} + \begin{bmatrix} Z' & Z'' \end{bmatrix}\begin{bmatrix} Y' & 0 \\ 0 & Y'' \end{bmatrix} \\
&= \begin{bmatrix} XZ' & XZ'' \end{bmatrix} + \begin{bmatrix} Z'Y' & Z''Y'' \end{bmatrix} \\
&= \begin{bmatrix} XZ' + Z'Y' & XZ'' + Z''Y'' \end{bmatrix} \\
&= \begin{bmatrix} \Delta_R f(X, Y')(Z') & \Delta_R f(X, Y'')(Z'') \end{bmatrix}
\end{aligned}$$

Properties of the Difference-Differential Operator

The Difference-Differential Operator has the following properties with respect to similarities,

$$\Delta_R f(TXT^{-1}, Y)(TZ) = T\Delta_R f(X, Y)(Z)$$
$$\Delta_R f(X, SYS^{-1})(ZS^{-1}) = \Delta_R f(X, Y)(Z)S^{-1}$$

Properties of the Difference-Differential Operator

For our function $\Delta_R f(X,Y)(Z) = XZ + ZY$,

$$\begin{aligned}\Delta_R f(TXT^{-1}, Y)(TZ) &= (TXT^{-1})(TZ) + (TZ)Y \\ &= TXZ + TZY = T(XZ + ZY) = T\Delta_R f(X,Y)(Z)\end{aligned}$$

and

$$\begin{aligned}\Delta_R f(X, SYS^{-1})(ZS^{-1}) &= X(ZS^{-1}) + (ZS^{-1})(SYS^{-1}) \\ &= XZS^{-1} + ZYS^{-1} = (XZ + ZY)S^{-1} = \Delta_R f(X,Y)(Z)S^{-1}\end{aligned}$$

Higher Order NC Functions

A function f for which

$$f(X^0, \ldots, X^k) : \mathcal{N}_1^{n_0 \times n_1} \times \ldots \times \mathcal{N}_k^{n_{k-1} \times n_k} \to \mathcal{N}_0^{n_0 \times n_k}$$

is a k-linear mapping over $\mathcal{R}$ is an nc function of order k if

NC Functions Respect Direct Sums

f respects direct sums:

$$f(X^{0'} \oplus X^{0''}, X^1, \ldots, X^k)\left(\begin{bmatrix} Z^{1'} \\ Z^{1''} \end{bmatrix}, Z^2, \ldots, Z^k\right) = \begin{bmatrix} f(X^{0'}, X^1, \ldots, X^k)\left(Z^{1'}, Z^2, \ldots, Z^k\right) \\ f(X^{0''}, X^1, \ldots, X^k)\left(Z^{1''}, Z^2, \ldots, Z^k\right) \end{bmatrix} \quad (1)$$

NC Functions Respect Direct Sums

$$f(X^0,\dots,X^{j-1},X^{j'}\oplus X^{j''},X^{j+1},\dots,X^k)$$
$$\left(Z^1,\dots,Z^{j-1},\begin{bmatrix} Z^{j'} & Z^{j''}\end{bmatrix},\begin{bmatrix} Z^{(j+1)'} \\ Z^{(j+1)''}\end{bmatrix},Z^{j+2},\dots,Z^k\right)$$
$$= f(X^0,\dots,X^{j-1},X^{j'},X^{j+1},\dots,X^k)\left(Z^1,\dots,Z^{j-1},Z^{j'},Z^{(j+1)'},Z^{j+2},\dots,Z^k\right)$$
$$+ f(X^0,\dots,X^{j-1},X^{j''},X^{(j+1)},\dots,X^k)$$
$$\left(Z^1,\dots,Z^{j-1},Z^{j''},Z^{(j+1)''},Z^{(j+2)},\dots,Z^k\right) \tag{2}$$

NC Functions Respect Direct Sums

and

$$
\begin{aligned}
& f(X^0, \ldots, X^{k-1}, X^{k'} \oplus X^{k''})\left(Z^1, \ldots, Z^{k-1}, \begin{bmatrix} Z^{k'} & Z^{k''} \end{bmatrix}\right) \\
&= \operatorname{row}\Big[f(X^0, \ldots, X^{k-1}, X^{k'})(Z^1, \ldots, Z^{k-1}, Z^{k'}) \\
&\qquad\qquad f(X^0, \ldots, X^{k-1}, X^{k''})(Z^1, \ldots, Z^{k-1}, Z^{k''}) \Big]
\end{aligned} \tag{3}
$$

NC Functions Respect Similarities

- f respects similarities:

$$f(S_0 X^0 S_0^{-1}, X^1, \dots, X^k)(S_0 Z^1, Z^2, \dots, Z^k) = S_0 f(X^0, \dots, X^k)(Z^1, \dots, Z^k), \tag{4}$$

$$f(X^0, \dots, X^{j-1}, S_j X^j S_j^{-1}, X^{j+1}, \dots, X^k)(Z^1, \dots, Z^{j-1}, Z^j S_j^{-1}, S_j Z^{j+1}, Z^{j+2}, \dots, Z^k) = f(X^0, \dots, X^k)(Z^1, \dots, Z^k) \tag{5}$$

$$f(X^0, \dots, X^{k-1}, S_k X^k S_k^{-1})(Z^1, Z^2, \dots, Z^k S_k^{-1}) = f(X^0, \dots, X^k)(Z^1, \dots, Z^k) S_k^{-1} \tag{6}$$

Order of an NC Function

By this definition $\Delta_R f(X,Y)(Z)$ is a first order function while f is considered a zero order function. In general, let

$$\mathcal{T}^k(\Omega^{(0)},\ldots,\Omega^{(k)};\mathcal{N}_{0,nc},\ldots,\mathcal{N}_{k,nc})$$

be the set of all nc functions of order k.

Generalization of Direct Sum

Proposition

Let

$$X^j = \bigoplus_{\alpha_j=1}^{m_j} X^j_{\alpha_j}, \qquad Z^j = [Z^j_{\alpha,\beta}]_{\alpha=1,\dots,m_{j-1}}^{\beta=1,\dots,m_j}$$

Then,

$$f(X^0,\dots,X^k)(Z^1,\dots,Z^k) = [f^{\alpha,\beta}]_{\alpha=1,\dots,m_0}^{\beta=1,\dots,m_k}$$

where,

$$f^{\alpha,\beta} = \sum_{\substack{\alpha_j=1,\dots,m_j \\ \alpha_0=\alpha,\alpha_k=\beta}} f(X^{0\alpha_0},\dots,X^{k\alpha_k})(Z^{1\alpha_0,\alpha_1},\dots,Z^{k\alpha_{k-1},\alpha_k})$$

Generalization of Direct Sum

Consider the function, $f(X^0, X^1, X^2)(Z^1, Z^2) = Z^1 X^1 Z^2$, we find,

$$f\left(\begin{bmatrix} X_1^0 & & \\ & \ddots & \\ & & X_{m_0}^0 \end{bmatrix}, \begin{bmatrix} X_1^1 & & \\ & \ddots & \\ & & X_{m_1}^1 \end{bmatrix}, \begin{bmatrix} X_1^2 & & \\ & \ddots & \\ & & X_{m_2}^2 \end{bmatrix}\right)$$
$$\left(\begin{bmatrix} Z_{11}^1 & \cdots & Z_{1,m_1}^1 \\ \vdots & \ddots & \vdots \\ Z_{m_0,1}^1 & \cdots & Z_{m_0,m_1}^1 \end{bmatrix}, \begin{bmatrix} Z_{11}^2 & \cdots & Z_{1,m_2}^2 \\ \vdots & \ddots & \vdots \\ Z_{m_1,1}^2 & \cdots & Z_{m_1,m_2}^2 \end{bmatrix}\right)$$

Generalization of Direct Sum

$$= \begin{bmatrix} Z^1_{11} & \cdots & Z^1_{1,m_1} \\ \vdots & \ddots & \vdots \\ Z^1_{m_0,1} & \cdots & Z^1_{m_0,m_1} \end{bmatrix} \begin{bmatrix} X^1_1 & & \\ & \ddots & \\ & & X^1_{m_1} \end{bmatrix} \begin{bmatrix} Z^2_{11} & \cdots & Z^2_{1,m_2} \\ \vdots & \ddots & \vdots \\ Z^2_{m_1,1} & \cdots & Z^2_{m_1,m_2} \end{bmatrix}$$

$$= \begin{bmatrix} Z^1_{11}X^1_1 Z^2_{11} + \cdots + Z^1_{1,m_1}X^1_{m_1}Z^2_{m_1,1} & \cdots & Z^1_{11}X^1_1 Z^2_{1,m_2} + \cdots + Z^1_{1,m_1}X^1_{m_1}Z^2_{m_1,m_2} \\ \vdots & \ddots & \vdots \\ Z^1_{m_0,1}X^1_1 Z^2_{11} + \cdots + Z^1_{m_0,m_1}X^1_{m_1}Z^2_{m_1,1} & \cdots & Z^1_{m_0,1}X^1_1 Z^2_{1,m_2} + \cdots + Z^1_{m_0,m_1}X^1_{m_1}Z^2_{m_1,m_2} \end{bmatrix}$$

Generalization of Direct Sum

$$= \begin{bmatrix} \sum_{\alpha_1=1}^{m_1} Z^1_{1,\alpha_1} X^1_{\alpha_1} Z^2_{\alpha_1,1} & \cdots & \sum_{\alpha_1=1}^{m_1} Z^1_{1,\alpha_1} X^1_{\alpha_1} Z^2_{\alpha_1,m_2} \\ \vdots & \ddots & \vdots \\ \sum_{\alpha_1=1}^{m_1} Z^1_{m_0,\alpha_1} X^1_{\alpha_1} Z^2_{\alpha_1,1} & \cdots & \sum_{\alpha_1=1}^{m_1} Z^1_{m_0,\alpha_1} X^1_{\alpha_1} Z^2_{\alpha_1,m_2} \end{bmatrix}$$

Which is a matrix where each entry has the form,

$$\sum_{\alpha_1=1}^{m_1} f(X^0_{\alpha_0}, X^1_{\alpha_1}, X^2_{\alpha_2})(Z^1_{\alpha_0,\alpha_1}, Z^2_{\alpha_1,\alpha_2})$$

Generalized Matrix Product

Our k-linear maps,

$$(Z^1, \dots, Z^k) \mapsto f(X^0, \dots, X^k)(Z^1, \dots, Z^k)$$

can also be written as linear maps on the corresponding tensor product, defined on elementary tensors as,

$$Z^1 \otimes \dots \otimes Z^k \mapsto f(X^0, \dots, X^k)(Z^1 \otimes \dots \otimes Z^k)$$

Generalized Matrix Product

We recall,

$$Z^1_{s_0,s_2} \odot_{s_1} \cdots_{s_{k-2},s_k} \odot_{s_{k-1}} Z^k := \left[\left(Z^1_{s_0,s_2} \odot_{s_1} \cdots_{s_{k-2},s_k} \odot_{s_{k-1}} Z^k \right)_{\alpha_0,\alpha_k} \right]_{\alpha_0=1,\ldots,m_0}^{\alpha_k=1,\ldots,m_k},$$

where,

$$\left(Z^1_{s_0,s_2} \odot_{s_1} \cdots_{s_{k-2},s_k} \odot_{s_{k-1}} Z^k \right)_{\alpha_0,\alpha_k} = \sum_{\substack{\alpha_j=1 \\ j=1,\ldots,k-1}}^{m_j} Z^1_{\alpha_0,\alpha_1} \otimes \ldots \otimes Z^k_{\alpha_{k-1},\alpha_k}$$

Rewriting Direct Sum Rule for Identical Summands

Proposition

Given,

$$X^j = \bigoplus_{\alpha_j=1}^{m_j} Y^j, \ \text{for } j = 0, \ldots, k$$

we rewrite the function as follows:

$$f(X^0, \ldots, X^k)(Z^1, \ldots, Z^k) = Z^1_{s_0,s_2} \odot_{s_1} \ldots_{s_{k-2},s_k} \odot_{s_{k-1}} Z^k f(Y^0, \ldots, Y^k),$$

where $f(Y^0, \ldots, Y^k)$ *acts entrywise on* $Z^1_{s_0,s_2} \odot_{s_1} \ldots_{s_{k-2},s_k} \odot_{s_{k-1}} Z^k$.

Rewriting Direct Sum Rule for Identical Summands

For our function $\Delta_R f(X^0, X^1, X^2)(Z^1, Z^2) = Z^1 Z^2$, if X^0, X^1 and X^2 are direct sums of Y^0, Y^1 and Y^2, then, as calculated above,

$$f\left(\begin{bmatrix} Y^0 & & \\ & \ddots & \\ & & Y^0 \end{bmatrix}, \begin{bmatrix} Y^1 & & \\ & \ddots & \\ & & Y^1 \end{bmatrix}, \begin{bmatrix} Y^2 & & \\ & \ddots & \\ & & Y^2 \end{bmatrix}\right) \left(\begin{bmatrix} Z^1_{11} & \cdots & Z^1_{1,m_1} \\ \vdots & \ddots & \vdots \\ Z^1_{m_0,1} & \cdots & Z^1_{m_0,m_1} \end{bmatrix}, \begin{bmatrix} Z^2_{11} & \cdots & Z^2_{1,m_2} \\ \vdots & \ddots & \vdots \\ Z^2_{m_1,1} & \cdots & Z^2_{m_1,m_2} \end{bmatrix}\right)$$

Rewriting Direct Sum Rule for Identical Summands

$$= \begin{bmatrix} \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{1,\alpha_1}, Z^2_{\alpha_1,1}) & \cdots & \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{1,\alpha_1}, Z^2_{\alpha_1,m_2}) \\ \vdots & \ddots & \vdots \\ \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{m_0,\alpha_1}, Z^2_{\alpha_1,m_2}) & \cdots & \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{m_0,\alpha_1}, Z^2_{\alpha_1,m_2}) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{1,\alpha_1} \otimes Z^2_{\alpha_1,1}) & \cdots & \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{1,\alpha_1} \otimes Z^2_{\alpha_1,m_2}) \\ \vdots & \ddots & \vdots \\ \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{m_0,\alpha_1} \otimes Z^2_{\alpha_1,m_2}) & \cdots & \sum_{\alpha_1=1}^{m_1} f(Y^0, Y^1, Y^2)(Z^1_{m_0,\alpha_1} \otimes Z^2_{\alpha_1,m_2}) \end{bmatrix}$$

Rewriting Direct Sum Rule for Identical Summands

$$= \begin{bmatrix} \sum_{\alpha_1=1}^{m_1} Z^1_{1,\alpha_1} \otimes Z^2_{\alpha_1,1} & \cdots & \sum_{\alpha_1=1}^{m_1} Z^1_{1,\alpha_1} \otimes Z^2_{\alpha_1,m_2} \\ \vdots & \ddots & \vdots \\ \sum_{\alpha_1=1}^{m_1} Z^1_{m_0,\alpha_1} \otimes Z^2_{\alpha_1,m_2} & \cdots & \sum_{\alpha_1=1}^{m_1} Z^1_{m_0,\alpha_1} \otimes Z^2_{\alpha_1,m_2} \end{bmatrix} f(Y^0, Y^1, Y^2)$$

$$= \left(\begin{bmatrix} Z^1_{11} & \cdots & Z^1_{1,m_1} \\ \vdots & \ddots & \vdots \\ Z^1_{m_0,1} & \cdots & Z^1_{m_0,m_1} \end{bmatrix}_{m_0,m_1} \odot_{m_2} \begin{bmatrix} Z^2_{11} & \cdots & Z^2_{1,m_2} \\ \vdots & \ddots & \vdots \\ Z^2_{m_1,1} & \cdots & Z^2_{m_1,m_2} \end{bmatrix} \right) f(Y^0, Y^1, Y^2)$$

$$= ({Z^1}_{m_0,m_1} \odot_{m_2} Z^2) f(Y^0, Y^1, Y^2)$$

Higher order Difference-Differential Operators

We extend the difference-differential operator to higher order functions as follows,

Proposition

Let $f \in \mathcal{T}^k(\Omega^{(0)}, \ldots, \Omega^{(k)}; \mathcal{N}_{0,nc}, \ldots, \mathcal{N}_{k,nc})$,

$$f\left(\begin{bmatrix} X^{0'} & Z \\ 0 & X^{0''} \end{bmatrix}, X^1, \ldots, X^k\right)\left(\begin{bmatrix} Z^{1'} \\ Z^{1''} \end{bmatrix}, Z^2, \ldots, Z^k\right)$$
$$= \begin{bmatrix} f(X^{0'}, X^1, \ldots, X^k)(Z^{1'}, Z^2, \ldots, Z^k) \\ \qquad +_0 \Delta_R f(X^{0'}, X^{0''}, X^1, \ldots, X^k)(Z, Z^{1''}, Z^2 \ldots, Z^k) \\ f(X^{0''}, X^1, \ldots, X^k)(Z^{1''}, Z^2, \ldots, Z^k) \end{bmatrix}$$

Higher order Difference-Differential Operators

Proposition

$$f\left(X^0,\dots,X^{j-1},\begin{bmatrix} X^{j'} & Z \\ 0 & X^{j''}\end{bmatrix},X^{j+1},\dots,X^k\right)$$

$$\left(Z^1,\dots,Z^{j-1},\begin{bmatrix} Z^{j'} & Z^{j''}\end{bmatrix},\begin{bmatrix} Z^{(j+1)'} \\ Z^{(j+1)''}\end{bmatrix},Z^{(j+2)},\dots,Z^k\right)$$

$$= f(X^0,\dots,X^{j-1},X^{j'},X^{j+1},\dots,X^k)(Z^1,\dots,Z^{(j-1)},Z^{j'},Z^{(j+1)'},Z^{(j+2)},\dots,Z^k)$$

$$+_j\, \Delta_R f(X^0,\dots,X^{j-1},X^{j'},X^{j''},X^{(j+1)},\dots,X^k)$$

$$(Z^1,\dots,Z^{j-1},Z^{j'},Z,Z^{(j+1)''},Z^{(j+2)},\dots,Z^k)$$

$$+ f(X^0,\dots,X^{j-1},X^{j''},X^{j+1},\dots,X^k)(Z^1,\dots,Z^{j-1},Z^{j''},Z^{(j+1)''},Z^{(j+2)},\dots,Z^k)$$

Higher order Difference-Differential Operators

Proposition

$$f(X^0,\ldots,X^{k-1},\begin{bmatrix} X^{k'} & Z \\ 0 & X^{k''} \end{bmatrix})(Z^1,\ldots,Z^{k-1},\begin{bmatrix} Z^{k'} & Z^{k''} \end{bmatrix})$$

$$= \Big[f(X^0,\ldots,X^{k-1},X^{k'})(Z^1,\ldots,Z^{k-1},Z^{k'}),$$

$$\qquad {}_k\Delta_R f(X^0,\ldots,X^{k-1},X^{k'},X^{k''})(Z^1,\ldots,Z^{k-1},Z^{k'},Z) + f(X^0,\ldots,X^{k-1},X^{k''})(Z^1,\ldots,Z^{k-1},Z^{k''}) \Big]$$

Higher order Difference-Differential Operators

As an example, consider the function $f(X^0, X^1, X^2)(Z^1, Z^2) = X^0 Z^1 X^1 Z^2 X^2$. Then,

$$f\left(\begin{bmatrix} X^{0'} & Z \\ 0 & X^{0''} \end{bmatrix}, X^1, X^2\right)\left(\begin{bmatrix} Z^{1'} \\ Z^{1''} \end{bmatrix}, Z^2\right) = \begin{bmatrix} X^{0'} & Z \\ 0 & X^{0''} \end{bmatrix}\begin{bmatrix} Z^{1'} \\ Z^{1''} \end{bmatrix} X^1 Z^2 X^2$$

$$= \begin{bmatrix} X^{0'} Z^{1'} + Z Z^{1''} \\ X^{0''} Z^{1''} \end{bmatrix} X^1 Z^2 X^2 = \begin{bmatrix} X^{0'} Z^{1'} X^1 Z^2 X^2 + Z Z^{1''} X^1 Z^2 X^2 \\ X^{0''} Z^{1''} X^1 Z^2 X^2 \end{bmatrix}$$

$$= \begin{bmatrix} f(X^{0'}, X^1, \ldots, X^k)(Z^{1'}, Z^2, \ldots, Z^k) \\ \qquad + {}_0\Delta_R f(X^{0'}, X^{0''}, X^1, \ldots, X^k)(Z, Z^{1''}, Z^2 \ldots, Z^k) \\ f(X^{0''}, X^1, \ldots, X^k)(Z^{1''}, Z^2, \ldots, Z^k) \end{bmatrix}$$

Thus, ${}_0\Delta_R f(X^{0'}, X^{0''}, X^1, X^2)(Z, Z^{1''}, Z^2) = Z Z^{1''} X^1 Z^2 X^2$.

Linearity of the Image of ${}_j\Delta_R f$

As for order 0 nc functions,

Proposition

For any nc function f on a right admissible, nc set Ω, ${}_j\Delta_R f(X^0, \ldots, X^{j-1}, X^{j'}, X^{j''}, X^{(j+1)}, \ldots, X^k)$, can be extended to a linear function on the $\mathcal{R}$-module $\mathcal{M}_j^{n_j' \times n_j''}$.

Difference Formulae for Higher Order NC Functions

Proposition

Let f be an nc function on the nc set $\Omega^{(0)} \times \ldots \times \Omega^{(k)}$, then,

$$
\begin{aligned}
&f(X^0, \ldots, X^k)(Z^1, \ldots, Z^k) - f(Y^0, \ldots, Y^k)(Z^1, \ldots, Z^k) \\
&= \sum_{\alpha_1=0}^{k} {}_{\alpha_1}\Delta_R f(Y^0, \ldots, Y^{\alpha_1}, X^{\alpha_1}, \ldots, X^k) \\
&\qquad (Z^1, \ldots, Z^{\alpha_1}, X^{\alpha_1} - Y^{\alpha_1}, Z^{\alpha_1+1}, \ldots, Z^k),
\end{aligned}
$$

Difference Formulae for Higher Order NC Functions

Applying this to the function, $f(X^0, X^1)(Z^1) = X^0 Z^1 X^1$,

$$\begin{aligned}(X^0 - Y^0)Z^1X^1 + Y^0Z^1(X^1 - Y^1) &= X^0Z^1X^1 - Y^0Z^1X^1 + Y^0Z^1X^1 - Y^0Z^1Y^1 \\ &= X^0Z^1X^1 - Y^0Z^1Y^1 \\ &= f(X^0, X^1)(Z^1) - f(Y^0, Y^1)(Z^1)\end{aligned}$$

Iterated Difference-Differential Operators

Recall that we found that for

$$f(X^0, X^1, X^2)(Z^1, Z^2) = X^0 Z^1 X^1 Z^2 X^2,$$

$$_0\Delta_R f(X^{0'}, X^{0''}, X^1, X^2)(Z, Z^{1''}, Z^2) = Z Z^{1''} X^1 Z^2 X^2.$$

If we want to find $_1\Delta_{R0}\Delta_R f$, should we take the derivative in the new position 1 or in the old position 1?

Since $X^{0''}$ does not appear in the expression and X^1 does, it is clear that these will give different results.

Iterated Difference-Differential Operators

We define,

$$_j\Delta^l{}_R := {}_j\Delta_R \ldots {}_j\Delta_R \qquad \text{for} \qquad 0 \leq j \leq k$$

Thus, $_j\Delta^l{}_R$ is calculated iteratively using 2×2 block upper triangular matrices.

Alternatively, it can be calculated in a single step.

Iterated Difference-Differential Operators

A necessary condition for integrability,

Theorem

Let $g \in \mathcal{T}^k(\Omega^{(0)}, \ldots, \Omega^{(k)}; \mathcal{N}_{0,nc}, \ldots, \mathcal{N}_{k,nc})$ *with* $\Omega^{(j)}$ *a right admissible nc set for all* $j = 0, \ldots, k$*. Let* $f = {}_j\Delta_R^l g$*. Then,* ${}_j\Delta_R f = {}_m\Delta_R f$ *for* $m = j, \ldots, j + l$*.*

Iterated Difference-Differential Operators

Coming back to our question from earlier, we now see that to find ${}_1\Delta_{R0}\Delta_R f$, we should take the derivative in the old position 1.

With this in mind, we define some new notation.

New Notation

Applying ${}_j\Delta_R$ to $f(X^0, \ldots, X^k)(Z^1, \ldots, Z^k)$, we now write

$$ {}_j\Delta_R f(X^0, \ldots, X^{j-1}, \vec{X}^j, X^{j+1}, \ldots, X^k)(Z^1, \ldots, Z^{j-1}, \vec{Z}^j, Z_2^{j+1}, \ldots, Z^k) $$

where

$$ \vec{X}^j = (X_0^j, X_1^j) $$

and

$$ \vec{Z}^j = (Z^{j,0}, Z^{j,1}) $$

.

If all entries of $\vec{X}^j$ are the same, X^j, denote it as $\widehat{X^j}$.

Taylor-Taylor Formula for Higher NC Functions

Theorem

For $f \in \mathcal{T}^k(\Omega^{(0)} \times \ldots \times \Omega^{(k)}; \mathcal{N}_{0,nc}, \ldots, \mathcal{N}_{k,nc})$, α_q the last nonzero α_j and an arbitrary integer N,

$$f(X^0, \ldots, X^k)(Z^1, \ldots, Z^k)$$
$$= \sum_{p=0}^{N} \sum_{\alpha_0 + \ldots + \alpha_k = p} {}_k\Delta_R^{\alpha_k} \ldots {}_0\Delta_R^{\alpha_0} f(\widehat{Y^0}, \ldots, \widehat{Y^k})$$
$$(\widehat{X^0 - Y^0}, Z^1, \widehat{X^1 - Y^1}, \ldots, Z^k, \widehat{X^k - Y^k})$$
$$+ \sum_{\alpha_0 + \ldots + \alpha_k = N+1} {}_q\Delta_R^{\alpha_q} \ldots {}_0\Delta_R^{\alpha_0} f(\widehat{Y^0}, \ldots, \widehat{Y^{q-1}}, \vec{Y}^q, X^{q+1}, \ldots, X^k)$$
$$(\widehat{X^0 - Y^0}, Z^1, \widehat{X^1 - Y^1}, \ldots, Z^k, \widehat{X^k - Y^k}),$$

Alternate Taylor-Taylor Formula

It is also possible to write the Taylor formula centered at $(Y^0, \ldots, Y^k) \in \Omega_{s_0}^{(0)} \times \ldots \times \Omega_{s_k}^{(k)}$ where for all j $n_j = m_j s_j$ for some positive integers m_j.

Theorem

Let $f \in \mathcal{T}^k(\Omega^{(0)} \times \ldots \times \Omega^{(k)}; \mathcal{N}_{0,nc}, \ldots, \mathcal{N}_{k,nc})$, for each $N \in \mathbb{N}$, α_q the last nonzero α_j and using the difference formula for higher order nc functions,

Alternate Taylor-Taylor Formula

Theorem

$$f(X^0,\dots,X^k)(Z^1,\dots,Z^k)$$

$$= \sum_{l=0}^{N} \sum_{\alpha_0+\cdots+\alpha_k=N} \Bigg(\left(X^0 - \bigoplus_{\beta_0=1}^{m_0} Y^0 \right)^{\odot_{s_0} \alpha_0} {}_{s_0,s_1} \odot_{s_0} Z^0{}_{s_0,s_2} \odot_{s_1} \left(X^1 - \bigoplus_{\beta_1=1}^{m_1} Y^1 \right)^{\odot_{s_1} \alpha_1} {}_{s_1,s_2} \odot_{s_1} \cdots$$

$$\cdots_{s_{k-2},s_k} \odot_{s_{k-1}} Z^k{}_{s_{k-1},s_k} \odot_{s_k} \left(X^k - \bigoplus_{\beta_k=1}^{m_k} Y^k \right)^{\odot_{s_k} \alpha_k} \Bigg)$$

$${}_k\Delta_R^{\alpha_k} \cdots {}_0\Delta_R^{\alpha_0} f(\widehat{Y^0},\dots,\widehat{Y^k})$$

Alternate Taylor-Taylor Formula

Theorem

$$+\sum_{\alpha_0+\ldots+\alpha_k=N+1}\left(\left(\left(X^0-\bigoplus_{\beta_0=1}^{m_0}Y^0\right)^{\odot_{s_0}\alpha_0}{}_{s_0,s_1}\odot_{s_0}Z^0{}_{s_0,s_2}\odot_{s_1}\left(X^1-\bigoplus_{\beta_1=1}^{m_1}Y^1\right)^{\odot_{s_1}\alpha_1}{}_{s_1,s_2}\odot_{s_1}\cdots\right.\right.$$

$$\left.\cdots{}_{s_{q-2},s_q}\odot_{s_{q-1}}Z^q{}_{s_{q-1},s_q}\odot_{s_q}\left(X^q-\bigoplus_{\beta_q=1}^{m_q}Y^q\right)^{\odot_{s_q}\alpha_q}\right)$$

$$\left.{}_{s_{q-1},s_{q+1}}\odot_{s_q}Z^{q+1}{}_{s_q,s_{q+2}}\odot_{s_{q+1}}\cdots{}_{s_{k-2},s_k}\odot_{s_{k-1}}Z^k\right)$$

$${}_k\Delta_R^{\alpha_k}\cdots{}_0\Delta_R^{\alpha_0}f\left(\widehat{Y^0},\ldots,\widehat{Y^{q-1}},\vec{Y}^q,X^{q+1},\ldots,X^k\right)$$

Current Research

I am currently studying the integration of nc functions in joint work with Dr. Victor Vinnikov and Dr. Dmitry Kaliushny-Verbotvetskyi. We have shown that as long as the modules involved are over rings of characteristic 0, then the necessary condition that ${}_j\Delta_R f = {}_m\Delta_R f$ for $m = j, \dots, j + l$, established above is also sufficient.

We have partial results in the case of finite characteristic.

Bibliography

- J. Agler and J. E. McCarthy. Global holomorphic functions in several noncommuting variables. Preprint, arXiv:1305.1636.
- J. Agler and N. J. Young. Symmetric functions of two noncommuting variables. Preprint, arXiv:1307.1588.
- I. Gelfand, S. Gelfand, V. Retakh, and R. L. Wilson. Quasideterminants. *Adv. Math.* 193(1):56-141, 2005.
- D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov. *Foundations of Free Noncommutative Function Theory*. Mathematical Surveys and Monographs, Vol. 199. American Mathematical Society, Providence, R. I., 2014.

Bibliography

- D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov. Noncommutative rational functions, their difference-differential calculus and realizations. *Multidimens. Syst. Signal Process.* 23 (2012), no. 1-2, 49-77.

- D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov. Noncommutative rational functions, their difference-differential calculus and realizations. *Multidimens. Syst. Signal Process.* 23 (2012), no. 1-2, 49-77.

- D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov. *Foundations of Free Noncommutative Function Theory*. Mathematical Surveys and Monographs, Vol. 199. American Mathematical Society, Providence, R. I., 2014.

Bibliography

- P.S. Muhly and B. Solel. Hardy Algebras, W^*-correspondances and interpolation theory. *Math. Ann. 330:353-415, 2004.*

- *P.S. Muhly and B. Solel. Progress in noncommutative function theory.* Sci. China Math. 54 (2011), no. 11, 2275-2294.

- P.S. Muhly and B. Solel. Schur class operator functions and automorphisms of Hardy algebras. *Doc. Math. 13:365-411, 2008.*

- *P.S. Muhly and B. Solel. Tensorial function theory: From Berezin transforms to Taylor's Taylor series and back. Integral Equations Operator Theory 76 (2013), no. 4, 463-508.*

Bibliography

J. L. Taylor. A general framework for a multi-operator functional calculus. *Advances in Math.*, **9**:183-252, 1972.

J. L. Taylor. Functions of several noncommuting variables. *Bull. Amer. Math. Soc.*, **79**:1 - 34, 1973.

D. V. Voiculescu, K. J. Dykema, and A. Nica. Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. CRM Monograph Series, 1. American Mathematical Society, Providence, RI, 1992. vi+70 pp.