22M:034 Engineer Math IV: Differential Equations

Midterm 2 Practice Problems

1. Solve the initial value problem:

$$y'' + 4y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Solution. Find first the roots of the characteristic equation

$$r^2 + 4r + 5 = 0.$$

They are

$$r_{1,2} = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.$$

Then the general solution of the equation is

$$y(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t.$$

From the condition $y(0) = 1$, we find that $c_1 = 1$. Since

$$y'(t) = -2e^{-2t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(c_1(-\sin t) + c_2 \cos t),$$

the initial condition $y'(0) = 0$ implies that $0 = -2c_1 + c_2$. Hence $c_2 = 2$.

Thus, the function

$$y(t) = e^{-2t}(\cos t + 2 \sin t)$$

is the solution of the initial value problem.

2. Find the general solution of the nonhomogeneous equation

$$y'' - 2y' - 3y = 3te^{2t}.$$

Solution. We first find the roots $r_1 = 3,\ r_2 = -1$ of the characteristic equation $r^2 - 2r - 3 = 0$.

The general solution of the corresponding homogeneous equation is

$$y_h(y) = c_1 e^{3t} + c_2 e^{-t}.$$

Since the function $3te^{2t}$ is not a solution of the homogeneous equation, a particular solution of the given nonhomogeneous equation can be found in the form

$$Y(t) = (At + B)e^{2t}.$$

Find $Y'(t)$ and $Y''(t)$ and substitute them into the equation to determine A and B.

$$Y'(t) = Ae^{2t} + 2(At + B)e^{2t} = (A + 2B)e^{2t} + 2Ate^{2t},$$

$$Y''(t) = (2A + 4B)e^{2t} + 2Ae^{2t} + 4Ate^{2t} = (4A + 4B)e^{2t} + 4Ate^{2t}.$$

Then

$$(4A + 4B)e^{2t} + 4Ate^{2t} - 2((A + 2B)e^{2t} + 2Ate^{2t}) - 3(At + B)e^{2t} = 3te^{2t}$$

or

$$(2A - 3B)e^{2t} - 3Ate^{2t} = 3te^{2t}.$$

Hence, $2A - 3B = 0$ and $A = -1$. Therefore $B = -2/3$. Finally,

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - (t + \frac{2}{3})e^{2t}$$

is the general solution of the nonhomogeneous equation.

Remark. The problem can be also solved by the method of variation of parameters.

3. Find a particular solution of the equation

$$ty'' - (1+t)y' + y = t^2 e^{2t} \qquad (t > 0).$$

Use the fact that the functions $y_1 = 1 + t,\ y_2 = e^t$ form a fundamental set of solutions to the corresponding homogeneous equation.

Solution. Rewrite the differential equation in the standard form

$$y'' - \frac{1+t}{t}y' + \frac{1}{t}y = te^{2t}.$$

A particular solution is sought in the form $y = u_1(t)y_1(t) + u_2(t)y_2(t)$ where

$$u_1 = -\int \frac{y_2 g}{W(y_1, y_2)}dt, \quad u_2 = \int \frac{y_1 g}{W(y_1, y_2)}dt.$$

Here $g(t) = te^{2t}$ and $W(y_1, y_2)(t)$ is the Wronskian of y_1 and y_2, that is

$$W(y_1, y_2)(t) = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = (1+t)e^t - e^t = te^t.$$

Thus,

$$u_1 = -\int \frac{e^t te^{2t}}{te^t}dt = -\int e^{2t}dt = -\frac{1}{2}e^{2t},$$

$$u_2 = \int \frac{(1+t)te^{2t}}{te^t}dt = \int e^t dt + \int te^t dt = e^t + te^t - e^t = te^t.$$

Finally,

$$y(t) = -\frac{1}{2}e^{2t}(1+t) + (te^t)e^t = -\frac{1}{2}e^{2t} + \frac{1}{2}te^{2t} = \frac{1}{2}(t-1)e^{2t}.$$

4. Find the *general solution* of the corresponding homogeneous equation, and determine the *form* of a particular solution. **Do not solve** for the coefficients in the particular solution.

(a)

$$y'' + y = 2\cos t,$$

The roots of the characteristic equation are i and $-i$. The general solution of the corresponding homogeneous equation is

$$y_h = c_1 \cos t + c_2 \sin t.$$

A particular solution should be found in the form

$$Y(t) = t(A\cos t + B\sin t).$$

The general solution of the nonhomogeneous equation is $y(t) = y_h(t) + Y(t)$.

(b)

$$y^{(5)} - 8y^{(4)} + 18y''' + 10y'' - 87y' + 90y = 3te^{3t} - e^{-2t} + \sin t.$$

Hint: The roots of the corresponding homogeneous equations are -2, 3, 3, $2+i$, and $2-i$ (root 3 is of multiplicity 2).

The general solution of the corresponding homogeneous equation is

$$y_h(t) = c_1e^{-2t} + c_2e^{3t} + c_3te^{3t} + e^{2t}(c_4\cos t + c_5\sin t).$$

A particular solution $Y(t)$ must be sought in the form:

$$Y(t) = t^2(At + B)e^{3t} + Cte^{-2t} + D\cos t + E\sin t.$$

The general solution of the nonhomogeneous equation is $y(t) = y_h(t) + Y(t)$.

1. Solve the following differential equation

$$y''' - y'' - y' + y = 0; \quad y(0) = 1, y'(0) = 0, y''(0) = 0. \tag{0.1}$$

Proof. Consider the characteristic equation $r^3 - r^2 - r + 1 = 0$. Solving for r we further get

$$r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r-1)(r^2-1) = (r-1)^2(r+1) = 0$$

So the roots are $r_1 = 1, r_2 = 1, r_3 = -1$. Notice that the root 1 is repeated two times then $\{e^t, te^t, e^{-t}\}$ forms a fundamental set of solutions for (0.1). Hence the general solution for (0.1) is

$$y(t) = c_1 e^t + c_2 te^t + c_3 e^{-t}. \tag{0.2}$$

Next we will find the constants c_1, c_2, c_3 for which (0.2) satisfies the initial data in (0.1). Plugging (0.2) in (0.1) we obtain that the following system of equations:

$$\begin{aligned}
y(0) = c_1 e^0 + c_2 0 \cdot e^0 + c_3 e^{-0} = c_1 + c_3 &= 1 \\
y'(0) = c_1 e^0 + c_2 e^0 + c_2 0 \cdot e^0 - c_3 e^{-0} = c_1 + c_2 - c_3 &= 0 \\
y''(0) = c_1 e^0 + 2c_2 e^0 + c_2 0 \cdot e^0 + c_3 e^{-0} = c_1 + 2c_2 + c_3 &= 0
\end{aligned}$$

Solving we obtain $c_1 = 3/4$, $c_2 = -1/2$, and $c_3 = 1/4$; thus the solution of (0.1) is $y(t) = 3/4e^t - 1/2te^t + 1/4e^{-t}$.

$\square$

2. Consider the following differential equation

$$y'' - 2y' + y = \frac{e^t}{1+t^2}. \tag{0.3}$$

a) Determine the general solution.

b) Find the solution that satisfies the initial values $y(0) = 1; y'(0) = 0$.

Proof. Consider the corresponding homogeneous equation, i.e.

$$y'' - 2y' + y = 0. \tag{0.4}$$

Its characteristic equation is $r^2 - 2r + 1 = 0$ whose solutions are $r_1 = r_2 = 1$. Therefore, $\{e^t, te^t\}$ forms a fundamental set of solutions for (0.4) and hence the general solution of (0.4) is

$$y(t) = c_1e^t + c_2te^t.$$

Next we proceed to finding a particular solution $Y(t)$ for equation (0.3). Applying the method of varying the parameter, assume the solution is of the form

$$Y(t) = u_1(t)e^t + u_2(t)te^t. \tag{0.5}$$

and we solve next for the functions u_1 and u_2. So we set up the system

$$\begin{bmatrix} u_1'(t)e^t & + & u_2'(t)te^t & = & 0 \\ u_1'(t)e^t & + & u_2'(t)(e^t + te^t) & = & e^t/(1+t^2) \end{bmatrix}$$

Solving the system using determinants, we get

$$u_1'(t) = \frac{\begin{vmatrix} 0 & te^t \\ e^t/(1+t^2) & e^t + te^t \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{-t}{1+t^2} \tag{0.6}$$

$$u_2'(t) = \frac{\begin{vmatrix} e^t & 0 \\ e^t & e^t/(1+t^2) \end{vmatrix}}{\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}} = \frac{1}{1+t^2} \tag{0.7}$$

So integrating in (0.6) and (0.7) we get

$$u_1(t) = \int \frac{-t}{1+t^2}dt = -\frac{1}{2}\int \frac{(1+t^2)'}{1+t^2}dt = -\frac{1}{2}\ln(1+t^2)$$

$$u_2(t) = \int \frac{1}{1+t^2} dt = \arctan(t)$$

In conclusion, the general solution of (0.3) is

$$y(t) = c_1 e^t + c_2 t e^t - \frac{1}{2}\ln(1+t^2)e^t + \arctan(t) t e^t \tag{0.8}$$

b) Using the initial data in the general solution (0.8), solving for c_1 and c_2 we get that

$$\begin{aligned}
1 = y(0) &= c_1 e^0 + c_2 0 e^0 - \frac{1}{2}\ln(1+0^2)e^0 + \arctan(0)0e^0 = c_1 \\
0 = y(0) &= c_1 e^0 + c_2 0 e^0 + c_2 e^0 - \frac{1}{2}\ln(1+0^2)e^0 - \frac{0}{1+0^2}e^0 + \arctan(0)0e^0 \\
&\quad + \arctan(0)e^0 + \frac{1}{1+0^2}0e^0 = c_1 + c_2
\end{aligned}$$

Thus $c_1 = 1$, $c_2 = -1$ and hence the solution for b) is

$$y(t) = e^t - te^t - \frac{1}{2}\ln(1+t^2)e^t + \arctan(t)te^t$$

□

3. A mass weighing 128 lb stretches a spring 96 in. If the mass is pulled down an additional one foot and then relesed, and if there is no damping determine the position u of the mass at any time t. Find the frequency, period, amplitude, and phase of the motion.

Proof. Since there is no damping and no external force the equation of the motion is

$$mu'' + ku = 0. \tag{0.9}$$

Next we proceed to finding m, k. First since $mg = w = 128$ we get that

$$m = \frac{w}{g} = \frac{128}{32} = 4\frac{\text{lb} \cdot \text{s}^2}{\text{ft}}$$

Notice that the enlongation is $L = 96in = 8\text{ft}$; We also have that $= 128w = mg = kL$ and hence get

$$k = \frac{w}{L} = \frac{128}{8} = 16\frac{\text{lb}}{\text{ft}}$$

Thus equation (0.9) become $4u'' + 16u = 0$, and dividing by 4 we get

$$u'' + 4u = 0. \tag{0.10}$$

The characteristic equation is $r^2 + 4 = 0$ and its solutions are the complex numbers $r_1 = 2i$ and $r_2 = -2i$. Thus the general solution of (0.10) is of the form

$$u(t) = A\cos(2t) + B\sin(2t) \tag{0.11}$$

Next we determine the constants A and B using the initial data. Since "the mass is pulled down an additional one foot" then $u(0) = 1$ and since the mass is "released " it means there is no initial acceleration so $u'(0) = 0$. Using the initial data in the general solution (0.11), solving for A and B we get that

$$1 = u(0) = A\cos(2 \cdot 0) + B\sin(2 \cdot 0) = A$$

$$0 = u(0) = -2A\sin(2 \cdot 0) + 2B\cos(2 \cdot 0) = 2B$$

Thus $A = 1$, $B = 0$ and hence the solution is $u(t) = \cos(2t)$.

Thus the amplitude is

$$R = \sqrt{A^2 + B^2} = \sqrt{1^2 + 0^2} = 1.$$

The period is

$$T = 2\pi\left(\frac{m}{k}\right)^{\frac{1}{2}} = 2\pi\left(\frac{4}{16}\right)^{\frac{1}{2}} = \pi.$$

The frequency is

$$\omega_0 = \left(\frac{k}{m}\right)^{\frac{1}{2}} = 2.$$

Finally, since the phase δ satisfies $\tan(\delta) = \frac{B}{A} = 0$ then $\delta = 0$.

$\square$

4. Find the general solution of the following differential equation

$$y'' - 3y' - 4y = e^{4t} + 4. \tag{0.12}$$

Proof. Consider the corresponding homogeneous equation, i.e.

$$y'' - 3y' - 4y = 0. \tag{0.13}$$

Its characteristic equation is $r^2 - 3r - 4 = 0$ whose solutions are $r_1 = 4, r_2 = -3$. Therefore, $\{e^{4t}, e^{-3t}\}$ forms a fundamental set of solutions for (0.13) and hence the general solution of (0.13) is

$$y(t) = c_1 e^{4t} + c_2 e^{-3t}.$$

Next we proceed to finding a particular solution $Y(t)$ for equation (0.12). Applying the method of undeterminated coefficients, assume the solution is of the form

$$Y(t) = Ate^{4t} + B. \tag{0.14}$$

and we solve next for A and B. So plugging $Y(t)$ in equation (0.12) we get

$$\begin{aligned}
(Ate^{4t} + B)'' - 3(Ate^{4t} + B)' - 4(Ate^{4t} + B) &= e^{4t} + 4 \\
(4Ate^{4t} + Ae^{4t})' - 3(4Ate^{4t} + Ae^{4t}) - 4(Ate^{4t} + B) &= e^{4t} + 4 \\
16Ate^{4t} + 5Ae^{4t} - 12Ate^{4t} - 3Ae^{4t} - 4Ate^{4t} - 4B &= e^{4t} + 4 \\
2Ae^{4t} - 4B &= e^{4t} + 4
\end{aligned}$$

Identifying the coefficients we get $2A = 1$ and $-4B = 4$ which further gives that $A = \frac{1}{2}$ and $B = -1$. Therefore using (0.14) we have $Y(t) = \frac{1}{2}te^{4t} - 1$ and the general solution of (0.12) is then

$$y(t) = c_1 e^{4t} + c_2 e^{-3t} + \frac{1}{2}te^{4t} - 1.$$

$\square$

5. Solve the following initial value problem

$$y'' - 6y' + 9y = 0; \quad y(0) = 0, y'(0) = 2, \tag{0.15}$$

and describe the behavior for increasing t.

Proof. The characteristic equation of (0.15) is $r^2 - 6r + 9 = 0$ and its solutions are $r_1 = r_2 = 3$. Therefore, $\{e^{3t}, te^{3t}\}$ forms a fundamental set of solutions for (0.15) and hence the general solution of (0.15) is

$$y(t) = c_1 e^{3t} + c_2 t e^{3t}. \tag{0.16}$$

Using the initial data in the general solution (0.16), solving for c_1 and c_2 we get that

$$0 = y(0) = c_1 e^{3\cdot 0} + c_2 \cdot 0 \cdot e^{3\cdot 0} = c_1$$
$$2 = y'(0) = 3c_1 e^{3\cdot 0} + 3c_2 \cdot 0 \cdot e^{3\cdot 0} + c_2 e^{3\cdot 0} = 3c_1 + c_2$$

Thus $c_1 = 0$, $c_2 = 2$ and hence the solution is $y(t) = 2te^{3t}$. Also notice that $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} 2te^{3t} = \infty$

$\square$

1. If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

Solution.

$$\begin{aligned}
W &= \begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix} = 3e^{4t} \\
e^{2t}(g'(t) - 2g(t)) &= 3e^{4t} \\
g'(t) - 2g(t) &= 3e^{2t}
\end{aligned}$$

Integrating factor: $\mu(t) = e^{\int -2dt} = e^{-2t}$

$$\begin{aligned}
(e^{2t}g(t))' &= 3 \\
e^{-2t}g(t) &= 3t + C \\
g(t) &= 3te^{2t} + Ce^{2t}
\end{aligned}$$

2. Solve the given initial value problems:

(a) $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = 1$.

(b) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$.

Solution.

(a) $y'' + y' - 2y = 0$, $y(0) = 0$, $y'(0) = 1$.

$r^2 + r - 2 = 0 \Leftrightarrow (r-1)(r+2) = 0$; $r_1 = 1$, $r_2 = -2$.

$y = c_1 e^t + c_2 e^{-2t}$.

$y(0) = 0 \Longrightarrow c_1 + c_2 = 1$.

$y' = c_1 e^t - 2c_2 e^{-2t}$; $y'(0) = 1 \Longrightarrow c_1 - 2c_2 = 1$.

Subtract the two: $3c_2 = 0$, so $c_2 = 0$; $c_1 = 1$.

$y = e^t$.

(b) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = 1$.

$r^2 + 4 = 0$; $r = \pm 2i$.

$y = c_1 \cos 2t + c_2 \sin 2t$

$y(0) = 0 \quad \Longrightarrow c_1 = 0$

$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$

$y'(0) = 1 \quad \Longrightarrow 2c_2 = 1,\ c_2 = \frac{1}{2}$

$$y = \frac{1}{2} \sin 2t$$

3. Solve the following differential equations:

(a) $2y'' + 3y' + y = t^2 + 3\sin t$.

(b) $y'' + 4y' + 4y = t^{-2}e^{-2t}$, $t > 0$.

Solution.

(a) $2y'' + 3y' + y = t^2 + 3\sin t$.

First: $2y'' + 3y' + y = t^2$. Try $y = At^2 + Bt + C$. $y' = 2At + B$; $y'' = 2A$.

$4A + 6At + 3B + At^2 + Bt + C = t^2$.

$$\begin{aligned} A &= 1 \\ 6A + B &= 0 \quad \Longrightarrow B = -6 \\ 4A + 3B + C &= 0 \quad \Longrightarrow C = -4 + 18 = 14 \end{aligned}$$

$y = t^2 - 6t + 14$

Second: $2y'' + 3y' + y = 3\sin t$.

$$\begin{aligned} y &= A\cos t + B\sin t \\ y' &= -A\sin t + B\cos t \\ y'' &= -A\cos t - B\sin t \end{aligned}$$

$-2A\cos t - 2B\sin t - 3A\sin t + 3B\cos t + A\cos t + B\sin t = 3\sin t$

$$\begin{cases} -A + 3B = 0 \\ -9A - 3B = 9 \end{cases}$$

$-10A = 9 \quad \Longrightarrow A = -\frac{9}{10}$; $B = -\frac{3}{10}$. Thus $y = -\frac{9}{10}\cos t - \frac{3}{10}\sin t$.

Combine the two: $y = t^2 - 6t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t$.

Also: $2y'' + 3y' + y = 0$; $2r^2 + 3r + 1 = 0$; $(r+1)(2r+1) = 0 \quad \Rightarrow r_1 = -1$, $r_2 = -\frac{1}{2}$.

$y = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t}$. So:

$$y = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t} + t^2 - 6t + 14 - \frac{9}{10}\cos t - \frac{3}{10}\sin t$$

(b) $y'' + 4y' + 4y = t^{-2t}e^{-2t}, \quad t > 0$

$r^2 + 4r + 4 = 0$; $r_{1,2} = -2 \quad \Longrightarrow y_1 = e^{-2t}, y_2 = te^{-2t}$.

Write solution $y = u_1(t)e^{-2t} + u_2(t)te^{-2t}$.

$$\begin{cases} u_1'(t)y_1 + u_2'(y)y_2 = 0 \\ -u_1'(t)y_1' + u_2'(y)y_2' = t^{-2}e^{-2t} \end{cases}$$

$$\begin{cases} u_1'(t)e^{-2t} + u_2'(y)te^{-2t} = 0 \\ -2u_1'(t)e^{-2t} + u_2'(y)(e^{-2t} - 2te^{-2t}) = t^{-2}e^{-2t} \end{cases}$$

equivalently

$$\begin{cases} 2u_1'(t) + 2u_2'(y)t = 0 \\ -2u_1'(t) + u_2'(y)(1-2t) = t^{-2} \end{cases}$$

Add and obtain: $u_2'(t) = t^{-2}$; $u_1'(t) = -t^{-1}$.

Integrate: $u_2(t) = -t^{-1} + c_2$; $u_1 = -\ln t + c_1$.

$$y = -e^{-2t}\ln t + c_1 e^{-2t} - e^{-2t} + c_2 t e^{-2t}$$

or

$$y = -e^{-2t}\ln t + d_1 e^{-2t} + d_2 t e^{-2t}.$$

1. Find the general solution for the following differential equation

$y'' - y' + y = 0.$

2. Find the solutions for the following differential equations

(a) $y'' + 4y' + 4y = \cos 2t;$

(b) $y'' - y' - 2y = e^{2t}.$

3. Let y_1 and y_2 be solutions of

$$y'' + 2ty' + \frac{1}{t+1}y = 0, \quad t \geq 0.$$

Let $W(t)$ be the Wronskian of $y_1(t)$ and $y_2(t)$. Given that $W(0) = 5$, find $W(t)$.

4. (a) Find the general solution for the forced vibration problem

$my'' + ky = \sin 4t$

with $m = 1$ and $k > 0$.

(b) For what value of $k > 0$ will the spring system be in resonance(so that the solution $Y(t)$ becomes unbounded as $t \to +\infty$)? Explain.

1. $y'' - y' + y = 0$

Solu: $y = e^{rt}$

$$r^2 - r + 1 = 0$$

$$r_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_1 = e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t, \quad y_2 = e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$$

$\Rightarrow$ general solu

$$y = c_1 e^{\frac{1}{2}t} \cos \frac{\sqrt{3}}{2} t + c_2 e^{\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$$

2, (a) $y'' + 4y' + 4y = \cos 2t$

Solution. First, $y'' + 4y' + 4y = 0$

$$y = e^{rt} \Rightarrow r^2 + 4r + 4 = 0$$

$$(r+2)(r+2) = 0, \quad r_1 = r_2 = -2$$

$$y_1 = e^{-2t}, \quad y_2 = te^{-2t}$$

$$y = c_1 e^{-2t} + c_2 t e^{-2t}$$

Now set $Y = A\cos 2t + B\sin 2t$ and plug into (a)

$$(-4A\cos 2t - 4B\sin 2t) + 4(-2A\sin 2t + 2B\cos 2t)$$
$$+ 4(A\cos 2t + B\sin 2t) = \cos 2t$$

$$\Rightarrow \begin{cases} (-4A + 8B + 4A)\cos 2t = \cos 2t \\ (-4B - 8A + 4B)\sin 2t = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A = 0 \\ B = \frac{1}{8} \end{cases}$$

$$\Rightarrow \quad Y = \frac{1}{8}\sin 2t$$

$$y = \frac{1}{8}\sin 2t + c_1 e^{-2t} + c_2 t e^{-2t}$$

2 (b) $\quad y'' - y' - 2y = e^{2t}$

Solu: $\quad y'' - y' - 2y = 0$

$$y = e^{rt}, \quad r^2 - r - 2 = 0, \ (r+1)(r-2) = 0$$

$$r_1 = -1, \quad r_2 = 2$$

$$y_1 = e^{-t}, \quad y_2 = e^{2t}$$

$Y = Ate^{2t}$, resonance case

$$Y' = Ae^{2t} + 2Ate^{2t}, \quad Y'' = 4Ae^{2t} + 4Ate^{2t}$$

plug into 2(b)

$$(4Ae^{2t} + 4Ate^{2t}) - (Ae^{2t} + 2Ate^{2t}) - 2Ate^{2t} = e^{2t}$$

$$(4A - A)e^{2t} = e^{2t}$$

$$3A = 1 \Rightarrow A = \frac{1}{3}$$

$$Y = \frac{1}{3}te^{2t} \Rightarrow y = \frac{1}{3}te^{2t} + c_1 e^{-t} + c_2 e^{2t}$$

3. $y'' + 2ty' + \frac{1}{t+1} y = 0, \quad t \geq 0$

Solu: Abel's Thm.

$$W(y_1, y_2)(t) = Ce^{-\int p_1(t)dt}$$

$$= Ce^{-\int 2t\,dt} = Ce^{-t^2}$$

$$W(y_1, y_2)(0) = C = 5$$

$$\Rightarrow W(y_1, y_2) = 5e^{-t^2}. \quad \#$$

4. (a) $my'' + ky = \sin 4t, \quad m=1, \quad k>0$

Solu: Let $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{k}$

$y'' + \omega_0^2 y = 0$

$y = e^{rt} \Rightarrow r^2 + \omega_0^2 = 0, \quad r^2 = -\omega_0^2$

$r_{1,2} = \pm\sqrt{-\omega_0^2} = \pm \omega_0 i, \quad \omega_0 = \sqrt{k}$

$y_1 = \cos \omega_0 t, \quad y_2 = \sin \omega_0 t$

(i) If $\omega_0 \neq 4$

$$Y = A\cos 4t + B\sin 4t$$

$$(-16A\cos 4t - 16B\sin 4t) + \omega_0^2 (A\cos 4t + B\sin 4t) = \sin 4t$$

$$\begin{cases} -16A + \omega_0^2 A = 0 & \Rightarrow A = 0 \\ -16B + \omega_0^2 B = 1 & \Rightarrow B = \frac{1}{\omega_0^2 - 16} \end{cases}$$

$$Y = \frac{1}{\omega_0^2 - 16} \sin 4t$$

$$y = \frac{1}{\omega_0^2 - 16} \sin 4t + C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

(ii) If $w_0 = 4 \Rightarrow Y = t(A\cos w_0 t + B\sin w_0 t)$

$$Y' = A\cos w_0 t + B\sin w_0 t - At w_0 \sin w_0 t + Bt w_0 \cos w_0 t$$

$$Y'' = -2Aw_0 \sin w_0 t + 2Bw_0 \cos w_0 t - At w_0^2 \cos w_0 t - Bt w_0^2 \sin w_0 t$$

Plug into the eq

$$-2Aw_0 \sin w_0 t + 2Bw_0 \cos w_0 t - At w_0^2 \cos w_0 t - Bt w_0^2 \sin w_0 t + w_0^2 (At\cos w_0 t + Bt\sin w_0 t) = \sin 4t$$

noticing $w_0 = 4$

$$\Rightarrow \begin{cases} -2Aw_0 = 1 \\ 2Bw_0 = 0 \end{cases} \Rightarrow \begin{array}{l} A = -\frac{1}{2w_0} = -\frac{1}{8} \\ B = 0 \end{array}$$

$$\Rightarrow Y = -\frac{1}{8} t\cos w_0 t = -\frac{1}{8} t\cos 4t$$

$$y = -\frac{1}{8} t\cos 4t + C_1 \cos 4t + C_2 \sin 4t \quad (*)$$

4(b) When $k = 16$, $w_0 = 4$, resonance occurs. In this case, y become unbounded as $t \to +\infty$ because the amplitude grows linearly with t, see $(*)$.

Midterm II-M34-101

Instructor: Jinghua Yao

1. (**The method of undetermined coefficients without degeneracy, and splitting**.) Find the general solution to the differential equation
$$y''(t) - y'(t) - 2y(t) = \sin(3t) + e^{4t} + t.$$

   Solution: The char equation is $r^2 - r - 2 = (r-2)(r+1) = 0$, which has two roots $r_1 = 2, r_2 = -1$. Hence the general solution to the homogeneous ODE is $y_c(t) = c_1 e^{2t} + c_2 e^{-t}$.

   By structural theorem to nonhomogeneous ODE, we need to find a special solution $Y(t)$. By splitting method, we can solve 3 special solutions $Y_1(t), Y_2(t), Y_3(t)$ to the following ODEs respectively: $y''(t) - y'(t) - 2y(t) = \sin(3t)$, $y''(t) - y'(t) - 2y(t) = e^{4t}$, $y''(t) - y'(t) - 2y(t) = t$.

   To solve for $Y_1(t)$, we can assume $Y_1(t) = A\cos 3t + B\sin 3t$ since 3 is not a root to the char equation or assume $\tilde{Y}(t) = \tilde{A}e^{3it}$ to solve $y''(t) - y'(t) - 2y(t) = e^{3it}$ then $Y_1(t) = Im\tilde{Y}(t)$; To solve for $Y_2(t)$, we can assume $Y_2(t) = Ce^{4t}$ since 4 is not a root to the char equation; To solve for $Y_3(t)$, we can assume $Y_3(t) = Dt + E$ since the ODE is 2nd order. $Y(t) = Y_1(t) + Y_2(t) + Y_3(t)$ by linearity.

   Finally, the general solution is $y(t) = y_c(t) + Y(t)$.

2. (**Linear dependence and linear independence**.) Let $f(x) = t^2|t|$ and $g(x) = t^3$. Show that $f(x), g(x)$ are linearly dependent on $-1 < t < 0$ and $0 < t < 1$ but linearly independent on $-1 < t < 1$.

   Solution. When t has a definite sign, we have $f(x) = -g(x)$ for $-1 < t < 0$ and $f(x) = g(x)$ for $0 < t < 1$, which tells they are linearly dependent. If $-1 < t < 1$, we can set their linear combination to be 0, i.e, $c_1 f(x) + c_2 g(x) = 0$. Then we easily conclude $c_1 = c_2 = 0$ by considering $t > 0$ and $t < 0$ in the prescribed interval $(-1, 1)$, which tells they are linearly independent in this case.

3. (**The method of variation of constants**.) Verify that a special solution to the differential equation $y''(t) + 2y'(t) - 3y(t) = g(t)$ is given by $Y(t) = \int_{t_0}^{t} \frac{1}{4}(e^{t-s} - e^{s-t})g(s)\,ds$ and $Y(t_0) = 0$ where t_0 is a constant in the interval of existence for solutions to the equation.

   Solution. Here one can solve this problem by the method of variation of parameters or one can directly verify this is a special solution. If one directly verifies this is a special solution, one may write the above expression as $Y(t) = e^t \int_{t_0}^{t} \frac{1}{4}e^{-s}g(s)\,ds - e^{-t}\int_{t_0}^{t} \frac{1}{4}e^{s}g(s)\,ds$, which will be helpful though unnecessary. $Y(t_0)$ is easily observed.

4. (**The method of undetermined coefficients with degeneracy**.) Let a, b, c be three real numbers such that $c \neq 0$ and $a \neq b$. Find a special solution to the differential equation $cy''(t) - c(a+b)y'(t) + abc\,y(t) = e^{at}$.

   Solution. The char equation is $Z(r) = c[r^2 - (a+b)r + ab] = c(r-a)(r-b) = 0$ and the two roots are $r_1 = a, r_2 = b$. To solve a special solution to the ODE, we need assume $Y(t) = Ate^{at}$ since a is a simple root of the char equation. Then one can run the procedure of the method of undetermined coefficients to get $A = \frac{1}{c(a-b)}$ hence $Y(t)$.

   Another solution: We know $A = \frac{1}{\frac{d}{dr}Z(r)\big|_{r=a}} = \frac{1}{c(a-b)}$ hence $Y(t)$.

5. (**The vibration**.) Write the vibration into the oscillation form $u(t) = R(t)\cos(\omega t - \delta)$, i.e., determine the amplitude $R(t)$, the frequency ω and the phase shift angle δ, for the motion $u''(t) + 2u'(t) + 5u = 0$ with initial conditions $u(0) = 3$ and $u'(0) = 5$. Note: we require the phase shift angle $\delta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ here.

   Solution. This problem is a variant of Example 3 in Page 201 of the textbook. We changed the data to make the computations easy. The general solution to the ODE is $u(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. One can determine $c_1 = 3, c_2 = 4$ by initial conditions. Hence $u(t) = 5e^{-t}\cos(2t - \delta)$ where $\tan\delta = 4/3$ hence $\delta = \arctan\frac{4}{3} \in (\frac{\pi}{4}, \frac{\pi}{2})$.

Midterm II (Practice)

Instructor: Seonguk Yoo

1. Solve the second order differential equation:

$$y'' - 2y' - 3y = t$$

Solution. The zeros of the characteristic equation $r^2 - 2r - 3 = 0$ are 3 and -1, which give us $y_1 = e^{3t}$ and $y_2 = e^{-t}$. Since the Wronskian of y_1 and y_2 is $-4e^{2t}$, it follows that y_1 and y_2 form a fundamental set of solution. Now, let the particular solution $Y(t) = At^3 + Bt^2 + Ct + D$. Then

$$Y'(t) = 3A^2 + 2Bt + C, \qquad Y'' = At + 2B.$$

Substitution gives the equation:

$$-3At^3 + (-6A - 3B)t^2 + (6A - 4B - 3C)t + (2B - 2C - 3D) = t.$$

Equivalently,

$$\begin{cases} -3A & = 0 \\ -6A - 3B & = 0 \\ 6A - 4B - 3C & = 1 \\ 2B - 2C - 3D & = 0 \end{cases}$$

The solution of the linear system is $A = B = 0$, $C = -\frac{1}{3}$, and $D = \frac{2}{9}$. Thus, the solution to the differential equation is

$$y = C_1 e^{3t} + C_2 e^{-t} - \frac{1}{3}t + \frac{2}{9} \qquad (C_1, \quad C_2 : \text{ some constants}).$$

2. Find the solution of the initial value problem:

$$y'' + 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 2$$

Solution. The zeros of the characteristic equation $r^2 + 2r + 5 = 0$ are $-1 \pm 2i$. Thus, the general solution to differential equation is

$$y = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) \qquad (C_1, \quad C_2 : \text{ some constants}).$$

To satisfy the first initial condition $y(0) = 1$, C_1 must be 1 and for the second initial condition $y'(0) = 2$,

$$y' = -e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + e^{-t}(-2C_1 \sin 2t + 2C_2 \cos 2t),$$

which means $y'(0) = -C_1 + 2C_2 = 2 \implies C_2 = \frac{3}{2}$. Therefore, the solution is

$$y = e^{-t}\left(\cos 2t + \frac{3}{2}\sin 2t\right).$$

3. Use the method of variation of parameters to find the general solution to the second order differential equation:

$$y'' + y = \sin t$$

Solution. The zeros of the character equation are $\pm i$, and so we take $y_1 = \cos t$ and $y_2 = \sin t$. The Wronskian is

$$W = \det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = 1 \neq 0.$$

By a theorem, the particular solution is obtained by

$$\begin{aligned} Y(t) &= -\cos t \int \frac{(\sin t)(\sin t)}{1} dt + \sin t \int \frac{(\cos t)(\sin t)}{1} dt \\ &= -\cos t \int \frac{1 - \cos 2t}{2} dt + \sin t \int \frac{\sin 2t}{2} dt \\ &= -\cos t \left(\frac{t}{2} - \frac{\sin 2t}{4} + C_3 \right) + \sin t \left(\frac{-\cos 2t}{4} \right) \\ &= -\frac{t \cos t}{2} + \frac{\sin t}{4} - C_3 \cos t. \end{aligned}$$

Notice that the last two terms can be absorbed in $y_c = C_1 \cos t + C_2 \sin t$ for some constants C_1 and C_2. Finally, the solution is

$$y = C_1 \cos t + C_2 \sin t - \frac{1}{2} t \cos t.$$

4. Consider a differential equation $yy'' + (y')^2 = 0$, $t > 0$.

   (1) Verify that $y_1 = 1$ and $y_2 = \sqrt{t}$ are solutions of the differential equation.

   (2) Calculate the Wronskian of y_1 and y_2. For what values of t, do they form a fundamental set of solutions?

   (3) Determine if $y = c_1 + c_2\sqrt{t}$ $(t > 0)$ can be, in general, a solution of the differential equation. Justify your answer.

Solution. (1) We can check

$$\begin{aligned} y_1 y_1'' + (y_1')^2 &= 1 \cdot 0 + 0^2 = 0 \\ y_2 y_2'' + (y_2')^2 &= \sqrt{t} \left(-\frac{1}{4t\sqrt{t}} \right) + \left(\frac{1}{2\sqrt{t}} \right)^2 = 0 \end{aligned}$$

(2) The Wronskian is

$$\det \begin{pmatrix} 1 & \sqrt{t} \\ 0 & \frac{1}{2\sqrt{t}} \end{pmatrix} = \frac{1}{2\sqrt{t}} \neq 0$$

for $t > 0$. Thus, we know y_1 and y_2 form a fundamental set of solutions.

(3) By a theorem, if y_1 and y_2 form a fundamental set of solutions, then the general solution to the differential equation is

$$y = C_1 \cdot 1 + C_2\sqrt{t} \qquad (t > 0).$$

5. Find the general solution of the differential equation:

$$y''' - y'' + y' - 1 = 0$$

Solution. The characteristic equation $r^3 - r^2 + r - 1 = 0$ is factored as

$$(r-1)(r^2+1) = 0 \implies r = 1, \quad \pm i.$$

Thus, the general solution is

$$y = C_1 e^t + C_2 \cos t + C_3 \sin t$$

for some constants C_1, C_2, and C_3.