Operator-valued Jacobi parameters.

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Theorem. (Darboux, Stieltjes, Chebyshev 1880s; Viennot, Flajolet 1980s)

$\mu =$ probability measure on $\mathbb{R}$;
$(\lambda_i, \alpha_i)_{i=1}^{\infty}$ two sequences of real numbers, with $\alpha_i \geq 0$.

The following are equivalent.

(1) The Cauchy transform $G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \, d\mu(x)$ has a continued fraction expansion

$$G_\mu(z) = \cfrac{1}{z - \lambda_1 - \cfrac{\alpha_1}{z - \lambda_2 - \cfrac{\alpha_2}{z - \lambda_3 - \cfrac{\alpha_3}{\ldots}}}}.$$

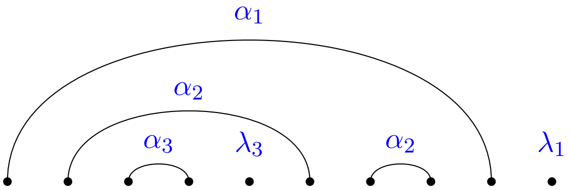
Theorem. (Darboux, Stieltjes, Chebyshev 1880s; Viennot, Flajolet 1980s)

The following are equivalent.

(2) The moments $m_n(\mu) = \int_{\mathbb{R}} x^n \, d\mu(x)$ have the expansion

$$m_n = \sum_{\pi \in NC_{1,2}(n)} \prod_{V \in \pi, |V|=1} \lambda_{d(V)} \prod_{V \in \pi, |V|=2} \alpha_{d(V)},$$

where $d(V)$ = depth of V in π.



$\leftrightarrow \ \alpha_1 \, \alpha_2^2 \, \alpha_3 \, \lambda_1 \, \lambda_3$.

Theorem. (Darboux, Stieltjes, Chebyshev 1880s; Viennot, Flajolet 1980s)

The following are equivalent.

(3) The monic orthogonal polynomials P_i with respect to μ satisfy a recursion

$$xP_i(x) = P_{i+1}(x) + \lambda_{i+1}P_i(x) + \alpha_i P_{i-1}(x).$$

In this case write

$$\mu = J\begin{pmatrix} \lambda_1, & \lambda_2, & \lambda_3, & \dots \\ \alpha_1, & \alpha_2, & \alpha_3, & \dots \end{pmatrix}$$

and call these the *Jacobi parameters of* μ.

$\mathcal{B}$-valued distributions.

Let $\mathcal{B} = C^*$-algebra.

$$\mathcal{B}\langle X\rangle = \mathrm{Span}\left(b_0 X b_1 X \ldots b_{n-1} X b_n : n \geq 0, b_i \in \mathcal{B}\right)$$

form a $*$-algebra.

Definition.

1. A $\mathcal{B}$-valued probability space is a triple $(\mathcal{A}, E, \mathcal{B})$, where $E : \mathcal{A} \to \mathcal{B}$ is a conditional expectation.
2. A $\mathcal{B}$-*valued distribution* is a completely positive (c.p.) conditional expectation $\mu : \mathcal{B}\langle X\rangle \to \mathcal{B}$. μ is determined by its moments
$$\mu[b_0 X b_1 X \ldots b_{n-1} X b_n].$$
3. μ is exponentially bounded if for some M,
$$\|\mu[b_0 X b_1 X \ldots X b_n]\| \leq M^n \|b_0\| \|b_1\| \ldots \|b_n\|.$$

$\mathcal{B}$-valued Jacobi parameters.

The mean

$$m(\mu) = \mu[X] \in \mathcal{B}^{sa}.$$

The variance

$$V(\mu) = \mu[XbX] - \mu[X]\, b\, \mu[X] : \mathcal{B} \to \mathcal{B} \quad \text{c.p.}$$

Replace $\lambda_i \in \mathbb{R}$ with

$$\lambda_i \in \mathcal{B}^{sa}.$$

Replace $\alpha_i \in \mathbb{R}$, $\alpha_i \geq 0$ with

$$\alpha_i \in \mathcal{CP}(\mathcal{B}).$$

Jacobi parameters ⇒ Distribution.

Question. Given $(\lambda_i, \alpha_i)_{i=1}^{\infty}$ such that $\lambda_i \in \mathcal{B}^{sa}$, $\alpha_i \in \mathcal{CP}(\mathcal{B})$, is there a $\mathcal{B}$-valued distribution μ with these Jacobi parameters?

More precisely, μ with

$$J(\mu) = \begin{pmatrix} \lambda_1, & \lambda_2, & \lambda_3, & \ldots \\ \alpha_1, & \alpha_2, & \alpha_3, & \ldots \end{pmatrix}$$

should satisfy

$$G_\mu(b) = \left(b - \lambda_1 - \alpha_1[G_{\mu'}(b)]\right)^{-1},$$

where

$$J(\mu') = \begin{pmatrix} \lambda_2, & \lambda_3, & \lambda_4, & \ldots \\ \alpha_2, & \alpha_3, & \alpha_4, & \ldots \end{pmatrix}$$

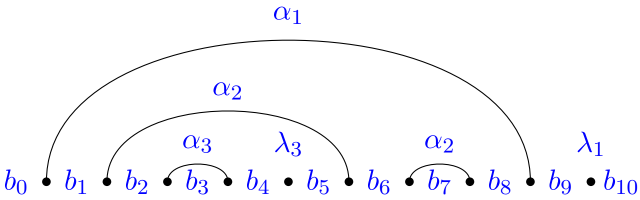
(coefficient stripping).

Jacobi parameters $\Rightarrow$ Distribution.

Answer. Yes, via a Fock space construction (cf. Speicher).

Proof goes through the equivalence between (2) and (3) and the operator-valued version of (3):

$$\mu[b_0 X b_1 X \ldots b_{n-1} X b_n] = \sum_{\pi \in NC_{1,2}(n)} (\lambda, \alpha)_\pi (b_0, b_1, \ldots, b_n).$$



$$(\lambda, \alpha)_\pi = b_0 \alpha_1 \Big[b_1 \alpha_2 \big[b_2 \alpha_3 [b_3] b_4 \lambda_3 b_5 \big] b_6 \alpha_2 [b_7] b_8 \Big] b_9 \lambda_1 b_{10}.$$

Properties of Jacobi parameters.

Let

$$\mu = J \begin{pmatrix} \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4, & \dots \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \dots \end{pmatrix}.$$

1. If all $\|\lambda_i\|_{i=1}^{\infty}$, $\|\alpha_i\|_{i=1}^{\infty}$ are uniformly bounded, then μ is an exponentially bounded non-commutative distribution.
2. Fix $d \in \mathbb{N}$. Denote

$$\widetilde{\lambda}_i = 1_d \otimes \lambda_i \in (M_d(\mathbb{C}) \otimes \mathcal{B})^{sa} \simeq (M_d(\mathcal{B}))^{sa},$$

$$\widetilde{\alpha}_i = I_d \otimes \alpha_i \in \mathcal{CP}(M_d(\mathcal{B})),$$

and

$$\widetilde{\mu} = I_d \otimes \mu : M_d(\mathcal{B})\langle X \rangle \to M_d(\mathcal{B}).$$

Then

$$\widetilde{\mu} = J \begin{pmatrix} \widetilde{\lambda}_0, & \widetilde{\lambda}_1, & \widetilde{\lambda}_2, & \widetilde{\lambda}_3, & \dots \\ \widetilde{\alpha}_1, & \widetilde{\alpha}_2, & \widetilde{\alpha}_3, & \widetilde{\alpha}_4, & \dots \end{pmatrix}.$$

Distribution $\Rightarrow$ Jacobi parameters.

Question. Given a $\mathcal{B}$-valued distribution μ, does it arise from some Jacobi parameters $\{\lambda_i \in \mathcal{B}, \alpha_i : \mathcal{B} \to \mathcal{B}\}$?

Answer. No.

Counterexample (A, Belinschi).

Let $x, y =$ be independent Gaussian variables in some $(\mathcal{A}, E)$.

Let

$$X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in M_2(\mathcal{A}).$$

Let $\mathcal{B} = M_2(\mathbb{C})$, so that X is in a $\mathcal{B}$-valued probability space

$$\left(M_2(\mathcal{A}), \begin{pmatrix} E & E \\ E & E \end{pmatrix}, M_2(\mathbb{C}) \right).$$

Define μ by

$$\mu[B_0 X B_1 X \ldots B_{n-1} X B_n] = \begin{pmatrix} E & E \\ E & E \end{pmatrix} (B_0 X B_1 X \ldots B_{n-1} X B_n).$$

Note that μ is symmetric, so its λ-Jacobi parameters are all zero.

Counterexample (continued).

Recall: if μ has Jacobi parameters $(\alpha_1, \alpha_2, \ldots)$, then

$$\mu[XBX] = \alpha_1[B].$$

In our case, for $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$,

$$\mu[XBX] = \begin{pmatrix} E & E \\ E & E \end{pmatrix} \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right) = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}.$$

$$\alpha_1[B] = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}.$$

Counterexample (continued).

Similarly,

$$\mu[XBXBXBX] = \alpha_1\left[B\,\alpha_2[B]\,B\right] + \alpha_1[B]\,B\,\alpha_1[B].$$

$$\mu[XBXBXBX] = \begin{pmatrix} 3b_{11}^3 + b_{12}b_{22}b_{21} & b_{12}b_{21}b_{12} + b_{11}b_{12}b_{22} \\ b_{22}b_{21}b_{11} + b_{21}b_{12}b_{21} & b_{21}b_{11}b_{12} + 3b_{22}^3 \end{pmatrix},$$

and so

$$\alpha_1\left[B\,\alpha_2[B]\,B\right] = \begin{pmatrix} 2b_{11}^3 + b_{12}b_{22}b_{21} & b_{12}b_{21}b_{12} \\ b_{21}b_{12}b_{21} & b_{21}b_{11}b_{12} + 2b_{22}^3 \end{pmatrix}.$$

But this is not diagonal. So no choice of α_2 can work.

Relation to non-commutative probability.

Theorem.

Let

$$X \sim J \begin{pmatrix} \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4, & \dots \\ \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \dots \end{pmatrix}$$

and

$$Y \sim J \begin{pmatrix} \tau_1, & \tau_2, & \tau_3, & \tau_4, & \dots \\ \beta_1, & \beta_2, & \beta_3, & \beta_4, & \dots \end{pmatrix}$$

be freely independent. Their joint moments are sums

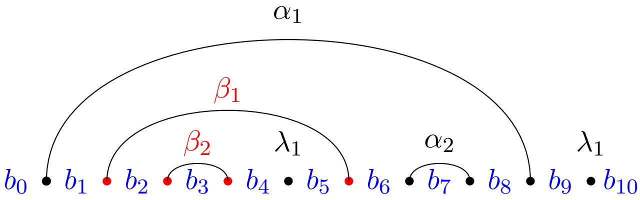
$$\sum_{\pi \in TCNC_{1,2}(n)} \begin{pmatrix} \lambda, \alpha \\ \tau, \beta \end{pmatrix}_\pi,$$

where the sum is over two-colored non-crossing partitions, colors in each π are consistent with the word in X and Y, and depth gets reset every time the color changes.

Example of a moment expansion.

$$\mu[b_0 X b_1 Y b_2 Y b_3 Y b_4 X b_5 Y b_6 X b_7 X b_8 X b_9 X b_{10}] = \sum_{\pi \in TCNC_{1,2}(10)} \binom{\lambda, \alpha}{\tau, \beta}_{\pi} (b_0, b_1, \dots, b_{10}).$$

Here one term in the sum is



Examples.

Example.

The limit in the $\mathcal{B}$-valued free Central Limit Theorem is the $\mathcal{B}$-valued semicircular distribution $\mathcal{S}(\alpha)$ with

$$J(\mathcal{S}(\alpha)) = \begin{pmatrix} 0, & 0, & 0, & \dots \\ \alpha, & \alpha, & \alpha, & \dots \end{pmatrix}.$$

Example.

The limit in the $\mathcal{B}$-valued free Poisson Limit Theorem has Jacobi parameters

$$J(\mu) = \begin{pmatrix} 0, & \lambda, & \lambda, & \lambda, & \dots \\ \alpha, & \alpha, & \alpha, & \alpha, & \dots \end{pmatrix}.$$

Examples (continued).

Example.

The limit in the $\mathcal{B}$-valued Boolean Central Limit Theorem is the $\mathcal{B}$-valued Bernoulli distribution $\mathrm{Ber}(\alpha)$ with

$$J(\mathrm{Ber}(\alpha)) = \begin{pmatrix} 0, & 0, & 0, & 0, & \ldots \\ \alpha, & 0, & 0, & 0, & \ldots \end{pmatrix}.$$

Example.

The limit in the $\mathcal{B}$-valued Boolean Poisson Limit Theorem is the $\mathcal{B}$-valued Bernoulli distribution $\mathrm{Ber}(\lambda, \alpha)$ with

$$J(\mathrm{Ber}(\lambda, \alpha)) = \begin{pmatrix} 0, & \lambda, & \lambda, & \lambda, & \ldots \\ \alpha, & 0, & 0, & 0, & \ldots \end{pmatrix}.$$

Examples (continued).

Example.

The limit in the $\mathcal{B}$-valued monotone Central Limit Theorem is the $\mathcal{B}$-valued arcsine distribution $\mathrm{Arc}(\alpha)$ with

$$J(\mathrm{Arc}(\alpha)) = \begin{pmatrix} 0, & 0, & 0, & \ldots \\ 2\alpha, & \alpha, & \alpha, & \ldots \end{pmatrix}.$$

Can use Jacobi parameters to show that

$$\mathrm{Arc}(\alpha) = \mathrm{Ber}(\alpha)^{\boxplus 2} = \mathcal{S}(\alpha)^{\uplus 2}.$$

Free Meixner distributions: examples.

All of these are particular cases of the free Meixner distributions:

Definition.

A (centered) *free Meixner distribution with parameters* $(\lambda, \alpha; \eta)$ is the distribution

$$\mathrm{fM}(\lambda, \alpha; \eta) = J \begin{pmatrix} 0, & \lambda, & \lambda, & \lambda, & \ldots \\ \eta, & \eta + \alpha, & \eta + \alpha, & \eta + \alpha, & \ldots \end{pmatrix}.$$

Quadratic equations.

Theorem.

Let μ is a free normalized Meixner distribution $\mathrm{fM}(\lambda, \alpha; I)$. Then

1. Its R-transform satisfies a quadratic relation

$$R_\mu(b) = b + b\lambda R_\mu(b) + \alpha[R_\mu(b)]R_\mu(b).$$

2. Its Boolean cumulant transform satisfies a quadratic relation

$$B_\mu(b) = b + b\lambda B_\mu(b) + (I + \alpha)[B_\mu(b)]B_\mu(b).$$

Recalling that $B_\mu(b) = b - G_\mu(b)^{-1}$, so does its Cauchy transform G_μ.

The same equations hold for the fully matrical extensions.

Free convolution: counterexample.

Proposition.

$$\mathrm{Ber}(\alpha) \boxplus \mathrm{Ber}(\eta)$$

in general does not arise from Jacobi parameters.

Proof. Can realize this inside the non-commutative probability space

$$\left(M_2(\mathbb{C}), E, \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right),$$

with $\alpha \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \right] = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ and $\eta = I$.

Free Meixner distributions: semigroups.

Theorem.

For fixed λ, α, free Meixner distributions form a free convolution semigroup with respect to parameter η: whenever $\alpha + \eta_1, \alpha + \eta_2 \in \mathcal{CP}(\mathcal{B})$,

$$\mathrm{fM}(\lambda, \alpha; \eta_1) \boxplus \mathrm{fM}(\lambda, \alpha; \eta_2) = \mathrm{fM}(\lambda, \alpha; \eta_1 + \eta_2)$$

and if $I + \alpha \in \mathcal{CP}(\mathcal{B})$, then $\mathrm{fM}(\lambda, \alpha; \eta) = \mathrm{fM}(\lambda, \alpha; I)^{\boxplus \eta}$.

Corollary.

If $G_\mu(b) = (b - \alpha[b^{-1}])^{-1}$ and $\eta \geq I$, then

$$G_{\mu^{\boxplus \eta}}(b) = (b - (\eta \circ \alpha)[G_\nu(b)])^{-1},$$

where $G_\nu(b) b = G_\nu(b)(\eta - I) \circ \alpha[G_\nu(b)] + b$.

Thank you!