Obstructions to lifting cocycles on groupoids and the associated C^*-algebras

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Groupoids

We regard a *groupoid* Γ as a small category with inverses.

- The object set of Γ is identified with the distinguished subset of units Γ^0.
- The range and source maps are denoted by $r, s : \Gamma \to \Gamma^0$.
- Composition is defined on the set of composable pairs

$$\Gamma^2 := \{(x, y) \in \Gamma \times \Gamma : s(x) = r(y)\};$$

- The inverse map $x \mapsto x^{-1}$ satisfies $r(x) = xx^{-1}$, $s(x) = x^{-1}x$.
- For $u \in \Gamma^0$, write $\Gamma^u = r^{-1}(u)$ and $\Gamma_u = s^{-1}(u)$.

Example: Transformation group groupoid

Example

Given a group action $H \curvearrowright X$ we form a groupoid $\Gamma := H \times X$ with $\Gamma^0 = \{e\} \times X$ (often identified with X) and

$$r(h,x) = (e, hx) \qquad s(h,x) = (e,x)$$
$$(h,x)^{-1} = (h^{-1}, hx) \qquad (hk,x) = (h,kx)(k,x)$$

Locally compact groupoids

Definition

- If Γ has a topology such that all the above maps are continuous, we say Γ is a topological groupoid.
- If in addition s is a local homeomorphism, Γ is said to be *étale*.

Haar systems on locally compact groupoids

Definition

A *left Haar system* for Γ is a family of measures $\{\lambda^u : u \in \Gamma^0\}$ s.t.

i. $\operatorname{supp} \lambda^u = \Gamma^u$,
ii. $u \mapsto \lambda(f)(u) := \int_\Gamma f \, d\lambda^u$ is continuous for all $f \in C_c(\Gamma)$,
iii. $\int_\Gamma f(xy) \, d\lambda^{s(x)}(y) = \int_\Gamma f(y) \, d\lambda^{r(x)}(y)$ for all $x \in \Gamma$, $f \in C_c(\Gamma)$.

If Γ is étale, counting measures on Γ^u constitute a left Haar system.

A right Haar system is defined similarly with $\operatorname{supp} \lambda_u = \Gamma_u$.

C^*-algebra of a groupoid

Definition

The Haar system provides a convolution operation on $C_c(\Gamma)$:

$$f * g(x) := \int_\Gamma f(y) g(y^{-1}x)\, d\lambda^{r(x)}(y).$$

An involution on $C_c(\Gamma)$ is defined via

$$f^*(x) = \overline{f(x^{-1})}.$$

Completed in the universal C^*-norm we obtain a C^*-algebra $C^*(\Gamma)$.

Cocycles

- Let Γ be a locally compact Hausdorff groupoid and let G be a locally compact abelian group.
- Let $Z_\Gamma(G)$ be the collection of continuous cocycles $\phi : \Gamma \to G$, that is

$$\phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \phi(\gamma_2).$$

- $Z_\Gamma(G)$ is an abelian group.

Twists

Definition

Let A be an abelian group and Γ a groupoid. A *twist* by A over Γ is a central groupoid extension

$$\Gamma^0 \times A \xrightarrow{j} \Sigma \xrightarrow{\pi} \Gamma,$$

where $\Sigma^0 = \Gamma^0$, j is injective, π is surjective, and $j(r(\sigma), a)\sigma = \sigma j(s(\sigma), a)$ for all $\sigma \in \Sigma$ and $a \in A$.

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Example

The *semi-direct* product of Γ and A, $\Gamma \times A$, is a twist of Γ by A called the *trivial twist*.

The group of twists

Definition

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Fact

If $f \in \mathrm{Hom}_\Gamma(A, B)$ and Σ is a twist by A, then there is a unique twist $f_* B$ by B and a twist morphism $f_* : \Sigma \to f_* \Sigma$ which is compatible with f.

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Fact

$T_\Gamma(A)$ is an abelian group under the operation

$$[\Sigma] + [\Sigma'] = \nabla^A_*[\Sigma *_\Gamma \Sigma'],$$

where $\nabla^A : A \oplus A \to A$ is $\nabla^A(a, a') = a + a'$.

Obstruction to lifting a cocycle

Example

- Let $p : B \to C$ be a homomorphism of abelian groups.
- Let $\phi \in Z_\Gamma(C)$.
- Define
$$\Sigma_\phi = \{(\gamma, b) \in \Gamma \times B \,:\, \phi(\gamma) = p(b)\}.$$

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Lemma

1. *Σ_ϕ is a twist of Γ by $A := \ker p$.*
2. *Σ_ϕ is trivial if and only if there is $\tilde{\phi} \in Z_\Gamma(B)$ such that $\phi = p_* \tilde{\phi}$.*
3. *If Γ is étale and A is discrete then Σ_ϕ is étale.*

Exact sequences

Lemma

If

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is a short exact sequence of abelian groups, then there is an exact sequence

$$0 \to Z_\Gamma(A) \xrightarrow{i_*} Z_\Gamma(B) \xrightarrow{p_*} Z_\Gamma(C) \xrightarrow{\delta} T_\Gamma(A),$$

where $\delta(\phi) := [\Sigma_\phi]$ *for* $\phi \in Z_\Gamma(C)$

Properties of $C^*(\Sigma_\phi)$

Theorem

Assume that Γ *is étale and* A *is discrete. Let* $\phi \in Z_\Gamma(C)$*. For each* $t \in \hat{B}$ *one can define a* $*$*-homomorphism* $\pi_t : C_c(\Sigma_\phi) \to C_c(\Gamma)$ *by*

$$\pi_t(f)(\gamma) = \sum_{\phi(\gamma)=p(b)} f(\gamma, b)\langle t, b\rangle$$

with the following properties

1. $\pi_t(f_1 * f_2)(\gamma) = (\pi_t(f_1) * \pi_t(f_2))(\gamma)$,
2. $\pi_t(f^*) = \pi_t(f)^*$, *and*
3. *for* $s \in \hat{C}$, $t \in \hat{B}$, $f \in C_c(\Sigma_\phi)$, *and* $\gamma \in \Gamma$ *we have*

$$\pi_{s+t}(f)(\gamma) = \alpha_s^\phi(\pi_t(f))(\gamma),$$

where $\alpha^\phi : \hat{C} \to \mathrm{Aut}\, C^*(\Gamma)$ *is defined via* $\alpha_t^\phi(f)(\gamma) = \langle t, \phi(\gamma)\rangle f(\gamma)$.

Main Theorem

Theorem

There is an injective $*$*-homomorphism* $\pi : C^*(\Sigma_\phi) \to C^b(\hat{B}, C^*(\Gamma))$ *such that*

1. *for all* $t \in \hat{B}$, $f \in C_c(\Sigma_\phi)$, $\pi(f)(t) = \pi_t(f)$
2. *for all* $t \in \hat{B}$, $s \in \hat{C}$, *and* $f \in C^*(\Sigma_\phi)$, $\pi(f)(s+t) = \alpha_s^\phi(\pi(f)(t))$.

Moreover, π *induces an isomorphism between* $C^*(\Sigma_\phi)$ *and* $\mathrm{Ind}_{\hat{C}}^{\hat{B}}(C^*(\Gamma), \alpha^\phi)$.

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Definition

If H is a closed subgroup of a locally compact group G and (A, H, α) is a dynamical system, the *induced algebra* is

$$\mathrm{Ind}_H^G(A, \alpha) = \{f \in C^b(G, A) : f(sh) = \alpha_h^{-1}(f(s)) \text{ and } sH \mapsto \|f(s)\| \in C_0(G/H)\}.$$

An Example

Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0.$$

If $\phi \in Z_\Gamma(\mathbb{T})$ then

$$\Sigma_\phi = \{(\gamma, t) \in \Gamma \times \mathbb{R} : \phi(\gamma) = e^{2\pi it}\}$$

Then $C^*(\Sigma_\phi)$ is $*$-isomorphic to the mapping torus

$$M_{\alpha^\phi} = \{f : [0,1] \to C^*(\Gamma) : f(1) = \alpha^\phi(f(0))\}.$$