Contractive determinantal representations of stable polynomials on a matrix polyball

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[1] Joint work with A. Grinshpan, V. Vinnikov, and H. J. Woerdeman

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$d = 1$: If $p(0) = 1$, then

$$p = (1 - a_1 z) \cdots (1 - a_n z) = \det(I - Kz),$$

where $a_i = 1/z_i$, $i = 1, \ldots, n$, the zeros z_i of p are counted according to their multiplicities, $K = \text{diag}[a_1, \ldots, a_n]$, and $n = \deg p$.

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If $\mathcal{D}$ is the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, then p is $\mathbb{D}$-stable (resp., strongly $\mathbb{D}$-stable) **iff** $\|K\| \leq 1$ (resp., $\|K\| < 1$).

$d = 2$: $p \in \mathbb{C}[z_1, z_2]$ is $\mathbb{D}^2$-stable (resp., strongly $\mathbb{D}^2$-stable) and $p(0) = 1$ **iff**

$$p = \det(I - KZ_n), \quad Z_n = z_1 I_{n_1} \oplus z_2 I_{n_2},$$

where $n = (n_1, n_2)$, $n_r = \deg_r p$, $r = 1, 2$, $K \in \mathbb{C}^{(n_1+n_2)\times(n_1+n_2)}$, and $\|K\| \leq 1$ (resp., $\|K\| < 1$) [Kummert, 1989], [Grinshpan, K-V, Vinnikov, Woerdeman, 2016].

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$d > 2$: Let $p \in \mathbb{C}[z_1, \ldots, z_d]$ be $\mathbb{D}^d$-stable (resp., strongly $\mathbb{D}^d$-stable) and $p(0) = 1$. Question: Is it always possible to write

$$p = \det(I - KZ_n), \quad Z_n = \bigoplus_{r=1}^{d} z_r I_{n_r},$$

where $n = (n_1, \ldots, n_d)$, $n_r = \deg_r p$, $r = 1, \ldots, d$, $K \in \mathbb{C}^{|n|\times|n|}$, $|n| = n_1 + \cdots + n_d$, and $\|K\| \le 1$ (resp., $\|K\| < 1$)?

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Answer: **YES** in some special cases; e.g., when p is linear, **NO** in general [Grinshpan, K-V, Woerdeman, 2013].

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Consider a *unit matrix polyball*

$$\mathcal{B} := \mathbb{B}^{\ell_1 \times m_1} \times \cdots \times \mathbb{B}^{\ell_k \times m_k}$$
$$= \Big\{ Z = (Z^{(1)}, \ldots, Z^{(k)}) \in \mathbb{C}^{\ell_1 \times m_1} \times \cdots \times \mathbb{C}^{\ell_k \times m_k}$$
$$: \|Z^{(r)}\| < 1, \ r = 1, \ldots, k \Big\}.$$

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- $k = 1$, $d = \ell m$, $\mathcal{B} = \mathbb{B}^{\ell \times m}$ (matrix unit ball a.k.a. Cartan's domain of type I). In particular, if $\ell = 1$, then $\mathcal{B} = \mathbb{B}^d = \{z \in \mathbb{C}^d : \sum_{i=1}^{d} |z_i|^2 < 1\}$ (unit ball).

Theorem (Main)

Let $p = \mathbb{C}[z_{ij}^{(r)}\colon r = 1, \ldots, k,\ i = 1, \ldots, \ell_r,\ j = 1, \ldots, m_r]$*, be strongly* $\mathcal{B}$*-stable, with* $p(0) = 1$*. Then there exist*

$n = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k$ *and* $K \in \mathbb{C}^{\sum_{r=1}^k m_r n_r \times \sum_{r=1}^k \ell_r n_r}$, $\|K\| < 1$, *so that*

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$$P(w, z) = \sum_{\lambda, \mu} P_{\lambda\mu} w^{\lambda} z^{\mu} \in \mathbb{C}^{\gamma \times \gamma}[w, z],$$

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$$P(T^*, T) := \sum_{\lambda,\, \mu} P_{\lambda\mu} \otimes T^{*\lambda} T^{\mu},$$

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2. $1 \in \mathcal{M}_1$.
3. For every $\gamma, \gamma' \in \mathbb{N}$, $P \in \mathcal{M}_\gamma$, and $F \in \mathbb{C}^{\gamma\times\gamma'}[z]$, one has $F^*(w)P(w,z)F(z) \in \mathcal{M}_{\gamma'}$.

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This generalizes the notion of a Hermitian quadratic module over $\mathbb{C}[z]$, where (1)–(3) hold with $\gamma = \gamma' = 1$ only.

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(iii) *For every $i = 1, \ldots, d$, one has $-w_i z_i \in \mathbb{R} + \mathcal{M}_1$.*

A matrix system $\mathcal{M} = \{\mathcal{M}_\gamma\}_{\gamma \in \mathbb{N}}$ of Hermitian quadratic modules over $\mathbb{C}[z]$ that satisfies any (and hence all) of properties (i)–(iii) in the Lemma is called *Archimedean*.

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Starting with polynomials $P_j \in \mathbb{C}^{\gamma_j \times \gamma_j}[w, z]_{\mathrm{h}}$, we introduce the sets $\mathcal{M}_\gamma$, $\gamma \in \mathbb{N}$, consisting of polynomials $P \in \mathbb{C}^{\gamma \times \gamma}[w, z]_{\mathrm{h}}$ for which there exist $H_j \in \mathbb{C}^{\gamma_j n_j \times \gamma}[z]$, for some $n_j \in \mathbb{N}$, $j = 0, \ldots, k$, such that

$$P(w, z) = H_0^*(w) H_0(z) + \sum_{j=1}^{k} H_j^*(w)(P_j(w, z) \otimes I_{n_j}) H_j(z).$$

Here $\gamma_0 = 1$.

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Here $\gamma_0 = 1$. We also assume that there exists a constant $c > 0$ such that $c^2 - w_i z_i \in \mathcal{M}_1$ for every $i = 1, \dots, d$. Then $\mathcal{M} = \mathcal{M}_{P_1,\dots,P_k} = \{\mathcal{M}_\gamma\}_{\gamma\in\mathbb{N}}$ is an Archimedean matrix system of Hermitian quadratic modules generated by $P_1, \dots, P_k$.

The following theorem is a matrix-valued generalization of the Hermitian Positivestellensatz [Putinar, 2006], [Helton, Putinar, 2007].

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Under the assumptions above, let $P \in \mathbb{C}^{\gamma\times\gamma}[w,z]$ *be such that for every* d*-tuple* $T = (T_1, \ldots, T_d)$ *of Hilbert space operators satisfying* $P_j(T^*, T) \geq 0$, $j = 1, \ldots, k$, *we have that* $P(T^*, T) > 0$. *Then* $P \in \mathcal{M}_\gamma$.

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The proof extends the one from [Helton, Putinar, 2007]. It uses the Minkowski–Eidelheit–Kakutani separation theorem and a special construction of T.

Step 2: Realization. Given $\mathbf{P} \in \mathbb{C}^{\ell \times m}[z]$, $z = (z_1 \ldots, z_d)$, let

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For $T \in \mathcal{T}_{\mathbf{P}}$, the Taylor joint spectrum $\sigma(T)$ lies in $\mathcal{D}_{\mathbf{P}}$, and therefore for an operator-valued function F holomorphic on $\mathcal{D}_{\mathbf{P}}$ one defines $F(T)$ by means of Taylor's functional calculus.

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We say that F belongs to the *Schur–Agler class* $\mathcal{SA}_{\mathbf{P}}(\mathcal{U}, \mathcal{Y})$ associated with $\mathbf{P}$, if $F \colon \mathcal{D}_{\mathbf{P}} \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is holomorphic and $\|F\|_{\mathcal{A},\mathbf{P}} \leq 1$.

By [Ambrozie, Timotin, 2003] and [Ball, Bolotnikov, 2004], $F \in \mathcal{SA}_{\mathbf{P}}(\mathcal{U}, \mathcal{Y})$ **iff** there exist a Hilbert space $\mathcal{X}$ and a unitary colligation

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} : (\mathbb{C}^m \otimes \mathcal{X}) \oplus \mathcal{U} \to (\mathbb{C}^\ell \otimes \mathcal{X}) \oplus \mathcal{Y}$$

such that

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This generalizes [Agler, 1990] from the case $\mathbf{P} = \text{diag}[z_1, \ldots, z_d]$ $\mathcal{D}_{\mathbf{P}} = \mathbb{D}^d$.

Theorem

Let $\mathbf{P} = \bigoplus_{r=1}^{k} \mathbf{P}^{(r)}$*, where* $\mathbf{P}^{(r)} \in \mathbb{C}^{\ell_r \times m_r}[z]$ *and*

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satisfy the Archimedean condition. Let $F = QR^{-1}$ *be a rational* $\alpha \times \beta$ *matrix-valued function which is regular on* $\overline{\mathcal{D}}_{\mathbf{P}}$ *and satisfies* $\|F\|_{\mathcal{A},\mathbf{P}} < 1$. *Then there exist* $n = (n_1, \ldots, n_k) \in \mathbb{Z}_+^k$ *and a contraction colligation matrix* $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *of size* $(\sum_{r=1}^{k} m_r n_r + \alpha) \times (\sum_{r=1}^{k} \ell_r n_r + \beta)$ *such that*

$$F = D + C\mathbf{P}_n(I - A\mathbf{P}_n)^{-1}B, \qquad \mathbf{P}_n = \bigoplus_{r=1}^{k}(\mathbf{P}^{(r)} \otimes I_{n_r}).$$

The proof uses the matrix-valued Hermitian Nullstellensatz which produces a decomposition

$$R^*(w)R(z) - Q^*(w)Q(z) = H_0^*(w)H_0(z) + \sum_{j=1}^{k} H_j^*(w)\Big((I_{m_r} - \mathbf{P}^{(r)*}(w)\mathbf{P}^{(r)}(z)) \otimes I_{n_j}\Big)H_j(z),$$

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Then a lurking contraction argument is applied to construct a colligation...

In a special case, when $\mathbf{P} = \mathbf{Z} = \bigoplus_{r=1}^{k} Z^{(r)}$, one has $\mathcal{D}_{\mathbf{P}} = \mathcal{B}$, the Archimedean condition holds, and

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Now we are going back to our main theorem...

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Therefore

$$cg = D + C(\rho^{-1}Z_n)(I - A(\rho^{-1}Z_n))^{-1}B = D + C'Z_n(I - A'Z_n)^{-1}B,$$

where $C' = \rho^{-1}C$ and $A' = \rho^{-1}A$ are strict contractions, and $\begin{bmatrix} A' & B \\ C' & D \end{bmatrix}$ is a contraction.

Step 3: NC lifting. Next we lift the rational function cg to a nc rational expression using the same realization formula,

$$R_0 = D + C' z_n (I - A' z_n))^{-1} B,$$

now with $z_n = \bigoplus_{r=1}^{k} (z^{(r)} \otimes I_{n_r})$ and the entries $z_{ij}^{(r)}$ of matrices $z^{(r)}$ being nc indeterminates, $r = 1, \ldots, k$, $i = 1, \ldots, \ell_r$, $j = 1, \ldots, m_r$.

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Step 4: Minimal compression. Using the result from [Ball, Groenewald, and Malakorn, 2005], one can compress the given structured noncommutative multidimensional noncommutative linear system to a minimal one associated with the colligation matrix $\begin{bmatrix} A_{\min} & B_{\min} \\ C_{\min} & D_{\min} \end{bmatrix}$, i.e., the one with minimal possible $n_r = (n_r)_{\min}$, $r = 1, \ldots, k$.

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Moreover, the realization of R_1^{-1} is minimal.

Step 6: NC singularities theorem. The *domain* of a scalar or matrix-valued nc rational expression R, $\operatorname{dom} R$, consists of d-tuples Z of $s \times s$ matrices, $s = 1, 2, \ldots$, for which all the matrix inversions in R are well-defined, so that $R(Z)$ makes sense.

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Theorem

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This generalizes an earlier result [K-V, Vinnikov, 2009] for $\mathbb{B}^d$ to $\mathcal{B}$.

In other words, the singularity set of $\mathfrak{R}$ is

$$\coprod_{s=1}^{\infty} \Big\{ Z = (Z^{(1)}, \ldots, Z^{(k)}) \in (\mathbb{C}^{s\times s})^{\ell_1 \times m_1} \times \cdots \times (\mathbb{C}^{s\times s})^{\ell_k \times m_k}$$

$$\cong (\mathbb{C}^{\ell_1 \times m_1} \times \cdots \times \mathbb{C}^{\ell_k \times m_k}) \otimes \mathbb{C}^{s\times s} \colon \ \det(I - A \odot Z_n) = 0 \Big\},$$

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where $A \odot Z_n \in \mathbb{C}^{\sum_{r=1}^k m_r n_r s \ \times \sum_{r=1}^k m_r n_r s}$ is a block $\sum_{r=1}^k m_r \times \sum_{r=1}^k m_r$ matrix with blocks

$$(A \odot Z_n)_{ij}^{(rr')} = \sum_{\kappa=1}^{\ell_{r'}} A_{i\kappa}^{(rr')} \otimes Z_{\kappa j}^{(r')} \in \mathbb{C}^{n_r \times n_{r'}} \otimes \mathbb{C}^{s\times s} \cong \mathbb{C}^{n_r s \times n_{r'} s},$$

$i = 1, \ldots, m_r$, $j = 1, \ldots, m_{r'}$.

Step 7: Back to commuting variables.

Corollary

The variety of singularities of a (commutative) $\alpha \times \beta$ matrix-valued rational function f which can be represented as a restriction of R from Theorem above to scalars $z_{ij}^{(r)}$ (i.e., to the case $s = 1$) is given by

$$\left\{ Z = (Z^{(1)}, \ldots, Z^{(k)}) \in \mathbb{C}^{\ell_1 \times m_1} \times \cdots \times \mathbb{C}^{\ell_k \times m_k} \colon \det(I - AZ_n) = 0 \right\},$$

where $Z_n = \bigoplus_{r=1}^{k} (Z^{(r)} \otimes I_{n_r})$.

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$$\begin{bmatrix} I - A_{\min} Z_{n_{\min}} & B_{\min} \\ -C_{\min} Z_{n_{\min}} & D_{\min} \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ -C_{\min} Z_{n_{\min}} (I - A_{\min} Z_{n_{\min}})^{-1} & I \end{bmatrix} \begin{bmatrix} I - A_{\min} Z_{n_{\min}} & 0 \\ 0 & c/p \end{bmatrix} \begin{bmatrix} I & (I - A_{\min} Z_{n_{\min}})^{-1} B \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} I & B_{\min}^{\times} \\ 0 & I \end{bmatrix} \begin{bmatrix} I - A_{\min}^{\times} Z_{n_{\min}} & 0 \\ 0 & D_{\min} \end{bmatrix} \begin{bmatrix} I & 0 \\ C_{\min}^{\times} Z_{n_{\min}} & I \end{bmatrix}$$

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we obtain that

$$\det \begin{bmatrix} I - A_{\min} Z_{n_{\min}} & B_{\min} \\ -C_{\min} Z_{n_{\min}} & D_{\min} \end{bmatrix} = \frac{c}{p} \det(I - A_{\min} Z_{n_{\min}})$$

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Corollary

Every strongly $\mathbb{D}^d$-stable polynomial p is an eventual Agler denominator, i.e., there exists $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$, $n \geq \deg p$, such that the rational inner function

$$\frac{z^n \bar{p}(1/z)}{p(z)}$$

is in the Schur–Agler class. Here for $z = (z_1, \ldots, z_d)$ we set $1/z = (1/z_1, \ldots, 1/z_d)$, $\bar{p}(z) = \overline{p(\bar{z}_1, \ldots, \bar{z}_d)}$, and $z^n = z_1^{n_1} \cdots z_d^{n_d}$.

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Indeed, by Main Theorem applied to $\mathbb{D}^d$, p has a strictly contractive determinantal representation. By [Grinshpan, K-V, Woerdeman, 2013], p is an eventual Agler denominator. Notice that $n = \deg p$ doesn't always work [GK-VW, 2013].

THANK YOU!