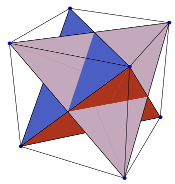


Chapter 1

Logic and Sets

1.1. Logical connectives

1.1.1. Unambiguous statements. Logic is concerned first of all with the logical structure of statements, and with the construction of complex statements from simple parts. A statement is a declarative sentence, which is supposed to be either true or false.

A statement must be made completely unambiguous in order to be judged as true or false. Often this requires that the writer of a sentence has established an adequate context which allows the reader to identify all those things referred to in the sentence. For example, if you read in a narrative: *"He is John's brother,"* you will not be able to understand this simple assertion unless the author has already identified John, and also allowed you to know who "*he*" is supposed to be. Likewise, if someone gives you directions, starting *"Turn left at the corner,"* you will be quite confused unless the speaker also tells you *what corner* and *from what direction* you are supposed to approach this corner.

The same thing happens in mathematical writing. If you run across the sentence $x^2 \geq 0$, you won't know what to make of it, unless the author has established what x is supposed to be. If the author has written, *"Let x be any real number. Then $x^2 \geq 0$,"* then you can understand the statement, and see that it is true.

A sentence containing variables, which is capable of becoming an an unambiguous statement when the variables have been adequately identified, is called a *predicate* or, perhaps less pretentiously, a *statement-with-variables*. Such a sentence is neither true nor false (nor comprehensible) until the variables have been identified.

It is the job of every writer of mathematics (you, for example!) to strive to abolish ambiguity. *The first rule of mathematical writing is this: any* *symbol you use, and any object of any sort to which you refer, must be adequately identified. Otherwise, what you write will be meaningless or incomprehensible.*

Our first task will be to examine how simple statements can be combined or modified by means of *logical connectives* to form new statements; the validity of such a composite statement depends only on the validity of its simple components.

The basic logical connectives are *and*, *or*, *not*, and *if...then*. We consider these in turn.

1.1.2. The conjunction *and*. For statements A and B, the statement "A and B" is true exactly when both A and B are true. This is conventionally illustrated by a *truth table*:

A	B	A and B
t	t	t
t	f	f
f	t	f
f	f	f

The table contains one row for each of the four possible combinations of truth values of A and B; the last entry of each row is the truth value of "A and B" corresponding to the given truth values of A and B.

For example:

- *"Julius Caesar was the first Roman emperor, and Wilhelm II was the last German emperor"* is true, because both parts are true.
- *"Julius Caesar was the first Roman emperor, and Peter the Great was the last German emperor"* is false because the second part is false.
- *"Julius Caesar was the first Roman emperor, and the Seventeenth of May is Norwegian independence day."* is true, because both parts are true, but it is a fairly ridiculous statement.
- "$2 < 3$, *and* π *is the area of a circle of radius 1*" is true because both parts are true.

1.1.3. The disjunction *or*. For statements A and B, the statement "A or B" is true when at least one of the component statements is true. Here is the truth table:

A	B	A or B
t	t	t
t	f	t
f	t	t
f	f	f

In everyday speech, “or” sometimes is taken to mean “one or the other, but not both,” but in mathematics the universal convention is that “or” means “one or the other or both.”

For example:

- *“Julius Caesar was the first Roman emperor, or Wilhelm II was the last German emperor”* is true, because both parts are true.
- *“Julius Caesar was the first Roman emperor, or Peter the Great was the last German emperor”* is true because the first part is true.
- *“Julius Caesar was the first Chinese emperor, or Peter the Great was the last German emperor.”* is false, because both parts are false.
- “$2 < 3$*, or* π *is the area of a circle of radius 2”* is true because the first part is true.

1.1.4. The negation *not*. The negation “not(A)” of a statement A is true when A is false and false when A is true.

A	not(A)
t	f
f	t

Of course, given an actual statement A, we do not generally negate it by writing “not(A).” Instead, we employ one of various means afforded by our natural language.

Examples:

- The negation of “ $2 < 3$” is “ $2 \geq 3$”.
- The negation of *“Julius Caesar was the first Roman emperor.”* is *“Julius Caesar was not the first Roman emperor.”*
- The negation of *“I am willing to compromise on this issue.”* is *I am unwilling to compromise on this issue.”*

1.1.5. Negation combined with conjunction and disjunction. At this point we might try to combine the negation “not” with the conjunction “and” or the disjunction “or.” We compute the truth table of “not(A and B),” as follows:

A	B	A and B	not(A and B)
t	t	t	f
t	f	f	t
f	t	f	t
f	f	f	t

Next, we observe that "not(A) or not(B)" has the same truth table as "not(A and B)."

A	B	not(A)	not(B)	not(A) or not(B)
t	t	f	f	f
t	f	f	t	t
f	t	t	f	t
f	f	t	t	t

We say that two *statement formulas* such as "not(A and B)" and "not(A) or not(B)" are *logically equivalent* if they have the same truth table; when we substitute actual statements for A and B in the logically equivalent statement formulas, we end up with two composite statements with exactly the same truth value; that is one is true if, and only if, the other is true.

What we have verified with truth tables also makes perfect intuitive sense: "A and B" is false precisely if not both A and B are true, that is when one or the other, or both, of A and B is false.

Exercise 1.1.1. Check similarly that "not(A or B)" is logically equivalent to "not(A) and not(B)," by writing out truth tables. Also verify that "not(not(A))" is equivalent to "A," by using truth tables.

The logical equivalence of "not(A or B)" and 'not(A) and not(B)" also makes intuitive sense. "A or B" is true when at least one of A and B is true. "A or B" is false when neither A nor B is true, that is when both are false.

Examples:

- The negation of *"Julius Caesar was the first Roman emperor, and Wilhelm II was the last German emperor"* is *"Julius Caesar was not the first Roman emperor, or Wilhelm II was not the last German emperor."* This is false.
- The negation of *"Julius Caesar was the first Roman emperor, and Peter the Great was the last German emperor"* is *"Julius Caesar was not the first Roman emperor, or Peter the Great was not the last German emperor."* This is true.
- The negation of *"Julius Caesar was the first Chinese emperor, or Peter the Great was the last German emperor" "Julius Caesar was not the first Chinese emperor, and Peter the Great was not the last German emperor."* This is true.

- The negation of "$2 < 3$, *or π is the area of a circle of radius 2*" is "$2 \geq 3$, *and π is not the area of a circle of radius 2.*" This is false, because the first part is false.

1.1.6. The implication *if...then*. Next, we consider the implication *"if A, then B"* or *"A implies B."* We *define* "if A, then B" to mean "not(A and not(B))," or, equivalently, "not(A) or B"; this is fair enough, since we want "if A, then B" to mean that one cannot have A without also having B. The negation of "A implies B" is thus "A and not(B)".

Exercise 1.1.2. Write out the truth table for "A implies B" and for its negation.

Definition 1.1.1. The *contrapositive* of the implication *"A implies B"* is *"not(B) implies not(A)."* The *converse* of the implication *"A implies B"* is *"B implies A"*.

The converse of a true implication may be either true or false. For example:

- The implication "*If* $-3 > 2$, *then* $9 > 4$" is true. The converse implication "*If* $9 > 4$, *then* $(-3) > 2$" is false.

However, the contrapositive of a true implication is always true, and the contrapositive of a false implication is always false, as is verified in Exercise 1.1.3.

Exercise 1.1.3. "A implies B" is equivalent to its *contrapositive* "not(B) implies not(A)." Write out the truth tables to verify this.

Exercise 1.1.4. Sometimes students jump to the conclusion that "A implies B" is equivalent to one or another of the following: "A and B," "B implies A,", or "not(A) implies not(B)." Check that in fact "A implies B" is not equivalent to any of these by writing out the truth tables and noticing the differences.

Exercise 1.1.5. Verify that "A implies (B implies C)" is logically equivalent to "(A and B) implies C," by use of truth tables.

Exercise 1.1.6. Verify that "A or B" is equivalent to 'if 'not(A), then B," by writing out truth tables. (Often a statement of the form "A or B" is most conveniently proved by assuming A does not hold, and proving B.)

The use of the connectives "and," and "not" in logic and mathematics coincide with their use in everyday language, and their meaning is clear. The use of "or" in mathematics differs only slightly from everyday use, in that we insist on using the *inclusive* rather than the *exclusive* or in mathematics.

The use of “if ... then” in mathematics, however, is a little mysterious. In ordinary speech, we require some genuine connection, preferably a causal connection between the “if” and the “then” in order to accept an “if ... then” statement as sensible and true. For example:

- *If you run an engine too fast, you will damage it.*
- *If it rains tomorrow, we will have to cancel the picnic.*
- $2 < 3$ *implies* $3/2 > 1$.

These are sensible uses of ‘if ... then” in ordinary language, and they involve causality: misuse of the engine will cause damage, rain will cause the cancellation of the picnic, and 2 being less than 3 is an explanation for 3/2 being greater than 1.

On the other hand, the implications:

- *If the Seventeenth of May is Norwegian independence day, then Julius Caesar was the first emperor of Rome.*
- $2 < 3$ *implies* $\pi > 3.14$.

would ordinarily be regarded as nonsense, as modern Norwegian history cannot have had any causal influence on ancient Roman history, and there is no apparent connection between the two inequalities in the second example. But according to our defined use of “if ... then,” both of these statements must be accepted as a true. Even worse:

- *If the Eighteenth of May is Norwegian independence day, then Julius Caesar was the last emperor of Germany.*
- *If* $2 > 3$ *then* $\sqrt{2}$ *is rational.*

are also true statements, according to our convention. However unfortunate these examples may seem, we find it preferable in mathematics and logic not to require any causal connection between the “if” and the “then,” but to judge the truth value of an implication “if A, then B” solely on the basis of the truth values of A and B.

1.1.7. Some logical expressions. Here are a few commonly used logical expressions:

- “A if B” means “B implies A.”
- “A only if B” means “A implies B.”
- “A if, and only if, B” means “A implies B, and B implies A.”
- “Unless” means “if not,” but “if not” is equivalent to “or.” (Check this!)
- Sometimes “but” is used instead of “and” for emphasis.

1.2. Quantified statements

1.2.1. Quantifiers. One frequently makes statements in mathematics which assert that all the elements in some set have a certain property, or that there exists at least one element in the set with a certain property. For example:

- For every real number x, one has $x^2 \geq 0$.
- For all lines L and M, if $L \neq M$ and $L \cap M$ is non-empty, then $L \cap M$ consists of exactly one point.
- There exists a positive real number whose square is 2.
- Let L be a line. Then there exist at least two points on L.

Statements containing one of the phrases "for every", "for all", "for each", etc. are said to have a *universal quantifier*. Such statements typically have the form:

- *For all* x, $P(x)$,

where $P(x)$ is some assertion about x. The first two examples above have universal quantifiers.

Statements containing one of the phrases "there exists," "there is," "one can find," etc. are said to have an *existential quantifier*. Such statements typically have the form:

- *There exists an* x *such that* $P(x)$,

where $P(x)$ is some assertion about x. The third and fourth examples above contain existential quantifiers.

One thing to watch out for in mathematical writing is the use of implicit universal quantifiers, which are usually coupled with implications. For example,

- If x is a non-zero real number, then x^2 is positive

actually means,

- For all real numbers x, if $x \neq 0$, then x^2 is positive,

or

- For all non-zero real numbers x, the quantity x^2 is positive.

1.2.2. Negation of Quantified Statements. Let us consider how to form the negation of sentences containing quantifiers. The negation of the assertion that every x has a certain property is that *some* x does not have this property; thus the negation of

- *For every* x, $P(x)$.

is

- *There exists an* x *such that not* $P(x)$.

For example the negation of the (true) statement

- For all non-zero real numbers x, the quantity x^2 is positive

is the (false) statement

- There exists a non-zero real numbers x, such that $x^2 \leq 0$.

Similarly the negation of a statement

- *There exists an x such that* $P(x)$.

is

- *For every x, not* $P(x)$.

For example, the negation of the (true) statement

- *There exists a real number x such that* $x^2 = 2$.

is the (false) statement

- *For all real numbers x,* $x^2 \neq 2$.

In order to express complex ideas, it is quite common to string together several quantifiers. For example

- *For every positive real number x, there exists a positive real number y such that* $y^2 = x$.
- *For every natural number m, there exists a natural number n such that* $n > m$.
- *For every pair of distinct points* p *and* q*, there exists exactly one line* L *such that* L *contains* p *and* q.

All of these are true statements.

There is a rather nice rule for negating such statements with chains of quantifiers: one runs through chain changing every universal quantifier to an existential quantifier, and every existential quantifier to a universal quantifier, and then one negates the assertion at the end.

For example, the negation of the (true) sentence

- *For every positive real number x, there exists a positive real number y such that* $y^2 = x$.

is the (false) statement

- *There exists a positive real number x such that for every positive real number y, one has* $y^2 \neq x$.

1.2.3. Implicit universal quantifiers. Frequently "if ... then" sentences in mathematics also involve the *universal quantifier* "for every".

- *For every real number x, if* $x \neq 0$*, then* $x^2 > 0$.

Quite often the quantifier is only implicitly present; in place of the sentence above, it is common to write

- *If x is a non-zero real number, then* $x^2 > 0$.

The negation of this is *not*

- x is a non-zero real number and $x^2 \leq 0$,

as one would expect if one ignored the (implicit) quantifier. Because of the universal quantifier, the negation is actually

- *There exists a real number x such that $x \neq 0$ and $x^2 \leq 0$.*

It might be preferable if mathematical writers made all quantifiers explicit, but they don't, so one must look out for and recognize implicit universal quantifiers in mathematical writing. Here are some more examples of statements with implicit universal quantifiers:

- *If two distinct lines intersect, their intersection contains exactly one point.*
- *If $p(x)$ is a polynomial of odd degree with real coefficients, then p has a real root.*

Something very much like the use of implicit universal quantifiers also occurs in everyday use of implications. In everyday speech, "if ... then" sentences frequently concern the uncertain future, for example:

$(*)$ *If it rains tomorrow, our picnic will be ruined.*

One notices something strange if one forms the negation of this statement. (When one is trying to understand an assertion, it is often illuminating to consider the negation.) According to our prescription for negating implications, the negation ought to be:

- *It will rain tomorrow, and our picnic will not be ruined.*

But this is surely not correct! The actual negation of the sentence $(*)$ ought to comment on the consequences of the weather without predicting the weather:

$(**)$ *It is possible that it will rain tomorrow, and our picnic will not be ruined.*

What is going on here? Any sentence about the future must at least implicitly take account of uncertainty; the purpose of the original sentence $(*)$ is to deny uncertainty, by issuing an absolute prediction:

- *Under all circumstances, if it rains tomorrow, our picnic will be ruined.*

The negation $(**)$ denies the certainty expressed by $(*)$.

1.2.4. Order of quantifiers. It is important to realize that the order of universal and existential quantifiers cannot be changed without utterly changing the meaning of the sentence. For example, if you start with the true statement:

- *For every positive real number x, there exists a positive real number y such that $y^2 = x$*

and reverse the two quantifiers, you get the totally absurd statement:

- *There exists a positive real number y such that for every positive real number x, one has $y^2 = x$.*

1.2.5. Negation of complex sentences. Here is a summary of rules for negating statements:

1. The negation of "A or B" is "not(A) and not(B)."
2. The negation of "A and B" is "not(A) or not(B)."
3. The negation of "For every x, P(x)" is "There exists x such that not(P(x))."
4. The negation of "There exists an x such that P(x)" is "For every x, not(P(x))."
5. The negation of "A implies B" is "A and not(B)."
6. Many statements with implications have implicit universal quantifiers, and one must use the rule (3) for negating such sentences.

The negation of a complex statement (one containing quantifiers or logical connectives) can be "simplified" step by step using the rules above, until it contains only negations of simple statements. For example, a statement of the form "For all x, if $P(x)$, then $Q(x)$ and $R(x)$" has a negation which simplifies as follows:

not(For all x, if $P(x)$, then $Q(x)$ and $R(x)$) $\equiv$
There exists x such that not(if $P(x)$, then $Q(x)$ and $R(x)$) $\equiv$
There exists x such that $P(x)$ and not($Q(x)$ and $R(x)$) $\equiv$
There exists x such that $P(x)$ and not($Q(x)$) or not($R(x)$) .

Let's consider a special case of a statement of this form:

- *For all real numbers x, if $x < 0$, then $x^3 < 0$ and $|x| = -x$.*

Here we have $P(x) : x < 0$, $Q(x) : x^3 < 0$ and $R(x) : |x| = -x$. Therefore the negation of the statement is:

- *There exists a real number x such that $x < 0$, and $x^3 \geq 0$ or $|x| \neq -x$.*

Here is another example

- *If L and M are distinct lines with non-empty intersection, then the intersection of L and M consists of one point.*

This sentence has an implicit universal quantifier and actually means:

- *For every pair of lines L and M, if L and M are distinct and have non-empty intersection, then the intersection of L and M consists of one point.*

Therefore the negation uses both the rule for negation of sentences with universal quantifiers, and the rule for negation of implications:

- *There exists a pair of lines L and M such that L and M are distinct and have non-empty intersection, and the intersection does not consist of one point.*

Finally, this can be rephrased as:

- *There exists a pair of lines L and M such that L and M are distinct and have at least two points in their intersection.*

Exercise 1.2.1. Form the negation of each of the following sentences. Simplify until the result contains negations only of simple sentences.

(a) Tonight I will go to a restaurant for dinner or to a movie.
(b) Tonight I will go to a restaurant for dinner and to a movie.
(c) If today is Tuesday, I have missed a deadline.
(d) For all lines L, L has at least two points.
(e) For every line L and every plane P, if L is not a subset of P, then $L \cap P$ has at most one point.

Exercise 1.2.2. Same instructions as for the previous problem Watch out for implicit universal quantifiers.

(a) If x is a real number, then $\sqrt{x^2} = |x|$.
(b) If x is a natural number and x is not a perfect square, then $\sqrt{x}$ is irrational.
(c) If n is a natural number, then there exists a natural number N such $N > n$.
(d) If L and M are distinct lines, then either L and M do not intersect, or their intersection contains exactly one point.

1.2.6. Deductions. Logic concerns not only statements but also deductions. Basically there is only one rule of deduction:

- *If A, then B. A. Therefore B.*

For quantified statements this takes the form:

- *For all x, if $A(x)$, then $B(x)$. $A(\alpha)$. Therefore $B(\alpha)$.*

Example:

- *Every subgroup of an abelian group is normal. $\mathbb{Z}$ is an abelian group, and $3\mathbb{Z}$ is a subgroup. Therefore $3\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$.*

If you don't know what this means, it doesn't matter: You don't *have* to know what it means in order to appreciate its form. Here is another example of exactly the same form:

- *Every car will eventually end up as a pile of rust. My brand new blue-green Miata is a car. Therefore it will eventually end up as a pile of rust.*

As you begin to read proofs, you should look out for the verbal variations which this one form of deduction takes, and make note of them for your own use.

Most statements requiring proof are "if ... then" statements. To prove "if A, then B," one has to assume A, and prove B under this assumption. To prove "For all x, $A(x)$ implies $B(x)$," one assumes that $A(\alpha)$ holds for a particular (but arbitrary) α, and proves $B(\alpha)$ for this particular α.

1.3. Sets

1.3.1. Sets and set operations. A *set* is a collection of (mathematical) objects. The objects contained in a set are called its *elements*. We write $x \in A$ if x is an element of the set A. Two sets are equal if they contain exactly the same elements. Very small sets can be specified by simply listing their elements, for example $A = \{1, 5, 7\}$. For sets A and B, we say that A is *contained* in B, and we write $A \subseteq B$ if each element of A is also an element of B. That is, if $x \in A$ then $x \in B$. (Because of the implicit universal quantifier, the negation of this is that there exists an element of A which is not an element of B.)

Two sets are *equal* if they contain exactly the same elements. This might seem like a quite stupid thing to mention, but in practice one often has two quite different descriptions of the same set, and one has to do a lot of work to show that the two sets contain the same elements. To do this, it is often convenient to show that each is contained in the other. That is, $A = B$ if, and only if, $A \subseteq B$ and $B \subseteq A$.

Subsets of a given set are frequently specified by a property or predicate; for example, $\{x \in \mathbb{R} : 1 \leq x \leq 4\}$ denotes the set of all real numbers between 1 and 4. Note that set containment is related to logical implication in the following fashion: If a property $P(x)$ implies a property $Q(x)$, then the set corresponding to $P(x)$ is contained in the set corresponding to $Q(x)$. For example, $x < -2$ implies that $x^2 > 4$, so $\{x \in \mathbb{R} : x < -2\} \subseteq \{x \in \mathbb{R} : x^2 > 4\}$.

The *intersection* of two sets A and B, written $A \cap B$, is the set of elements contained in both sets. $A \cap B = \{x : x \in A \quad \text{and} \quad x \in B\}$. Note the relation between intersection and the logical conjunction. If $A = \{x \in C : P(x)\}$ and $B = \{x \in C : Q(x)\}$, then $A \cap B = \{x \in C : P(x) \quad \text{and} \quad Q(x)\}$.

The *union* of two sets A and B, written $A \cup B$, is the set of elements contained in at least one of the two sets. $A \cup B = \{x : x \in A \quad \text{or} \quad x \in B\}$. Set union and the logical disjunction are related as are set intersection and logical conjunction. If $A = \{x \in C : P(x)\}$ and $B = \{x \in C : Q(x)\}$, then $A \cup B = \{x \in C : P(x) \quad \text{or} \quad Q(x)\}$.

Given finitely many sets, for example, five sets A, B, C, D, E, one similarly defines their intersection $A \cap B \cap C \cap D \cap E$ to consist of those elements which are in all of the sets, and the union $A \cup B \cup C \cup D \cup E$ to consist of those elements which are in at least one of the sets.

There is a unique set with no elements at all which is called the *empty set*, or the *null set* and usually denoted $\emptyset$.

Proposition 1.3.1. *The empty set is a subset of every set.*

Proof. Given an arbitrary set A, we have to show that $\emptyset \subseteq A$; that is, for every element $x \in \emptyset$, one has $x \in A$. The negation of this statement is that there exists an element $x \in \emptyset$ such that $x \notin A$. But this negation is false, because there are no elements at all in $\emptyset$! So the original statement is true. □

If the intersection of two sets is the empty set, we say that the sets are *disjoint*, or *non-intersecting*.

Here is a small theorem concerning the properties of set operations.

Proposition 1.3.2. *For all sets A, B, C,*

(a) $A \cup A = A$, *and* $A \cap A = A$.
(b) $A \cup B = B \cup A$, *and* $A \cap B = B \cap A$.
(c) $(A \cup B) \cup C = A \cup B \cup C = A \cup (B \cup C)$, *and* $(A \cap B) \cap C = A \cap B \cap C = A \cap (B \cap C)$.
(d) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

The proofs are just a matter of checking definitions.

Given two sets A and B, we define the *relative complement* of B in A, denoted $A \setminus B$, to be the elements of A which are not contained in B. That is, $A \setminus B = \{x \in A : x \notin B\}$.

In general, all the sets appearing in some particular mathematical discussion are subsets of some “universal” set U; for example, we might be discussing only subsets of the real numbers $\mathbb{R}$. (However, there is no universal set once and for all, for all mathematical discussions; the assumption of a “set of all sets” leads to contradictions.) It is customary and convenient to use some special notation such as $C(B)$ for the complement of B relative to U, and to refer to $C(B) = U \setminus B$ simply as *the complement of* B. (The notation $C(B)$ is not standard.)

Exercise 1.3.1. The sets $A \cap B$ and $A \setminus B$ are disjoint and have union equal to A.

Exercise 1.3.2 (de Morgan's laws)**.** For any sets A and B, one has:

$$C(A \cup B) = C(A) \cap C(B),$$

and

$$C(A \cap B) = C(A) \cup C(B).$$

Exercise 1.3.3. For any sets A and B, $A \setminus B = A \cap C(B)$.

Exercise 1.3.4. For any sets A and B,

$$(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).$$

1.3.2. Functions. We recall the notion of a *function from* A *to* B and some terminology regarding functions which is standard throughout mathematics. A function f from A to B is a rule which gives for each element of $a \in A$ an "outcome" in $f(a) \in B$. A is called the *domain* of the function, B the *co-domain*, $f(a)$ is called the *value* of the function at a, and the set of all values, $\{f(a) : a \in A\}$, is called the *range* of the function.

In general, the range is only a subset of B; a function is said to be *surjective*, or *onto*, if its range is all of B; that is, for each $b \in B$, there exists an $a \in A$, such that $f(a) = b$. Figure 1.3.1 exhibits a surjective function. Note that the statement that a function is surjective has to be expressed by a statement with a string of quantifiers.

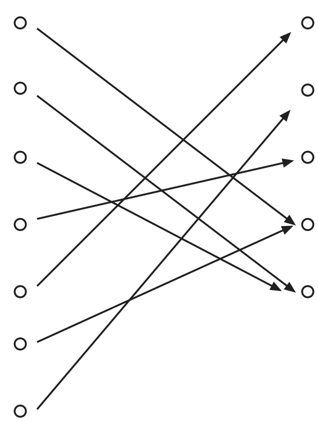


Figure 1.3.1. A Surjection

A function f is said to be *injective*, or *one-to-one*, if for each two distinct elements a and a' in A, one has $f(a) \neq f(a')$. Equivalently, for all $a, a' \in A$, if $f(a) = f(a')$ then $a = a'$. Figure 1.3.2 displays an injective and a non- injective function.

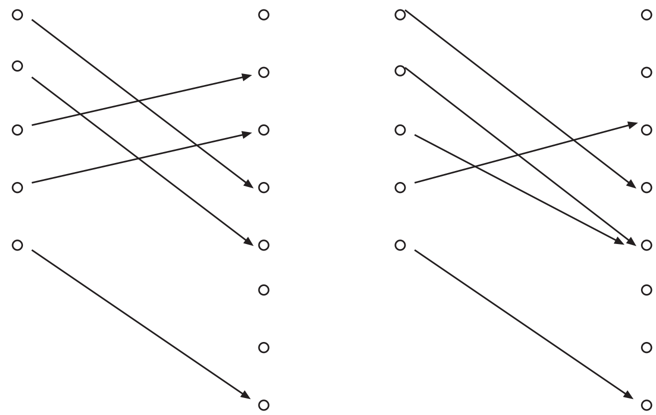


Figure 1.3.2. Injective and Non-injective functions

Finally f is said to be *bijective* if it is both injective and surjective. A bijective function (or *bijection*) is also said to be a *one-to-one correspondence* between A and B, since it matches up the elements of the two sets one-to-one. When f is bijective, there is an *inverse function* f^{-1} defined by $f^{-1}(b) = a$ if, and only if, $f(a) = b$. Figure 1.3.3 displays a bijective function.

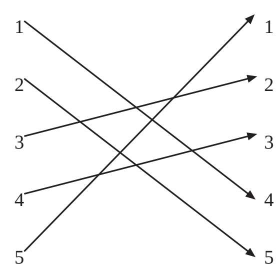
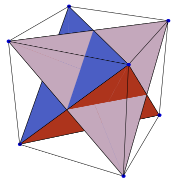


Figure 1.3.3. A Bijection

If $f : X \to Y$ is a function and A is a subset of X, we write $f(A)$ for $\{f(a) : a \in A\} = \{y \in Y : \text{ there exists } a \in A \text{ such that } y = f(a)\}$. We refer to $f(A)$ as the *image of A under f*. If B is a subset of Y, we write $f^{-1}(B)$ for $\{x \in X : f(x) \in B\}$. We refer to $f^{-1}(B)$ as the *preimage of B under f*.



Chapter 2

Elements of Geometry

2.1. First concepts

The geometry which we will study consists of a set S, called *space*. The elements of the set are called *points*. Furthermore S has certain distinguished subsets called *lines* and *planes*. A little later we will introduce other special types of subsets of S, for example, *circles*, *triangles*, *spheres*, etc.

On the one hand, we want to picture these various types of subsets according to our usual conceptions of them: Lines, planes, and so forth are idealizations of objects known from experience of the physical world. For example, a line is an idealization of a piece of string stretched tightly between two points. (But it is supposed to extend indefinitely in both directions, and, of course, we do not have any direct physical experience with anything of indefinite extent.) Similarly a plane is supposed to be a flat surface, like a table-top, but also is supposed to extend indefinitely in all directions. (Sort of like Nebraska, but larger. Again, we don't have any direct physical experience with flat surfaces of indefinite extent.) We want to use our intuition and experience with physical space to suggest the results which should hold true in our geometry, and to guide our assumptions.

On the other hand, it is a fundamental goal of a logical treatment of geometry to make all of our assumptions quite explicit. We want to try to be very careful not to use in any proof any hidden assumptions about geometric objects. Only in this way can we be sure that our arguments are correct, and that we can trust our results.

We will allow ourselves the use of the real numbers, and all of their usual properties.

Axiom I-1 Given two *distinct* points, there is exactly one line containing them.

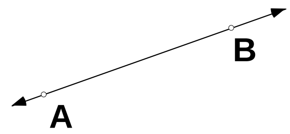


Figure 2.1.1. Axiom I-1

Remember, a line is a set of points, and containment here means containment as elements. We denote by $\overleftrightarrow{PQ}$ the line containing distinct points P and Q

We call any collection of points which lie on one line colinear and any collection of points which lie on one plane coplanar

Axiom I-2 Given three *non-colinear* points, there is exactly one plane containing them.

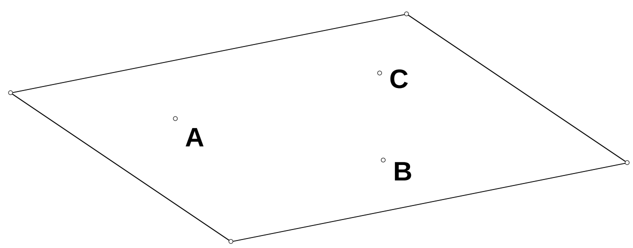


Figure 2.1.2. Axiom I-2

Axiom I-3 If two distinct points lie in a plane P, then the line containing them is a subset of P.

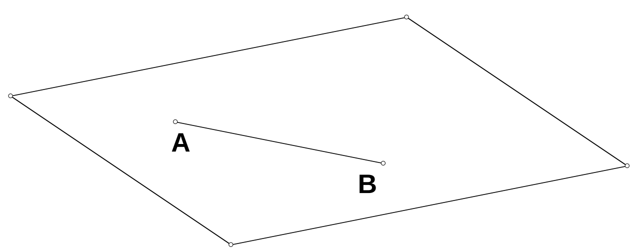


Figure 2.1.3. Axiom I-3

Axiom I-4 If two planes intersect, then their intersection is a line.

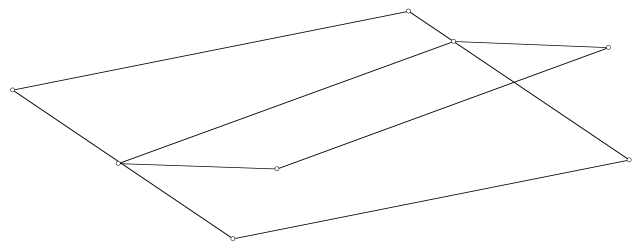


Figure 2.1.4. Axiom I-4

Theorem 2.1.1. *If two distinct lines intersect, then their intersection consists of exactly one point.*

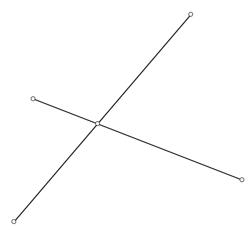


Figure 2.1.5. Intersection of Two LInes

Proof. We could rephrase the statement thus: if two lines are distinct, then their intersection does not contain two distinct points. The contrapositive is: If two lines L and M contain two distinct points in their intersection, then $L = M$. We prove this contrapositive statement.

Suppose L and M are lines (possibly the same, possibly distinct), and P and Q are two different points in their intersection. Since P, Q are elements of L, it follows from Axiom I-1 that $L = \overleftrightarrow{PQ}$. Likewise, since P, Q are elements of M, it follows from Axiom I-1 that $M = \overleftrightarrow{PQ}$. But then $M = \overleftrightarrow{PQ} = L$ ☐

Theorem 2.1.2. *If a line L intersects a plane $\mathcal{P}$ and L is not a subset of $\mathcal{P}$ then the intersection of L and $\mathcal{P}$ consists of exactly one point.*

Proof. Exercise. ☐

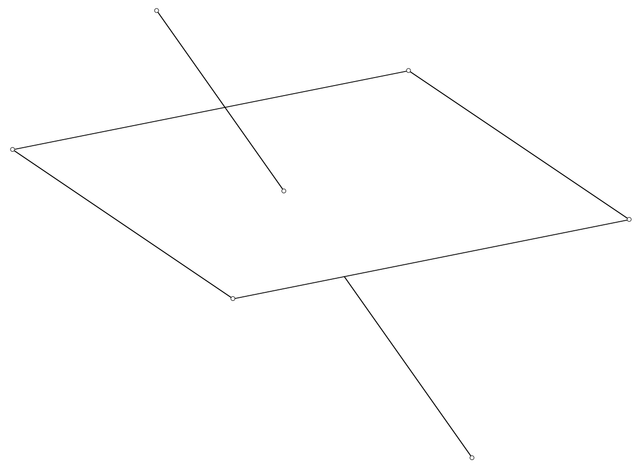


Figure 2.1.6. Intersection of a Line and a Plane

So far, all the axioms (and two theorems) would be valid for a geometry with only one point P with $\{P\}$ begin both a line and an plane! So clearly the axioms so far do not force us to be talking about the geometry which we expect to talk about! Very shortly, I will give axioms which ensure that space has lots of points, but in the meanwhile let us at least assume the following:

Axiom I-5 Every line has at least two points. Every plane has at least 3 non-colinear points. And $\mathcal{S}$ has at least 4 non-coplanar points.

Theorem 2.1.3. *If L is a line, and P is a point not in L, then there is exactly one plane $\mathcal{P}$ containing $L \cup \{P\}$.*

Proof. Exercise. □

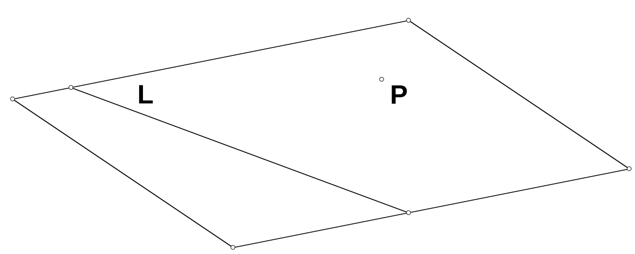


Figure 2.1.7. Plane determined by a Line and a Point

Theorem 2.1.4. *If L and M are two distinct lines which intersect, then there is exactly one plane containing $L \cup M$.*

Proof. Exercise. □

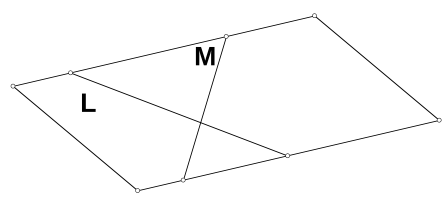


Figure 2.1.8. Plane determine by Two Lines

2.2. Distance

A familiar notion in geometry is that of distance. The distance between two points is the length of the line segment connecting them. In order to get things into logical order, we will actually introduce the notion of distance first, and use it to establish the notion of line segment!

Axiom D-1 For every pair of points A, B there is a number $d(A, B)$, called the *distance from* A *to* B. Distance satisfies the following properties:

1. $d(A, B) = d(B, A)$
2. $d(A, B) \geq 0$**, and** $d(A, B) = 0$ **if, and only if,** $A = B$**.**

Definition 2.2.1. A *coordinate function* on a line L is a bijective (one-to-one and onto) function f from L to the real numbers $\mathbb{R}$ which satisfies $|f(A) - f(B)| = d(A, B)$ for all $A, B \in L$. Given a coordinate function f, the number $f(A)$ is called the coordinate of the point $A \in L$.

Axiom D-2 Every line has at least one coordinate function.

It follows immediately that every line contains infinitely many points, because $\mathbb{R}$ is an infinite set, and a coordinate function is a one-to-one correspondence of the line with $\mathbb{R}$. Any coordinate function makes a line into a "number line" or "ruler".

Lemma 2.2.2. *If L is a line and $f : L \to \mathbb{R}$ is a coordinate function, then $g(A) = -f(A)$ is also a coordinate function.*

Proof. Exercise. □

Lemma 2.2.3. *If L is a line and $f : L \to \mathbb{R}$ is a coordinate function, then for any real number s, $h(A) = f(A) + s$ is also a coordinate function.*

Proof. Exercise. □

Lemma 2.2.4. *Let L be a line and A and B distinct points on the line L. Then there is a coordinate function f on L satisfying $f(A) = 0$ and $f(B) > 0$. Furthermore, if g is any coordinate function on L then f can be taken to have the form*

$$f(P) = \pm g(P) + s$$

for some $s \in \mathbb{R}$.

Proof. By Axiom D-2, L has a coordinate function g. Let $s = g(A)$, and define $f_1(P) = g(P) - s$. By Lemma 2.3, f_1 is also a coordinate function, and $f_1(A) = g(A) - s = 0$. Now $|f_1(B)| = |f_1(B) - f_1(A)| = d(A, B) > 0$, since $A \neq B$. If $f_1(B) > 0$, we take $f(P) = f_1(P)$. Otherwise, we take $f(P) = -f_1(P)$, which is also a coordinate function by Lemma 2.2. □

Theorem 2.2.5. *Let L be a line and A and B distinct points on the line L. There is exactly one coordinate function f on L satisfying $f(A) = 0$ and $f(B) > 0$.*

Proof. The previous lemma says that there is at least one such function. We have to show that there is only one. So let f, g be two coordinate functions on L satisfying $f(A) = g(A) = 0$ and $f(B) > 0, g(B) > 0$. We have to show that $f(C) = g(C)$ for all $C \in L$. In any case, we have $|f(C)| = |f(C) - f(A)| = d(A, B) = |g(C) - g(A)| = |g(C)|$. So in case $f(C)$ and $g(C)$ are both non-negative or both non-positive, they are equal. In particular, $f(B) = g(B) = d(A, B)$.

If $f(C), g(C)$ satisfy $f(C) \leq f(B)$ and $g(C) \leq g(B)$, then $g(B) - g(C) = d(B, C) = f(B) - f(C)$. Therefore, $f(C) - g(C) = f(B) - g(B) = 0$, or $f(C) = g(C)$.

The only remaining case to consider is that for some $C \in L$, one of $f(C), g(C)$ is negative and one is greater than $f(B) = g(B)$. Without loss of generality, assume $g(C) < 0$ and $g(B) < f(C)$. Then we have

$$\begin{aligned} d(C, B) &= g(B) - g(C) \\ &= (g(B) - g(A)) + (g(A) - g(C)) \\ &= d(A, B) + d(A, C), \end{aligned}$$

since $g(C) < g(A) < g(B)$. Using the coordinate function f instead, we have

$$\begin{aligned} d(A, C) &= f(C) - f(A) \\ &= (f(C) - f(B)) + (f(B) - f(A)) \\ &= d(B, C) + d(A, B), \end{aligned}$$

since $f(A) < f(B) < f(C)$. Adding the two displayed equations gives

$$d(C, B) + d(A, C) = d(A, B) + d(A, C) + d(B, C) + d(A, B),$$

and canceling like quantities on the two sides gives

$$0 = 2d(A, B).$$

But this is false, because $A \neq B$. This contradiction shows that the case under consideration cannot occur. So we always have $f(C) = g(C)$. □

Theorem 2.2.6. *Let f, g be two coordinate functions on a line L. Then*

$$f(P) = \pm g(P) + s,$$

for some $s \in \mathbb{R}$.

Proof. Let $A = f^{-1}(0)$, so $f(A) = 0$. Furthermore, let $B = f^{-1}(1)$, so $f(B) = 1$. According to Lemma 2.4, there is a coordinate function h of the form $h(P) = \pm g(P) + s$ which satisfies $h(A) = 0$ and $h(B) > 0$. But according to Theorem 2.5, $h = f$, so f has the desired form. □

2.3. Betweenness, segments, and rays

Definition 2.3.1. Let x, y, and z be three different real numbers. We say that y is *between* x and z if $x < y < z$ or $z < y < x$. We denote this relation by $x — y — z$

Note that $x — y — z$ is equivalent to $z — y — x$.

Lemma 2.3.2. *Let x, y, and z be three different real numbers. Let s be a real number. The following are equivalent:*
(a) $x — y — z$.
(b) $(x+s) — (y+s) — (z+s)$.
(c) $(-x) — (-y) — (-z)$.
(d) $(-x+s) — (-y+s) — (-z+s)$.

Proof. This is true because addition of a number to both sides of an inequality preserves the inequality, while multiplying both sides of an inequality by (-1) reverses the order of the inequality. ☐

Lemma 2.3.3. *Let L be a line, and let f, g be two coordinate functions on L. Let A, B, C be distinct points on L. The following are equivalent:*
(a) $f(A) — f(B) — f(C)$.
(b) $g(A) — g(B) — g(C)$.

Proof. Note that all the quantities $f(P)$, $g(P)$ for P a point in L are real numbers. So the two conditions concern betweenness for real numbers.

According to Theorem 2.2.6, there is an $\epsilon \in \{\pm 1\}$ and a real number s such that for all points P on L, $f(P) = \epsilon g(P) + s$. Then according to Lemma 2.3.2, the two conditions (a) and (b) are equivalent. ☐

Definition 2.3.4. Let L be a line, and let A, B, C be distinct points on L. We say that B is *between* A and C if for some coordinate function f on L, one has $f(A) — f(B) — f(C)$. We denote this relation by $A — B — C$.

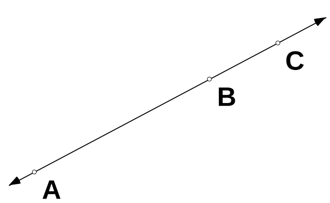


Figure 2.3.1. B is between A and C

According to Lemma 2.3.2, if $f(A) — f(B) — f(C)$ for *one* coordinate function f, then $f(A) — f(B) — f(C)$ for *all* coordinate functions f. So the concept of betweenness for points on a line does not depend on the choice of a coordinate function. By convention, when we assert that three points A, B, C satisfy A — B — C, we implicitly assert that the three points are distinct and colinear.

The next two theorems are very easy:

Theorem 2.3.5. *A — B — C if, and only if, C — B — A.*

Proof. Exercise. □

Theorem 2.3.6. *Given three distinct points on a line, exactly one of them is between the other two.*

Proof. Exercise. □

Definition 2.3.7. Let A and B be two distinct points. The *line segment* $\overline{AB}$ is the subset of the line $\overleftrightarrow{AB}$ consisting of A, B, and the set of points C which are between A and B.

$$\overline{AB} = \{C : A \text{ —— } C \text{ —— } B\} \cup \{A, B\}$$

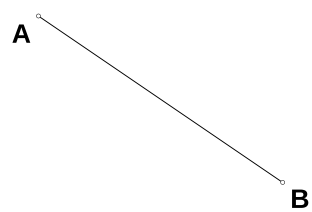


Figure 2.3.2. A Segment

Theorem 2.3.8. *Let A, B be distinct points and let f be a coordinate system on $\overleftrightarrow{AB}$ such that $f(A) < f(B)$. Then*

$$\overline{AB} = \{C \in \overleftrightarrow{AB} : f(A) \leq f(C) \leq f(B)\}.$$

Proof. Exercise. □

Theorem 2.3.9. *A line segment determines its endpoints. That is, if segments $\overline{AB}$ and $\overline{A'B'}$ are equal, then $\{A, B\} = \{A', B'\}$.*

Proof. Exercise. □

Definition 2.3.10. The *length* of a line segment $\overline{AB}$ is $d(A, B)$. The length is sometimes denoted by $\ell(\overline{AB})$. Two segments are said to be *congruent* if they have the same length. One denotes congruence of segments by $\overline{AB} \cong \overline{CD}$.

Note that the definition of the length of a segment makes sense *because of Theorem 2.3.9.*

Theorem 2.3.11. *A line segment has a unique midpoint. That is, given a segment $\overline{AB}$ there is a unique point $C \in \overline{AB}$ satisfying $d(A, C) = D(C, B) = (1/2)d(A, B)$.*

Proof. Exercise. □

Definition 2.3.12. Let A and B be two distinct points. The *ray* $\overrightarrow{AB}$ is is the subset of the line $\overleftrightarrow{AB}$ consisting of A, B, and the set of points C such that $A — C — B$ or $A — B — C$.

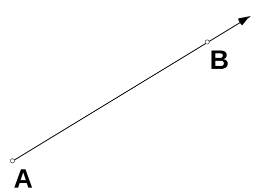


Figure 2.3.3. A Ray

Theorem 2.3.13. *Let A and B be two distinct points. The ray $\overrightarrow{AB}$ consists of those points $C \in \overleftrightarrow{AB}$ such that C* does not *satisfy $C — A — B$.*

Proof. Exercise. □

Theorem 2.3.14. *Let A and B be two distinct points. Let f be a coordinate function on $\overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B) > 0$. Then the ray $\overrightarrow{AB}$ consists of those points $C \in \overleftrightarrow{AB}$ such that $f(C) \geq 0$.*

Proof. Exercise. □

Theorem 2.3.15. *A ray determines its endpoint. That is, if rays $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are equal, then $A = A'$.*

Proof. Exercise. □

Corollary 2.3.16. *Let A and B be two distinct points, and let $r > 0$ be a positive real number. Then there is exactly one point C on the ray $\overrightarrow{AB}$ such that $d(A, C) = r$.*

Proof. Let f be a coordinate function on $\overleftrightarrow{AB}$ such that $f(A) = 0$ and $f(B) > 0$. For all points $D \in \overrightarrow{AB}$, one has $f(D) \geq 0$, by the Theorem, and therefore $d(A, D) = |f(D) - f(A)| = f(D)$. Let $r > 0$. Since f is one-to-one, there can be at most one point $C \in \overrightarrow{AB}$ such that $d(A, C) = f(C) = r$. Since f is onto, there is a point C on $\overleftrightarrow{AB}$ such that $f(C) = r$, and again by the Theorem, $C \in \overrightarrow{AB}$. □

Theorem 2.3.17. *A ray is determined by its endpoint and any other point on the ray. That is, if $C \in \overrightarrow{AB}$ and $C \neq A$, then $\overrightarrow{AC} = \overrightarrow{AB}$.*

Proof. Exercise. □

Definition 2.3.18. An *angle* is the union of two rays with the same endpoint, not contained in one line. The two rays are called the sides of the angle. The common endpoint is called the vertex of the angle. The angle $\overrightarrow{AB} \cup \overrightarrow{AC}$ is denoted $\angle BAC$ (or equally well $\angle CAB$.)

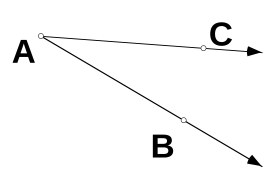


Figure 2.3.4. An Angle

Remark 2.3.19. The union of two *distinct* rays with a common endpoint, which *do* lie on one line, *is* the line. (Proof?) So we will sometimes call a line with a distinguished point on the line a *straight angle*.

Definition 2.3.20. Let A, B, C be non-colinear points. The *triangle* $\triangle ABC$ is the union of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. The three segments are called the *sides of the triangle*.

The angles $\angle ABC$, $\angle BCA$, $\angle CAB$ are called the *angles of the triangle*. One often denotes these angles by $\angle A$, $\angle B$, and $\angle C$, respectively. One says that $\angle C$ and side $\overline{AB}$ are *opposite*, and similarly for the other angles and sides.

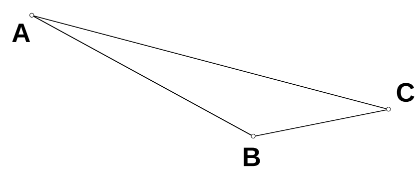


Figure 2.3.5. A Triangle

Theorem 2.3.21. *A triangle determines its vertices. That is, if* $\triangle ABC = \triangle DEF$*, then* $\{A, B, C\} = \{D, E, F\}$.

Proof. The proof of this is surprisingly tricky, and requires several steps. We will skip it, but the ambitious reader may wish to prove it. □

The next result is slightly technical. It gives a characterization of betweenness (and therefore of line segments).

Theorem 2.3.22. *Let* A, B, C *be distinct points on a line. The following are equivalent:*

(a) $A — B — C$.
(b) $d(A, C) = d(A, B) + d(B, C)$.

Proof. It is possible to choose a coordinate function f on L such that $f(A) < f(B)$, by Theorem 2.2.5.

Suppose $A — B — C$. Then $f(A) < f(B) < f(C)$, so

$$\begin{aligned} d(A,C) &= f(C) - f(A) \\ &= (f(C) - f(B)) + (f(B) - f(A)) \\ &= d(B,C) + d(A,B). \end{aligned}$$

Thus we have (a) implies (b).

Suppose now that (b) holds. According to Theorem 2.3.6, exactly one of the conditions is satisfied:

1. $B — A — C$.
2. $A — C — B$.
3. $A — B — C$.

Our strategy is to eliminate the first two possibilities, leaving only the third.

Suppose we have $B — A — C$. It follows that

$$d(B,C) = d(B,A) + d(A,C), \tag{2.3.1}$$

by the (already proved) implication (a) implies (b).

Now adding this equation and the equation in condition (b), and then canceling like terms on the two sides gives

$$0 = 2d(B,A), \tag{2.3.2}$$

so that $A = B$ by Axiom D-1. This contradicts our original assumptions, so it cannot be true that $B — A — C$.

The second possibility is eliminated in exactly the same way. This leaves only the third possibility, and proves the implication (b) implies (a). □

This theorem gives us a not so obvious characterization of line segments:

Corollary 2.3.23. *Let A and C be distinct points, and let B be a third point on* $\overleftrightarrow{AC}$*, possibly equal to one of A, C. The following are equivalent:*

(a) *B is on the line segment* $\overline{AC}$.
(b) $d(A,C) = d(A,B) + d(B,C)$.

Theorem 2.3.24. (Segment addition and subtraction) *Suppose* A, B, C *are colinear with* $A — B — C$ *and* A', B', C' *are colinear with* $A' — B' — C'$.

(a) *If* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{BC} \cong \overline{B'C'}$*, then* $\overline{AC} \cong \overline{A'C'}$.
(b) *If* $\overline{AB} \cong \overline{A'B'}$ *and* $\overline{AC} \cong \overline{A'C'}$*, then* $\overline{BC} \cong \overline{B'C'}$.

Proof. This is immediate from the definition of congruence and the implication (a) implies (b) in Theorem 2.3.22. □

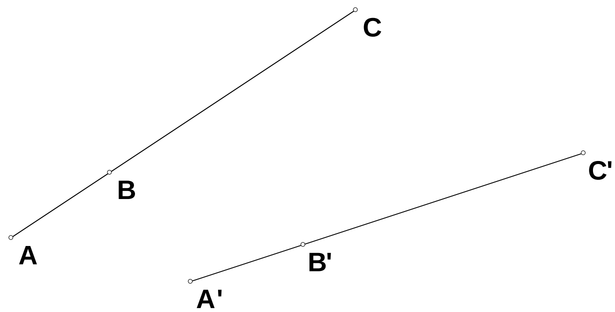


Figure 2.3.6. Segment Addition Theorem

Exercise 2.3.1. Given two distinct points A and B on a line L, prove that there is a point M on L such that $A — M — B$ and that there is a point E on L such that $A — B — E$.

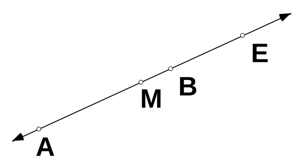


Figure 2.3.7. Exercise 2.3.1

Given 4 distinct points A, B, C, D on a line, we write $A — B — C — D$ if all the relations hold: $A — B — C$, $A — B — D$, $A — C — D$, and $B — C — D$.

Exercise 2.3.2. Prove that any four points on a line can be named in exactly one order A, B, C, D such that $A — B — C — D$.

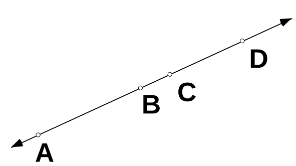


Figure 2.3.8. Exercise 2.3.2

Proof. Exercise, using a coordinate function. □

2.4. Separation of a plane by a line

According to our usual conception of lines and planes, a line L contained in a plane P divides the plane into two "halves," one one each "side" of the line. Given two points on one side of the line, it is possible to trace a curve from one point to the other which does not cross the line L. But given two points on opposite sides of the line, any curve from one to the other will cross the line. These statements do not follow from our previous axioms, so we need to assert them as a new axiom.

First we need a definition:

Definition 2.4.1. A set S is *convex* if, for each two distinct points $A, B \in S$, the line segment $\overline{AB}$ is a subset of S.

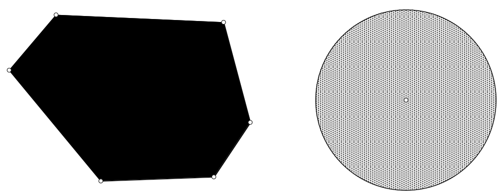


Figure 2.4.1. Convex sets

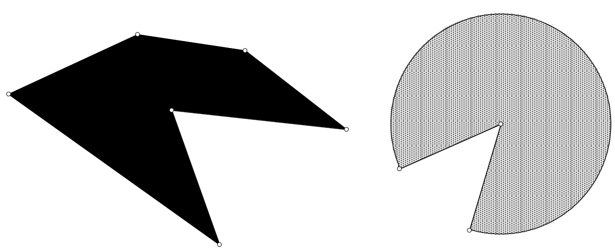


Figure 2.4.2. Non-convex sets

Exercise 2.4.1.

(a) Every line is convex.

(b) Every line segment is convex.
(c) Every ray is convex.
(d) Every plane is convex.

Axiom PS (Plane separation axiom) Let L be a line and $\mathcal{P}$ a plane containing L. Then $\mathcal{P} \setminus L$ (the set of points on $\mathcal{P}$ which are not on L) is the union of two sets H_1 and H_2 with the properties:

1. **H_1 and H_2 are non-empty and convex.**
2. **Whenever P and Q are points such that $P \in H_1$ and $Q \in H_2$, the segment $\overline{PQ}$ intersects L.**

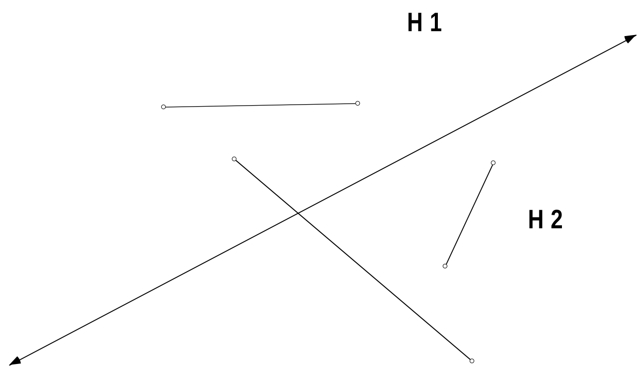


Figure 2.4.3. Plane Separation Axiom

One calls H_1 and H_2 the two *half-planes determined by* L. One says that two points both contained in one of the half-planes *are on the same side of* L, and that two points contained in different half-planes *are on opposite sides of* L. One calls L the *boundary* of each of the half-planes. The union of either of the half-planes with L is called a *closed half-plane.*

Exercise 2.4.2. Prove: Let L be a line in a plane $\mathcal{P}$, and let A, B be points of $\mathcal{P}$ which are not on L. Then L intersects the segment $\overline{AB}$ if, and only if, A and B are on opposite sides of L.

Exercise 2.4.3. Prove: Let L be a line in a plane $\mathcal{P}$, and let A, B, C be points of $\mathcal{P}$ which are not on L. If A and B are on opposite sides of L, and C and B are on opposite sides of L, then A and C are one the same side of L.

Theorem 2.4.2. *(Pasch's Axiom) Let $\triangle ABC$ be a triangle in a plane $\mathcal{P}$. Let $L \neq \overleftrightarrow{AB}$ be a line in $\mathcal{P}$ which intersects the segment $\overline{AB}$ at a point between A and B. Then L intersects one of the other two sides of the triangle.*

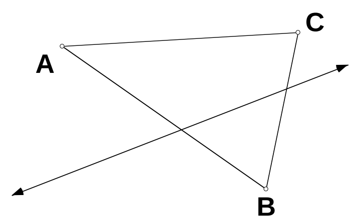


Figure 2.4.4. Pasch's Axiom

Proof. Since L intersects the segment $\overline{AB}$ at a point between A and B, A and B lie on opposite sides of L, by the previous exercise. Suppose that L does not intersect $\overline{AC}$; then A and C are on the same side of L, again, by the previous exercise. It follows that C and B are on opposite sides of L, and therefore L intersects $\overline{CB}$ by the Plane Separation Axiom. □

Remark 2.4.3. This is called Pasch's *axiom* because it was introduced by Pasch as an axiom, in place of the Plane Separation Axiom. For us, it is a theorem.

Theorem 2.4.4. *Let $\triangle ABC$ be a triangle in a plane $\mathcal{P}$. Let L be a line in $\mathcal{P}$ which does not contain any of the vertices A, B, C of the triangle. Then L does not intersect all three sides of the triangle.*

Proof. Refer to the figure for Pasch's axiom. Suppose L intersects two of the sides of the triangle, say $\overline{AB}$ and $\overline{BC}$. It has to be show that L does not intersect $\overline{AC}$. Because L intersects $\overline{AB}$, it follows that A and B are on opposite sides of L. Similarly, C and B are on opposite sides of L. Therefore, A and C are on the same side of L, so L does not intersect $\overline{AC}$. □

Theorem 2.4.5. *Let $\mathcal{P}$ be a plane, and let L be a line in $\mathcal{P}$. Let $M \neq L$ be another line in $\mathcal{P}$ which intersects L. Then M intersects both half-planes of $\mathcal{P}$ determined by L.*

Proof. Let A be the unique point of intersection of L and M (using Theorem 1.1). Let f be a coordinate function on M and let B and C be points on M such that $f(B) < f(A) < f(C)$. Then we have $B — A — C$. Suppose B and C are on the same side of L, and let H denote the half-plane which contains both of them. Since H is convex, the segment $\overline{BC}$ is a subset of H. Since $A \in \overline{BC}$, it follows that $A \in H$. But A is also in L, so $A \in H \cap L = \emptyset$. This contradiction shows that B and C are on opposite sides of L, and thus M intersects both half-planes determined by L. □

Lemma 2.4.6. *The set of points on a ray, other than the endpoint, is convex.*

Proof. Let $\overrightarrow{AB}$ be a ray, and let S denote $\overrightarrow{AB} \setminus \{A\}$. It must be shown that S is convex. Write M for $\overleftrightarrow{AB}$. Let f be a coordinate function on M such that $f(A) = 0$ and $f(B) > 0$ (Theorem 2.2.5). Then the ray $\overrightarrow{AB}$ is the set of points C on M such that $f(C) \geq 0$ (Theorem 2.3.14) and S is the set of points C on M such that $f(C) > 0$. Let C and D be two distinct points in S, and suppose without loss of generality that $0 < f(C) < f(D)$. If $C — X — D$, then $f(C) < f(X) < f(D)$. But then $f(X) > 0$, so $X \in S$. □

Theorem 2.4.7. *Let $\mathcal{P}$ be a plane, let L be a line in $\mathcal{P}$. Let H be one of the half-planes of $\mathcal{P}$ determined by L. Let A be a point on L and let B be a point in H. Then every point of the $\overrightarrow{AB}$ other than A is an element of H. That is, $\overrightarrow{AB} \setminus \{A\} \subseteq H$. Moreover, $\overleftrightarrow{AB} \cap H = \overrightarrow{AB} \setminus \{A\}$.*

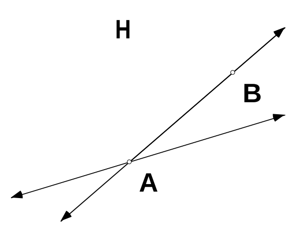


Figure 2.4.5. Theorem 2.4.7

Proof. Let S denote $\overrightarrow{AB} \setminus \{A\}$. It must be shown that $S = \overleftrightarrow{AB} \cap H$.

Write M for $\overleftrightarrow{AB}$; since $B \notin L$, we know $M \neq L$, and therefore A is the unique point on $M \cap L$. It follows that $S \cap L = \emptyset$.

Let H' denote the half-plane opposite to H. Suppose (in order to reach a contradiction) that $S \cap H'$ contains a point C. According to the previous lemma, S is convex; since both B and C are in S, one has $\overline{BC} \subseteq S$, so $\overline{BC} \cap L \subseteq S \cap L = \emptyset$. On the other hand, by the Plane Separation Axiom, $\overline{BC} \cap L \neq \emptyset$. This contradiction shows that $S \cap H' = \emptyset$. It follows that $S \subseteq H$, so $S \subseteq H \cap \overleftrightarrow{AB}$.

To finish the proof, it must be shown that $H \cap \overleftrightarrow{AB} \subseteq S$, or, equivalently, $\overleftrightarrow{AB} \setminus S \subseteq P \setminus H$. So let $X \in \overleftrightarrow{AB} \setminus S$. If $X = A$, then $X \in L \subseteq P \setminus H$. If $X \neq \overrightarrow{AB}$, then one has $X — A — B$. But then L intersects $\overline{XB}$ at A, so X and B are on opposite sides of L. Hence $X \notin H$. □

Definition 2.4.8. Consider an angle $\angle ABC$ in a plane P. The point B lies in one half-plane H of P determined by $\overleftrightarrow{AC}$. Similarly, the point C lies in one half-plane K of P determined by $\overleftrightarrow{AB}$. The intersection $H \cap K$ of these two half-planes is called the *interior of the angle.* We will call the union of the angle and its interior *the closed wedge determined by the angle*. See Figure 2.4.6.

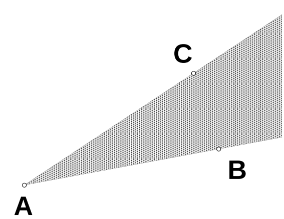


Figure 2.4.6. Angle interior

Definition 2.4.9. The *interior* of the triangle is the intersection of the interiors of the three angles of the triangle. See Figure 2.4.7

Theorem 2.4.10. *Consider an angle $\angle BAC$, and let D be a point in the interior of the angle. Then every point of the ray $\overrightarrow{AD}$, except for the endpoint A, lies in the interior of the angle. That is, $\overrightarrow{AD} \setminus \{A\}$ lies in the interior of the angle. Moreover, the intersection of the line $\overleftrightarrow{AD}$ and the interior of the angle is $\overrightarrow{AD} \setminus \{A\}$.*

Proof. This follows from two applications of Theorem 2.4.7. See Figure 2.4.8. □

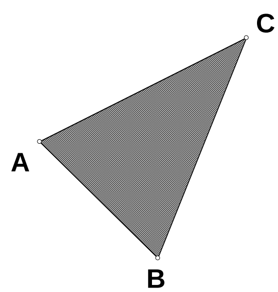


Figure 2.4.7. Triangle interior

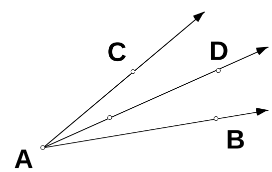


Figure 2.4.8. Theorem 2.4.10

Theorem 2.4.11. *Consider a triangle $\triangle ABC$. All the points of the segment $\overline{BC}$, except for the endpoints, lie in the interior of the angle $\angle BAC$.*

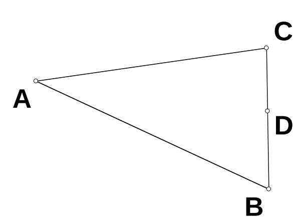


Figure 2.4.9. Theorem 2.4.11

Proof. See Figure 2.4.9. Let D be a point between B and C. Then D and C are on the same side of line $\overleftrightarrow{AB}$ because that line intersects $\overleftrightarrow{CD}$ at B, which is not between C and D. Similarly, D and B are on the same side of line $\overleftrightarrow{AC}$. But this means that D is in the interior of angle $\angle CAB$. □

Theorem 2.4.12. (Crossbar Theorem) *Let $\triangle ABC$ be a triangle, and let D be a point in the interior of the angle $\angle A$. Then the ray $\overrightarrow{AD}$ intersects the side $\overline{BC}$ of the triangle opposite to $\angle A$.*

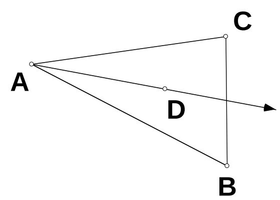


Figure 2.4.10. Crossbar Theorem

Proof. Refer to Figure 2.4.10 for the theorem statement and Figure 2.4.11 for the proof. This is pretty tricky, and the reader is invited to skip it for now, unless possessed by particular zeal.

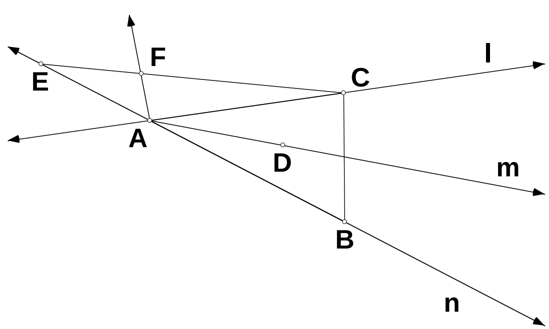


Figure 2.4.11. Crossbar Proof

Designate the lines $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{AB}$ by ℓ, m, and n respectively. Let E be a point on line n such that E — A — B (Exercise 2.3.1). Let F be a point on the segment $\overline{EC}$ such that E — F — C (Exercise 2.3.1).

We make several observations:

1. E and B are on opposite sides of ℓ because ℓ intersects $\overline{EB}$ at A.
2. E and F are on the same side of ℓ because ℓ intersects $\overleftrightarrow{EF}$ at C, which is not between E and F.
3. D and B are on the same side of ℓ because D is in the interior of the angle $\angle CAB$.
4. Therefore F and D are on opposite sides of ℓ.

5. D and C are on the same side of n since D is in the interior of the angle $\angle CAB$.
6. C and F are on the same side of n because n intersects $\overleftrightarrow{FC}$ at E, which is not between F and C.
7. Therefore F and D are on the same side of n.

Since F and D lie on opposite sides of ℓ, the segment $\overline{FD}$ intersects ℓ at some point A'. Since F and D lie on the same side of n, the point A' is not on n, and in particular $A' \neq A$. If F were on line m, then $\overleftrightarrow{FD}$ would be equal to m. But this cannot be so, because $\overleftrightarrow{FD}$ intersects ℓ at A' while m intersects ℓ at A.

Thus we conclude that m does not intersect $\overline{EC}$ at any point F between E and C. Thus E and C are on the same side of m. But E and B are on opposite sides of m because m intersects $\overline{EB}$ at A and E — A — B. Therefore C and B are on opposite sides of m, and m must intersect $\overline{CB}$ at some point X between C and B.

It remains only to show that X is on the ray $\overrightarrow{AD} \subseteq m$. But according to Theorem 2.4.11, X is in the interior of the angle $\angle BAC$, and according to Theorem 2.4.10, the intersection of the interior of the angle and the line m is contained in the ray $\overrightarrow{AD}$. Therefore X is on the ray $\overrightarrow{AD}$. □

Theorem 2.4.13. *Let $\triangle ABC$ be a triangle, and let Y be a point in the interior of angle $\angle A$. Let T be the point of intersection of $\overrightarrow{AY}$ and side $\overline{BC}$ (using the Crossbar Theorem). Let Z be a point on the segment $\overline{AT}$ with A — Z — T. Then Z is in the interior of the triangle $\triangle ABC$. See Figure 2.4.12.*

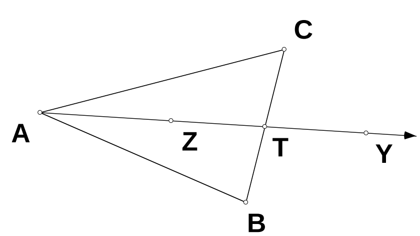


Figure 2.4.12. Theorem 2.4.13

Proof. Exercise. □

Exercise 2.4.4. Prove: The intersection of two convex sets is convex. The intersection of several convex sets is convex.

Exercise 2.4.5. Prove: The interior of an angle is convex.

Exercise 2.4.6. Prove: The interior of a triangle is convex.

In the following, L is a line in a plane $\mathcal{P}$, and H_1 and H_2 are the two half-planes of $\mathcal{P}$ determined by L.

Exercise 2.4.7. Prove: The closed half-plane $H_1 \cup L$ is convex.

Exercise 2.4.8. Prove: H_1 contains at least 3 non-colinear points.

Exercise 2.4.9. Prove: $\mathcal{P}$ is the unique plane containing H_1.

Exercise 2.4.10. Suppose that segments $\overline{AB}$ and $\overline{CD}$ intersect at a point X such that $A — X — B$, and $C — X — D$. Show that B is in the interior of angle $\angle CAD$, C is in the interior of angle $\angle ADB$, A is in the interior of the angle $\angle DBC$, and D is in the interior of angle $\angle BCA$.

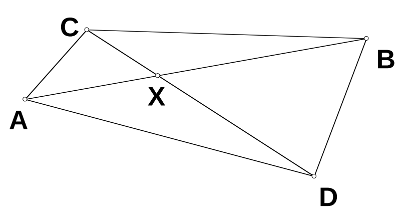


Figure 2.4.13. Exercise 2.4.10

2.5. Angular Measure

You are no doubt familiar with measuring angles using a protractor. The common unit of angle measure is the *degree*; a straight angle is divided into 180 degrees, a right angle into 90 degrees. Protractor measurement is codified in the following additional axioms for geometry:

Axiom AM-1 There is a function m from the set of all angles to the set of real numbers in the open interval $(0, 180)$.

Definition 2.5.1. The value of the function m on an angle is called the *measure* of the angle. Two angles are *congruent* if they have the same measure. Congruence of angles is denoted by

$$\angle ABC \cong \angle EFG$$

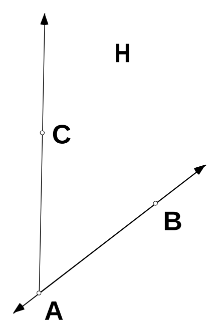


Figure 2.5.1. Angle construction

Axiom AM-2 (Angle Construction). Fix a ray $\overrightarrow{AB}$, and one half plane H determined by the line $\overleftrightarrow{AB}$. For each number $r \in (0, 180)$, there is exactly one ray $\overrightarrow{AP}$ with endpoint A and with $P \in H$, such that $m\angle PAB = r$.

Using the notion of congruence, this axiom translates to the following statement: Let $\angle XYZ$ be an angle. Fix a ray $\overrightarrow{AB}$, and one half plane H determined by the line $\overleftrightarrow{AB}$. Then there is exactly one ray $\overrightarrow{AP}$ with endpoint A and with $P \in H$, such that $\angle XYZ \cong \angle PAB$.

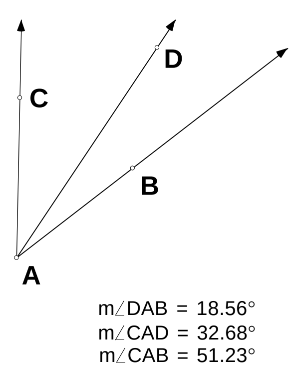


Figure 2.5.2. Angle addition

Axiom AM-3 (Angle Addition). Suppose that D is a point in the interior of angle $\angle BAC$. Then

$$m\angle BAC = m\angle BAD + m\angle DAC.$$

Using the notion of congruence, one immediately obtains the following statement:

Theorem 2.5.2. *Let D be a point in the interior of $\angle ABC$, and let D' be a point in the interior of $\angle A'B'C'$.*

(a) *If $\angle ABD \cong \angle A'B'D'$ and $\angle CBD \cong \angle C'B'D'$, then $\angle ABC \cong \angle A'B'C'$.*

(b) *If $\angle ABD \cong \angle A'B'D'$ and $\angle ABC \cong \angle A'B'C'$, then $\angle CBD \cong \angle C'B'D'$.*

Proof. Exercise. □

Definition 2.5.3. Two angles $\angle DAC$ and $\angle DAB$ form a *linear pair* in case:

1. Rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are opposite rays on a line ; i.e. C, A, B are collinear, and $C — A — B$; and
2. D is not on the line $\overleftrightarrow{AB}$.

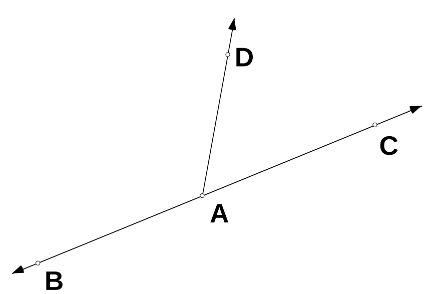


Figure 2.5.3. Linear pair

Axiom AM-4 (Linear Pair Axiom). If angles $\angle DAC$ and $\angle DAB$ form a linear pair, then the sum of their measures is 180,

$$m\angle DAC + m\angle BAD = 180.$$

Definition 2.5.4. Two angles are called *supplementary* if the sum of their measures is 180. Two angles are called *complementary* if the sum of their measures is 90.

Thus the Linear Pair Axiom says that *two angles forming a linear pair are supplementary.*

Definition 2.5.5. An angle is called a *right angle* if its measure is 90. An angle is called *acute* if its measure is less than 90, and is called *obtuse* if its measure is more than 90.

Theorem 2.5.6. *An angle $\angle BAD$ is a right angle if, and only if, there is a point C on line $\overleftrightarrow{AB}$ such that angles $\angle BAD$ and $\angle DAC$ are congruent and form a linear pair.*

Proof. Exercise. □

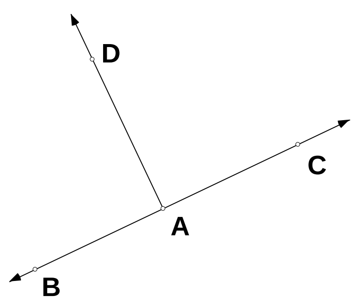


Figure 2.5.4. Right angles

Definition 2.5.7. The two rays of a right angle are said to be *perpendicular.* Two lines which intersect forming a right angle (and thus four right angles) are said to be *perpendicular.*

Line segments are said to be perpendicular if the lines containing them are perpendicular; the segments themselves are not required to intersect; they must only lie on perpendicular intersecting lines. The same term is used for a line segment and a line, a ray and a line segment, etc. which lie on perpendicular lines. The symbol $\perp$ is used to denote perpendicularity. Thus $\overleftrightarrow{AB} \perp \overline{BC}$ means that the line and the segment are perpendicular.

Theorem 2.5.8. *Let L be a line, let A be a point on L, and let $\mathcal{P}$ be a plane containing L Then there exists one and only one line M in $\mathcal{P}$ intersecting L at A such that $M \perp L$.*

Proof. Exercise. □

Theorem 2.5.9. *Let $\overline{AB}$ be a line segment, let P be the midpoint of $\overline{AB}$, and let $\mathcal{P}$ be a plane containing $\overline{AB}$ Then there exists one and only one line M in $\mathcal{P}$ intersecting $\overline{AB}$ at P such that $M \perp \overline{AB}$.*

Proof. Exercise. □

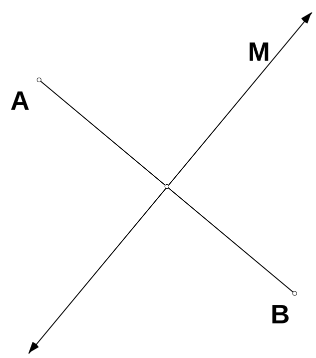


Figure 2.5.5. Perpendicular bisector

Definition 2.5.10. The line M in $\mathcal{P}$ intersecting the segment $\overline{AB}$ at its midpoint and perpendicular to $\overline{AB}$ is called the *perpendicular bisector of* $\overline{AB}$ *in* $\mathcal{P}$.

When two distinct lines intersect, they form four angles. Namely, suppose that two lines intersect at A, that B, C are points one one of the lines with $B — A — C$, and that $B'C'$ are points on the other line such that $B' — A — C'$. Then one has four angles, $\angle BAC'$, $\angle C'AC$, $\angle CAB'$, and $\angle B'AB$.

Definition 2.5.11. Two angles formed by a pair of intersecting straight lines is called a *vertical pair* if they do not share a common ray.

Note that when two lines intersect, there are two vertical pairs among the four angles which they form. In the notation used above, the pair $\angle BAC'$, $\angle B'AC$ is a vertical pair, and the pair $\angle BAB'$, $\angle CAC'$ is a vertical pair.

Theorem 2.5.12. *Two angles forming a vertical pair are congruent.*

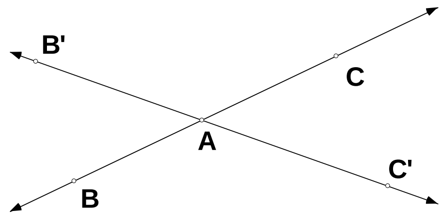


Figure 2.5.6. Vertical pair

Proof. Suppose that two distinct lines intersect at A, that B, C are points one one of the lines with B — A — C, and that $B'C'$ are points on the other line such that B' — A — C'. We have to show that $\angle BAC' \cong \angle B'AC$. Note that the pair of angles $\angle BAC'$, $\angle BAB'$ is a linear pair, so the angles are supplementary by Axiom AM-4. Likewise the pair $\angle BAB'$, $\angle B'AC$ is a linear pair, so the angles are supplementary by Axiom AM-4. Thus we have

$$m\angle BAC' + m\angle BAB' = 180, \text{ and}$$

$$m\angle B'AC + m\angle BAB' = 180,$$

so $m\angle BAC' = m\angle B'AC$, as was to be shown. ☐

Theorem 2.5.13. *If one of the angles formed by a pair of distinct intersecting lines is a right angle, then all four angles are right angles.*

Proof. Exercise. ☐

Exercise 2.5.1. Let $\angle BAC$ be an angle in a plane P and let D be a point in P on the same side of $\overleftrightarrow{AC}$ as B. If $m\angle DAC < m\angle BAC$, then D is in the interior of the angle $\angle BAC$.

2.6. Congruence of Triangles

I have previously introduced the notions of congruence of line segments and of angles: Two line segments are congruent if they have the same length (distance between endpoints) and two angles are congruent if they have the same angular measure. I will now define a notion of congruence for triangles: in brief, two triangles are congruent if they can be "matched up" so that all the "corresponding parts" are congruent. This concept requires a detailed explanation:

Consider two triangles $\triangle ABC$ and $\triangle DEF$. A *correspondence* between the two triangles is a bijection (one-to-one and onto function) between the two sets of vertices $\{A, B, C\}$ and $\{D, E, F\}$. For example, one correspondence is

$$\begin{aligned} A &\longleftrightarrow E \\ B &\longleftrightarrow D \\ C &\longleftrightarrow F. \end{aligned}$$

We abbreviate this correspondence by

$$ABC \longleftrightarrow EDF.$$

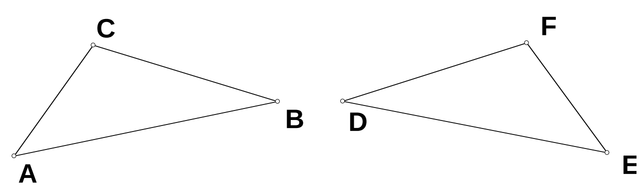


Figure 2.6.1. Congruent triangles

There are six possible correspondences between two triangles. (Exercise) A correspondence between two triangles induces bijections between the sets of sides of the two triangles and between the sets of angles of the two triangles. For example, the correspondence above induces the bijection

$$\begin{aligned} \angle A &\longleftrightarrow \angle E \\ \angle B &\longleftrightarrow \angle D \\ \angle C &\longleftrightarrow \angle F \end{aligned}$$

between the sets of angles, and the bijection

$$\begin{aligned} \overline{AB} &\longleftrightarrow \overline{ED} \\ \overline{BC} &\longleftrightarrow \overline{DF} \\ \overline{AC} &\longleftrightarrow \overline{EF} \end{aligned}$$

between the sets of sides of the two triangles.

Definition 2.6.1. A correspondence

$$ABC \longleftrightarrow EDF.$$

between two triangles $\triangle ABC$ and $\triangle EDF$ is called a *congruence* if all pairs of corresponding angles are congruent and all pairs of corresponding sides are congruent, that is

$$\angle A \cong \angle E, \quad \angle B \cong \angle D, \quad \angle C \cong \angle F,$$

and

$$\overline{AB} \cong \overline{ED}, \quad \overline{BC} \cong \overline{DF}, \quad \overline{AC} \cong \overline{EF}.$$

If the correspondence $ABC \longleftrightarrow EDF$ is a congruence, we write $\triangle ABC \cong \triangle EDF$.

In Figure 2.6.1, $\triangle ABC \cong \triangle EDF$.

When we write $\triangle ABC \cong \triangle EDF$, we mean that the particular correspondence $ABC \longleftrightarrow EDF$ is a congruence. This is a different assertion from $\triangle ABC \cong \triangle DEF$, which says that a different correspondence between the same triangles is a congruence. However, when we say in words that two triangles $\triangle ABC$ and $\triangle EDF$ are congruent, we mean only that at least one of the six possible correspondences between the two triangle is a congruence.

Exercise 2.6.1. (Transitivity of Congruence) Prove: If $\triangle ABC \cong \triangle DEF$ and $\triangle DEF \cong \triangle GHJ$, then $\triangle ABC \cong \triangle GHJ$.

The basic axiom concerning congruence of triangles is the Side-Angle-Side axiom:

Axiom SAS Consider a correspondence between two triangles. If two pairs of corresponding sides are congruent, and if the angles formed by these sides are congruent, then the correspondence is a congruence.

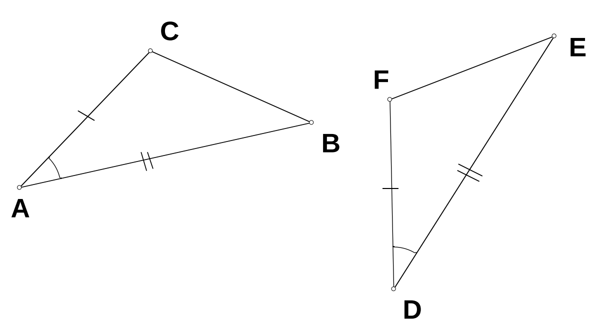


Figure 2.6.2. Side Angle Side Axiom

Let us express this with symbols: Consider a correspondence $ABC \longleftrightarrow DEF$ between triangles. Suppose that $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$. Then $\triangle ABC \cong \triangle DEF$.

In Figure 2.6.2 the corresponding markings on pairs of sides and angles indicates congruence of these parts of the triangles. Such markings are a convenient device for keeping track of congruent parts.

Definition 2.6.2. A triangle is called *isosceles* if it has two congruent sides. It is called *equilateral* if it has all three sides congruent. It is called *equiangular* if it has all three angles congruent.

Theorem 2.6.3 (Isosceles Triangle Theorem)**.** *If two sides of a triangle are congruent, then the angles opposite to these two sides are congruent.*

Let us restate the Theorem in symbols: Suppose in a triangle $\triangle ABC$ that $\overline{AB} \cong \overline{AC}$. Then $\angle B \cong \angle C$.

Proof. We consider a correspondence between $\triangle ABC$ *and itself*, namely $ABC \longleftrightarrow ACB$. Under this correspondence, $\overline{AB} \longleftrightarrow \overline{AC}$, $\overline{AC} \longleftrightarrow \overline{AB}$, and $\angle A \longleftrightarrow \angle A$. These corresponding parts are congruent, by hypothesis. Hence, by the SAS axiom, one has $\triangle ABC \cong \triangle ACB$. Since all corresponding parts of congruent triangles are congruent, we conclude $\angle B \cong \angle C$. □

Corollary 2.6.4. *An equilateral triangle is equiangular.*

Proof. Exercise. □

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Theorem 2.6.5 (Angle-Side-Angle Theorem)**.** *Consider a correspondence between two triangles. Suppose that two angles and the side connecting these two angles in one triangle are congruent to the corresponding parts of the other triangle. Then the correspondence is a congruence.*

We restate the Theorem in symbols. Consider a correspondence $ABC \longleftrightarrow DEF$ between triangles. Suppose that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\overline{AB} \cong \overline{DE}$. Then $\triangle ABC \cong \triangle DEF$. Refer to Figure 2.6.3

Proof. Refer to Figure 2.6.4. According to Corollary 2.3.16, there is a unique point G on the ray $\overrightarrow{DF}$ such that $\overline{DG} \cong \overline{AC}$.

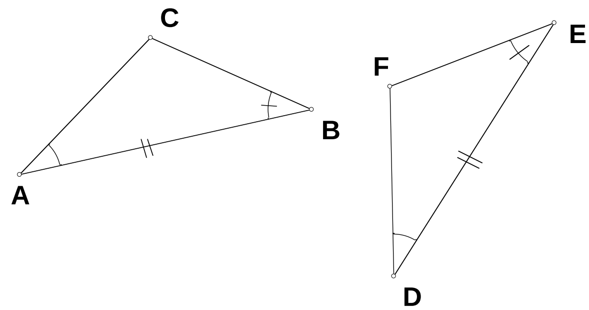

Figure 2.6.3. Angle-Side-Angle Theorem

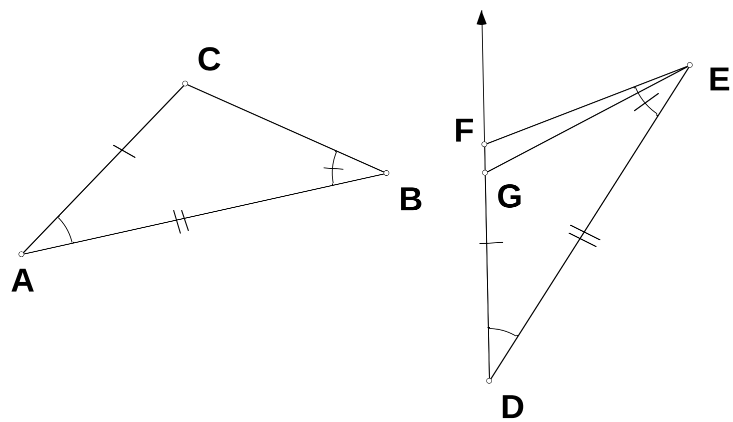

Figure 2.6.4. Proof of Angle-Side-Angle Theorem

Then we have

$$\overline{AC} \cong \overline{DG}, \quad \angle A \cong \angle D, \quad \text{and } \overline{AD} \cong \overline{DE},$$

so the SAS axiom $\triangle ABC \cong \triangle DEG$. But then

$$\angle DEG \cong \angle B \cong \angle DEF,$$

where the first congruence follows from the congruence of triangles $\triangle ABC \cong \triangle DEG$ and the second by hypothesis. Points F and G are on the same side of line $\overleftrightarrow{DE}$. (Why?) Hence by the uniqueness assertion in Axiom AM-2, the rays $\overrightarrow{EF}$ and $\overrightarrow{EG}$ coincide. Since F and G are both points of intersection of this ray with the line $\overleftrightarrow{DG}$, we have $F = G$ by Theorem 2.1.1. But then $\triangle ABC \cong \triangle DEF$, as was to be shown. □

Theorem 2.6.6. *If two angles in a triangle are congruent, then the sides opposite them are congruent.*

Proof. Exercise. □

Corollary 2.6.7. *If a triangle is equiangular, it is equilateral.*

Proof. Exercise. □

Combining Corollaries 2.6.4 and 2.6.7, we have:

Theorem 2.6.8. *A triangle is equilateral, if, and only if, it is equiangular.*

The final congruence criterion for triangles is the Side-Side-Side criterion.

Theorem 2.6.9 (Side-Side-Side Theorem)**.** *Consider a correspondence between two triangles. Suppose that all three pairs of corresponding sides are congruent. Then the correspondence is a congruence.*

We restate the theorem in symbols: Consider an correspondence of triangles $ABC \longleftrightarrow DEF$. Suppose $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\overline{BC} \cong \overline{EF}$. Then $\triangle ABC \cong \triangle DEF$. Refer to Figure 2.6.5

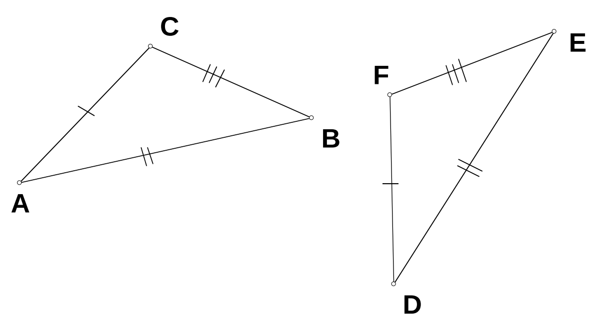

Figure 2.6.5. Side-Side-Side Theorem

Proof. This is fairly complicated, so the reader may wish to skip it on the first reading. The first step in the proof is to construct a (congruent) copy of $\triangle DEF$ sharing one side with $\triangle ABC$. By Axiom AM-2, there is a unique ray $\overrightarrow{AX}$ such that X and C are on opposite sides of $\overleftrightarrow{AB}$ and $\angle XAB \cong \angle FED$. By Corollary 2.3.16, there is a unique point C' on $\overrightarrow{AX}$ such that $\overline{AC'} \cong \overline{DF}$.

Now, by the SAS axiom we have $\triangle DFE \cong \triangle AC'B$. Refer to Figure 2.6.6. It will now suffice to prove that $\triangle ACB \cong \triangle AC'B$, since the transitivity of congruences will give the desired result.

Since C and C' are on opposite sides of $\overleftrightarrow{AB}$, the segment joining them meets $\overleftrightarrow{AB}$ at a point M. There are three possibilities to consider:

1. $A — M — B$.
2. M is one of the endpoints of $\overline{AB}$.
3. M is not on the segment $\overline{AB}$.

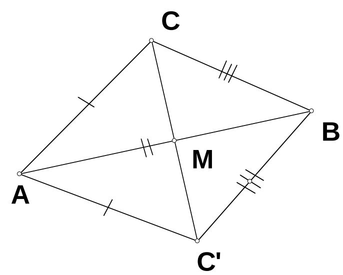

Figure 2.6.6. Proof Case 1

We start with case (1). See Figure 2.6.6. By the Isosceles Triangle Theorem 2.6.3 applied to $\triangle ACC'$m one has $\angle ACC' \cong \angle AC'C$, and by the same theorem applied to $\triangle BCC'$, one has $\angle BCC' \cong \angle BC'C$. Since $A — M — B$, it follows that M is in the interior of angles $\angle ACB$ and $\angle AC'B$, by Theorem 2.4.11. Therefore, two applications of the Angle Addition Axiom AM-3 give $m\angle ACB = m\angle ACM + m\angle MCB = m\angle ACC' + m\angle C'CB$, and likewise $m\angle AC'B = m\angle AC'M + m\angle MC'B = m\angle AC'C + m\angle CC'B$. Taking into account the angle congruences obtained above from the Isosceles Triangle Theorem, one then has $m\angle AC'B = m\angle ACB$. But then $\triangle AC'B \cong \triangle ACB$, by the SAS axiom.

Case (2) is easier. One can assume without loss of generality that $M = A$. Refer to Figure 2.6.7. Applying the Isosceles Triangle Theorem to $\triangle BCC'$ gives $\angle BCC' \cong \angle BC'C$. Now an application of the SAS axiom gives $\triangle ACB \cong \triangle AC'B$.

For case (3), refer to Figure 2.6.8. We can assume without loss of generality that $M — A — B$. Use of the Isosceles Triangle Theorem twice, as in case (1) gives the congruences $\angle ACC' \cong \angle AC'C$ and $\angle BCC' \cong \angle BC'C$. This time, however, we have A interior to angles $\angle BCX = \angle BCC'$ and $\angle BC'X = \angle BC'C$. (Why?) The Angle Addition Axiom gives, therefore, $m\angle BCA = m\angle BCC' - m\angle ACC'$, and $m\angle BC'A = m\angle BC'C - m\angle AC'C$.

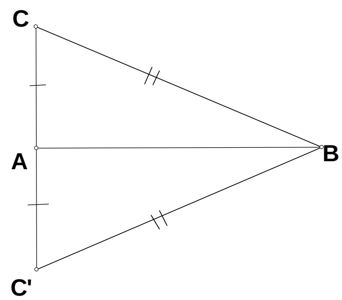

Figure 2.6.7. Proof Case 2

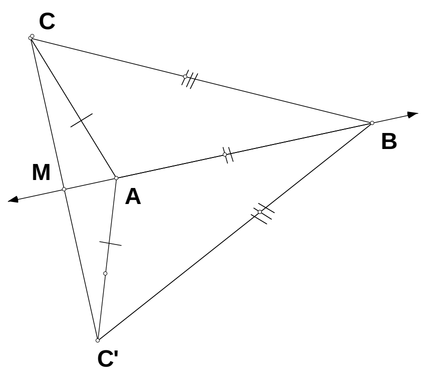

Figure 2.6.8. Proof Case 3

Using the congruences obtained from the Isosceles Triangle Theorem then gives $m\angle BCA = m\angle BC'A$. Now, as before, the SAS axiom implies that $\triangle AC'B \cong \triangle ACB$. □

Exercise 2.6.2. This is more a project than a simple exercise. Here is the outline for an alternative proof of the SSS Theorem. If we knew that $\angle A \cong \angle D$, then the desired congruence would follow from the SAS axiom.

In order to reach a contradiction, suppose that $m\angle A \neq m\angle D$. We can suppose without loss of generality that $m\angle A > m\angle D$. (Otherwise, just exchange the roles of the two triangles.)

As a first step, construct a triangle $\triangle ABY$ such that $\triangle ABY \cong \triangle DEF$, and Y is on the *same side* of $\overleftrightarrow{AB}$ as C. Show that Y is in the interior of the angle $\angle CAB$

There are three cases to consider

(a) Y is on $\overline{CB}$.

(b) Y and A are on opposite sides of $\overleftrightarrow{CB}$.
(c) Y and A are on the same side of $\overleftrightarrow{CB}$.

The first case is easily disposed of. If Y is on $\overline{CB}$, then $Y = C$, as both are on ray $\overrightarrow{BC}$ and both have the same distance to point B. (Use 2.3.16.) But then $\angle CAB = \angle YAB = \angle XAB \cong \angle FDE$, in contradiction to our assumption.

Show that the other two cases also lead to contradictions. See Figure 2.6.9 for case (3), where the segment $\overline{CY}$ is drawn in. Use the Isosceles Triangle theorem to get congruence of certain angles, and eventually derive a contradiction. Handle case (2) similarly.

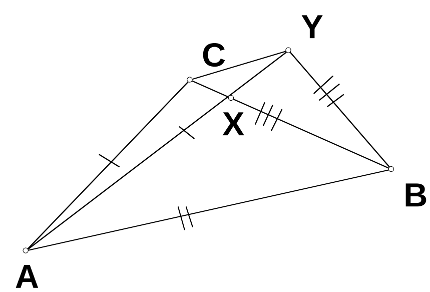

Figure 2.6.9. Alternative Proof: Case (b)

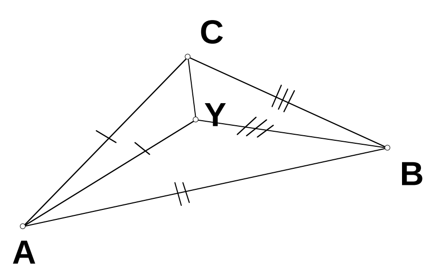

Figure 2.6.10. Alternative Proof: Case (a)

Exercise 2.6.3. Show that there are six correspondences between two triangles $\triangle ABC$ and $\triangle DEF$.

Exercise 2.6.4. Suppose that a triangle $\triangle ABC$ is isosceles, but not equilateral. How many different congruences are there *between the triangle and itself?*

Exercise 2.6.5. Suppose that a triangle $\triangle ABC$ is equilateral. How many different congruences are there *between the triangle and itself?*

Exercise 2.6.6. In Figure 2.6.11, suppose that point X is the midpoint of segments $\overline{AB}$ and $\overline{CD}$. Which triangles in the figure are congruent? Prove your assertion.

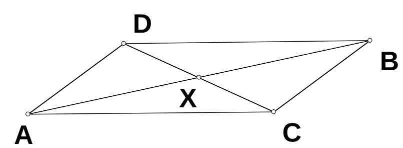

Figure 2.6.11. Exercise 2.6.6

Exercise 2.6.7. In Figure 2.6.12, suppose that point X is the midpoint of segment $\overline{AB}$, and that $\overline{CD} \perp \overline{AB}$. Which triangles in the figure are congruent? Prove your assertion.

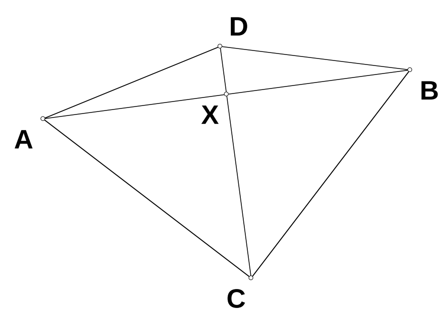

Figure 2.6.12. A Kite

Exercise 2.6.8. Refer again to Figure 2.6.12. Suppose now that point X is the midpoint of segment $\overline{AB}$, and that $\angle CAX \cong \angle CBX$. Which triangles in the figure are congruent? Prove your assertion.

Exercise 2.6.9. Refer again to Figure 2.6.12. Suppose now that $\angle ACX \cong \angle BCX$ and $\overline{AC} \cong \overline{BC}$. Which triangles in the figure are congruent? Prove your assertion.

2.7. Some geometric constructions

In this section, we will discuss some geometric constructions (constructions with straightedge and compass). I am putting these constructions here in the text because they make nice illustrations of the use of congruent triangles. The constructions depend, however, on two "self-evident" facts, which we will be able to prove only later.

First we recall the definition of a circle:

Definition 2.7.1. Let P be a plane, $A \in P$ a point, and $r > 0$ a positive number. The *circle with center A and radius r in the plane P* is the set of all points $X \in P$ satisfying $d(A, X) = r$.

FACT A: Let L be a line and A a point not contained on the line L let P be the plane containing L and A. If $r > 0$ is sufficiently large, then the circle of radius r about A in P intersects L in exactly two points.

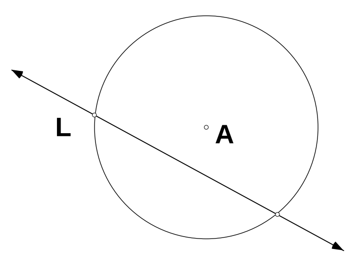

Figure 2.7.1. Circle and Line

FACT B: Let A and B be two points in the plane P with $d(A, B) = c$. Let $r > 0$ and let S be the circle in P centered at A with radius r. Let R be any positive number between $|c - r|$ and $c + r$, and let T be the circle in P of radius R centered at B. The S and T have exactly two point of intersection, and furthermore, the two points of intersection are on opposite sides of the line $\overleftrightarrow{AB}$.

The proofs of these two facts will depend on the Pythagorean theorem, and will be given in in Section xxxx. So logically, this section belongs after Section xxxx. **The material in this section may not be used, except within this section, until the proofs of FACTS A and B are obtained.**

Since the two FACTS depend on the Pythagorean theorem, it won't do any harm to allow the use of this theorem in this section as well.

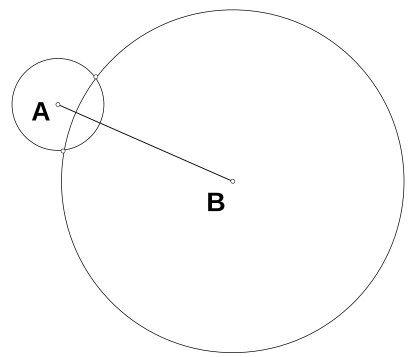

Figure 2.7.2. Two Circles

Theorem 2.7.2 (Pythagorean theorem)**.** *Let $\triangle ABC$ be a right triangle, with right angle at A. Let x, y, z denote the lengths of the sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. Then $z^2 = x^2 + y^2$.*

Construction 2.7.3 (Midpoint of a line segment)**.** *Consider a line segment $\overline{AB}$ in a plane $\mathcal{P}$. We will construct the midpoint of the line segment.*

Let c denote $d(A, B)$. Draw the circles S and T in $\mathcal{P}$ of radius c centered at A and B respectively. Let X and Y be the two points of intersection of the two circles (which exist according to FACT B). Since X and Y are on opposite sides of $\overleftrightarrow{AB}$, the segment $\overline{XY}$ intersects line AB at a point M. It is asserted that M is the midpoint of $\overline{AB}$.

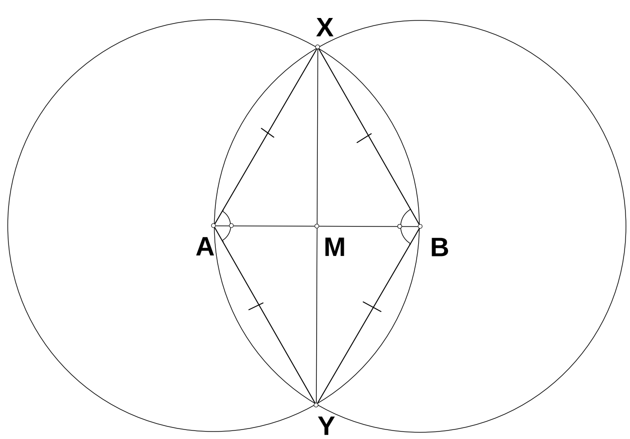

Figure 2.7.3. Midpoint Construction

Proof. Refer to Figure 2.7.3. First of all, one can show, using the Pythagorean theorem that M is between A and B. (One has to consider two cases: M is one of the endpoints of $\overline{AB}$, and M is not contained in the segment $\overline{AB}$, and show that each of these cases is impossible. This is left as an exercise.)

By construction all of the segments $\overline{AX}$, $\overline{AY}$, $\overline{AB}$, $\overline{BX}$, $\overline{BY}$ are congruent, with length c. By the Isosceles Triangle Theorem 2.6.3 applied to triangles $\triangle AXB$ and $\triangle AYB$ we have the following congruences of angles:

$$\angle XAB \cong \angle XBA, \quad \angle YAB \cong \angle YBA, \tag{2.7.1}$$

Because the diagonals of the quadrilateral $AYBX$ cross, each vertex of the quadrilateral is in the interior of the opposite angle (Exercise **??**). Therefore the Angle Addition Axiom AM-3 tells us that

$$\begin{aligned} m\angle XAB + m\angle YAB &= m\angle XAY, \\ m\angle XBA + m\angle YBA &= m\angle XBY. \end{aligned} \tag{2.7.2}$$

Combining this with Equation 2.7.1 gives

$$\angle XAY \cong \angle XBY. \tag{2.7.3}$$

Now the SAS axiom implies that $\triangle XAY \cong \triangle XBY$, and therefore $\angle AXM \cong \angle BXM$. Since $\overline{AX} \cong \overline{BX}$ and $\overline{MX} \cong \overline{MX}$, another use of the SAS axiom tells us that $\triangle XAM \cong \triangle XBM$. Consequently, $\overline{AM} \cong \overline{MB}$, as was to be shown. □

This finishes the proof of the construction. Note that we also may conclude that $\angle AMX \cong \angle BMX$. But these two angles form a linear pair, so it follows that they are right angles. *Thus we have also constructed a perpendicular line segment meeting $\overline{AB}$ at the midpoint.*

Construction 2.7.4 (Perpendicular to a line, at a point on the line)**.** *Let L be a line and P a point on the line, and let $\mathcal{P}$ be a plane containing L. We will construct a line segment in $\mathcal{P}$ perpendicular to L and meeting L at P.*

Draw a circle in $\mathcal{P}$ of arbitrary positive radius r; this circle will meet L at two points A and B such that $A — P — B$. Now, of course, P is the midpoint of the segment $\overline{AB}$.

Do the midpoint construction (Construction 2.7.3) on segment $\overline{AB}$, obtaining a segment $\overline{XY}$ meeting $\overline{AB}$ at P.

Then $\overline{XY} \perp L$.

Proof. Let f be a coordinate function on L satisfying $f(P) = 0$. Take A to be the point with $f(A) = r$ and B the point with $f(B) = -r$. Then $A — P — B$, and $d(P, A) = d(P, B) = r$, so A and B are on the circle.

Now the remarks following the proof of the midpoint construction show that the segment $\overline{XY}$ obtained by the midpoint construction is perpendicular to the line L at P. ☐

Construction 2.7.5 (Perpendicular segment from a point to a line)**.** *Refer to Figure 2.7.4. Let L be a line and P a point not on the line. We will construct a perpendicular line segment from P to a point on L.*

Let $\mathcal{P}$ denote the plane containing L and P. Draw a sufficiently large circle in the plane $\mathcal{P}$ centered at P, obtaining two points of intersection A and B of the circle with the line L. (FACT A.)

Now construct the midpoint M of the segment $\overline{AB}$ on L.

It is asserted that $\overline{PM}$ is perpendicular to L.

Proof. Exercise. ☐

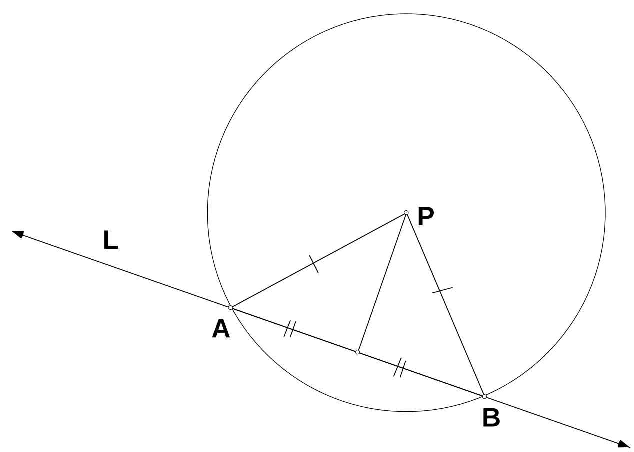

Figure 2.7.4. Perpendicular Construction

Exercise 2.7.1. Prove that $\overline{PM}$ is perpendicular to L.

Construction 2.7.6 (Bisector of an angle)**.** *Refer to Figure 2.7.5. Let $\angle BAC$ be an angle in a plane $\mathcal{P}$. Draw a circle of arbitrary positive radius r in the plane $\mathcal{P}$ centered at A. The circle intersects rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at points X and Y respectively.*

Now construct the midpoint M of the segment $\overline{XY}$ and draw the ray $\overrightarrow{AM}$. It is asserted that $\overrightarrow{AM}$ bisects the angle $\angle BAC$.

In more detail, the assertion is that the ray $\overrightarrow{AM}$ lies in the interior of the angle $\angle BAC$, and $\angle BAM \cong \angle CAM$.

Proof. Exercise. □

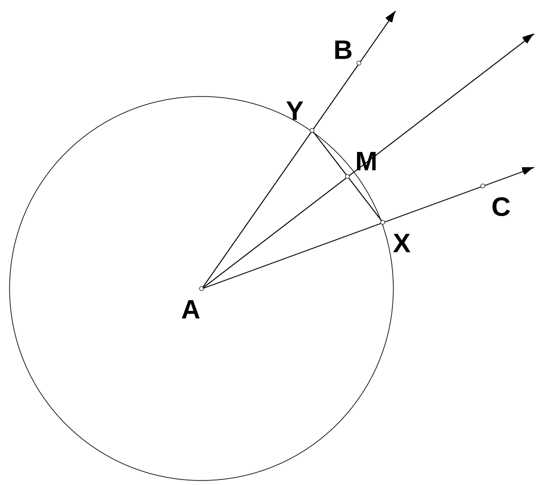

Figure 2.7.5. Angle Bisector

Exercise 2.7.2. Prove that $\overrightarrow{AM}$ bisects the angle $\angle BAC$, and that it is the only ray bisecting the angle.

2.8. Bisectors and Perpendiculars

We are **NOT** going to use the material from the previous section in this section, because that material depended on as yet unproved statements.

Recall that we already have the following theorems (whose proofs were left as exercises.)

Theorem 2.8.1. *Every line segment has a unique midpoint.*

Theorem 2.8.2. *Let L be a line, let A be a point on L, and let $\mathcal{P}$ be a plane containing L Then there exists one and only one line M in $\mathcal{P}$ intersecting L at A such that $M \perp L$.*

Theorem 2.8.3. *Let $\overline{AB}$ be a line segment, let P be the midpoint of $\overline{AB}$, and let $\mathcal{P}$ be a plane containing $\overline{AB}$. Then there exists one and only one line M in $\mathcal{P}$ intersecting $\overline{AB}$ at P such that $M \perp \overline{AB}$.*

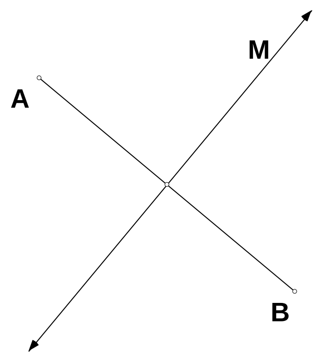

Figure 2.8.1. Perpendicular bisector

Now that we have a theory of congruent triangles, we can characterize the perpendicular bisector of a line segment as follows:

Theorem 2.8.4 (Perpendicular Bisector Theorem)**.** *Let $\overline{AB}$ be a line segment, let P be the midpoint of $\overline{AB}$, and let $\mathcal{P}$ be a plane containing $\overline{AB}$. Let M denote the perpendicular bisector of $\overline{AB}$ in $\mathcal{P}$. The following are equivalent for a point $X \in \mathcal{P}$:*

(a) *X is on the perpendicular bisector M.*
(b) *X is equidistant from A and B; that is, $d(X, A) = d(X, B)$.*

See Figure 2.8.2.

Proof. Let X be a point on the perpendicular bisector M. Consider the triangles $\triangle APX$ and $\triangle BPX$. One has $\overline{AP} \cong \overline{BP}$, since P is the midpoint of $\overline{AB}$, $\angle APX \cong \angle BPX$, since both are right angles, and $\overline{PX} \cong \overline{PX}$. Therefore, by the SAS congruence axiom, $\triangle APX \cong \triangle BPX$, and in particular $\overline{AX} \cong \overline{BX}$. So X is equidistant from A and B.

Conversely, let X be a point in the plane $\mathcal{P}$ which is equidistant from A and B. Again consider the triangles $\triangle APX$ and $\triangle BPX$. One has $\overline{AP} \cong \overline{BP}$, since P is the midpoint of $\overline{AB}$, $\overline{AX} \cong \overline{BX}$, by hypothesis, and $\overline{PX} \cong \overline{PX}$. Therefore, by the SSS congruence theorem, $\triangle APX \cong \triangle BPX$, and in particular $\angle APX \cong \angle BPX$. Since these two angles form a linear pair, both are right angles. But that means that X lies on the perpendicular bisector of $\overline{AB}$. □

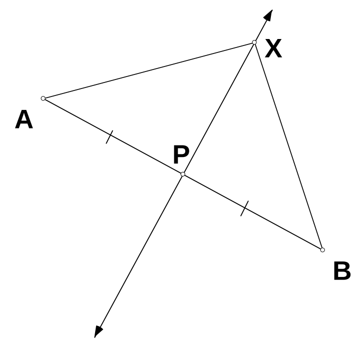

Figure 2.8.2. Perpendicular Bisector Theorem

Continuing with this theme of bisectors and perpendiculars, we now show:

Theorem 2.8.5. *Every angle has a unique bisector. That is, given an angle $\angle BAC$ in a plane $\mathcal{P}$, there is a unique ray $\overrightarrow{AD}$ in $\mathcal{P}$ such that $\overrightarrow{AD}$ is contained in the interior of angle $\angle BAC$ and $\angle DAB \cong \angle DAC$.*

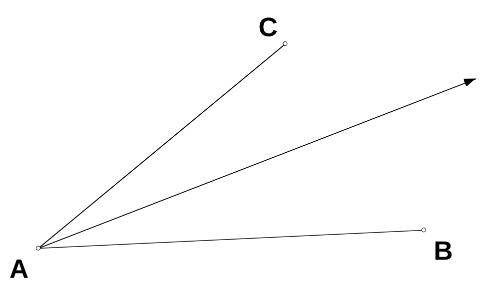

Figure 2.8.3. Angle bisector

Proof. The proof follows the line of the construction given in the previous section (but, of course, we do not use the construction of the midpoint of a segment given there.) Refer to Figure 2.7.5.

Let r be any positive number, and let X and Y be points on $\overrightarrow{AC}$ and $\overrightarrow{AB}$ respectively, such that $d(A,X) = d(A,Y) = r$ (Corollary 2.3.16). Consider the segment $\overline{XY}$ (which lies in $\mathcal{P}$ by Axiom I-3. Let M be the midpoint of this segment. I claim that $\overrightarrow{AM}$ is *a* bisector of angle $\angle BAC$.

First, M is in the interior of $\angle BAC$ by Theorem 2.4.11, and $\overrightarrow{AM}$ is in the interior of $\angle BAC$ by Theorem 2.4.10.

By the SSS Congruence Theorem, $\triangle AMX \cong \triangle AMY$. (Why are all the corresponding sides congruent?) It follows that $\angle XAM \cong \angle YAM$, as was to be shown.

Uniqueness follows from the uniqueness statement in Axiom AM-2, or from the following exercise. □

Theorem 2.8.6 (Angle Bisector Theorem)**.** *Let $\angle BAC$ be an angle in a plane $\mathcal{P}$. and let M be a point in the interior of the angle. Let X and Y be points on the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively such that $\overline{AX} \cong \overline{AY}$. Prove that the following are equivalent:*

(a) *M is on the bisector of the angle $\angle BAC$.*
(b) *M is equidistant from X and Y. See Figure 2.8.4.*

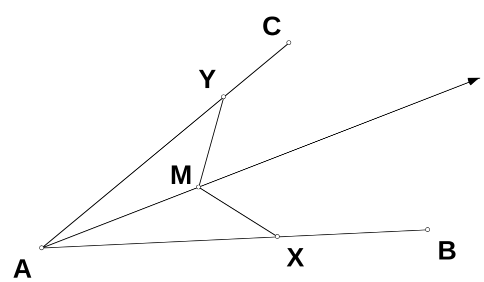

Figure 2.8.4. Angle Bisector Theorem

Proof. Exercise. □

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Exercise 2.8.1. Refer to Figure 2.7.5. In that figure, assume that $\overline{AX} \cong \overline{AY}$, that ray $\overrightarrow{AM}$ lies in the interior of $\angle YAX$, that M is the intersection of $\overrightarrow{AM}$ with segment $\overline{XY}$, and that $\angle XAM \cong \angle YAM$. Prove that M is the midpoint of $\overline{XY}$.

Using the theory of congruent triangles, we can also now prove the existence of a perpendicular line from a given point to a given line not containing the point:

Theorem 2.8.7 (Existence of Perpendiculars). *Let L be line and let P be a point not on the line L. Then there is a line containing P a perpendicular to L.*

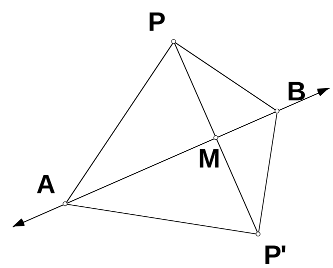

Figure 2.8.5. Existence of Perpendiculars

Remark 2.8.8. One would certainly expect that the perpendicular is unique, and this is so, but we have to wait a little to prove this fact.

Proof. See Figure 2.8.5. Choose any two points A and B on the line L, and consider the triangle $\triangle PAB$. First we construct another point P' on the opposite side of L from P such that $\triangle PAB \cong \triangle P'AB$. (Compare the proof of the SSS congruence theorem.) This is done as follows:

By Axiom AM-2, there is a unique ray $\overrightarrow{AX}$ such that X and P are on opposite sides of L and $\angle XAB \cong \angle PAB$. By Corollary 2.3.16, there is a unique point P' on $\overrightarrow{AX}$ such that $\overline{AP'} \cong \overline{AP}$. Now, by the SAS axiom we have $\triangle PAB \cong \triangle P'AB$.

The segment PP' intersects line L at some point M since P and P' are on opposite sides of the line.

If M happens (by extraordinary luck) to coincide with A, then we already have $\triangle PMB \cong \triangle P'MB$, and in particular $\angle PMB \cong \angle P'MB$. But these two angles form a linear pair, and therefore are right angles. Hence $\overleftrightarrow{PP'}$ is perpendicular to L.

If M is not equal to A, we consider the triangles $\triangle PAM$ and $\triangle P'AM$, and show that they are congruent. We have $\overline{PA} \cong \overline{P'A}$ and $\angle PAB \cong \angle P'AB$ by the congruence $\triangle PAB \cong \triangle P'AB$. Since also $\overline{AM}$ is congruent to itself, the SAS congruence axiom gives $\triangle PAM \cong \triangle P'AM$. But then in particular, $\angle AMP \cong \angle AMP'$. As these two angles form a linear pair, both are right angles. Hence $\overleftrightarrow{PP'}$ is perpendicular to L. $\square$

2.9. Exterior Angles, Transversals, and Parallels

Consider a triangle $\triangle ABC$ and extend the side $\overline{BA}$ beyond vertex A, as shown in Figure 2.9.1. The angle $\angle CAX$ is called an *exterior angle* of triangle $\triangle ABC$ at vertex A.

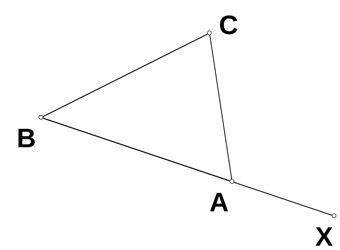

Figure 2.9.1. Exterior Angle

A second exterior angle at A is formed by extending the side $\overline{CA}$ beyond A. The two exterior angles at A form a vertical pair, so they have the same angular measure. Furthermore, either exterior angle forms a linear pair with angle $\angle A$, so $\angle A$ and an exterior angle at A are supplementary.

The following inequality is crucial to the theory of parallel lines:

Theorem 2.9.1. *Let $\triangle ABC$ be a triangle, and let α be the measure of the exterior angle at vertex A. Then $\alpha > m(\angle B)$ and $\alpha > m(\angle C)$.*

Proof. The proof is slightly tricky, so we will skip it for now. □

Corollary 2.9.2. *Two angles of a triangle cannot be supplementary. In particular, a triangle cannot have two right angles.*

Proof. Suppose triangle $\triangle ABC$ has supplementary angles at vertices A and B, $m(\angle A) + m(\angle B) = 180$. Since angle $\angle A$ is also supplementary to an exterior angle at vertex A, it follows that the measure of this exterior angle is equal to $m(\angle B)$. But this contradicts the previous theorem. □

Corollary 2.9.3. *(Uniqueness of perpendiculars) Let L be a line, and let P be a point not on the line. Then there is exactly one line M which contains P and which is perpendicular to L.*

Proof. Existence of such line was shown in Theorem 2.8.7. Now suppose there were two such lines, intersecting L at points Q and R. Then triangle $\triangle PQR$ has two right angles, at vertices Q and R, contradicting the previous corollary. ☐

Now consider two (distinct) lines L and L' in a plane $\mathcal{P}$. Let T be a third line in $\mathcal{P}$ which intersects L at A and L' at A'. Line T is said to be a *transversal* to lines L and L'.

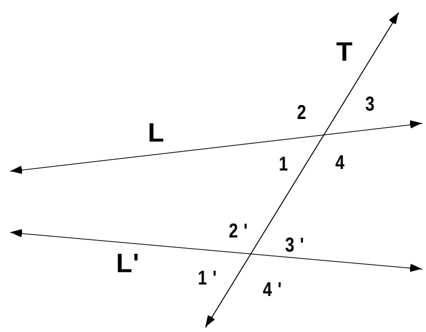

Figure 2.9.2. Transversal

T forms four angles with L at A, and four angles with L' at A'. Refer to Figure 2.9.2, where we have labeled the four angles at A as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, and the four angles at A' as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$. One says that the pairs $(\angle 1, \angle 1')$, $(\angle 2, 2')$, etc. are *corresponding angles*.

One says that the pairs $(\angle 2', \angle 4)$ and $(\angle 3', \angle 1)$ are pairs of *alternate interior angles*.

Exercise 2.9.1. Give a definition of corresponding angles which does not depend on pointing at a diagram.

Exercise 2.9.2. Give a definition of alternating interior angles which does not depend on pointing at a diagram.

Exercise 2.9.3. Let L and L' be two (distinct) lines in plane and let T be a transversal to L and L'. Prove: If one pair of corresponding angles is congruent, then all four pairs of corresponding angles are congruent.

Exercise 2.9.4. Let L and L' be two (distinct) lines in plane and let T be a transversal to L and L'. Prove: If one pair of alternate interior angles is congruent, then both pairs of alternate interior angles are congruent.

Exercise 2.9.5. Let L and L' be two (distinct) lines in plane and let T be a transversal to L and L'. Prove: A pair of alternate interior angles is congruent if, and only if, a pair of corresponding angles is congruent.

Definition 2.9.4. Two distinct lines are parallel if they are coplanar and have empty intersection. Any line is parallel to itself.

Theorem 2.9.5. *Let L and L' be two (distinct) lines in plane and let T be a transversal to L and L'. Suppose that a pair of alternate interior angles is congruent (or equivalently, a pair of corresponding angles is congruent. Then the lines L and L' are parallel.*

Proof. It follows from the exercises that all four pairs of corresponding angles are congruent, and both pairs of alternate interior angles are congruent. Let A and A' denote the points of intersection of T with L and L' respectively.

Suppose that L and L' are not parallel, and let P be the unique point of intersection of L and L'. Let R be a point on L such that $P — A — R$. Then angles $\angle RAA'$ and $\angle PAA'$ are a pair of alternate interior angles, so are, by hypothesis, congruent. But angles $\angle RAA'$ and $\angle PAA'$ form a linear pair, so they are supplementary. It follows that angles $\angle PAA'$ and $\angle PA'A$ are supplementary. But these are two angles of the triangle $\triangle PAA'$, so they cannot be supplementary by Corollary 2.9.2. This contradiction shows that the two lines L and L' are parallel. □

Corollary 2.9.6. *Let L be a line and P a point not on the line L. Then there is (at least one) line in the plane $\mathcal{P}$ determined by L and P which contains the point P and is parallel to L.*

Proof. Let T be the unique line containing P which intersects L and is perpendicular to L. Let L' be the unique line in $\mathcal{P}$ which contains P and is perpendicular to T. Then T is transversal to L and L', and is perpendicular to both L and L'. It follows that all pairs of corresponding angles are congruent, so the lines L and L' are parallel. □

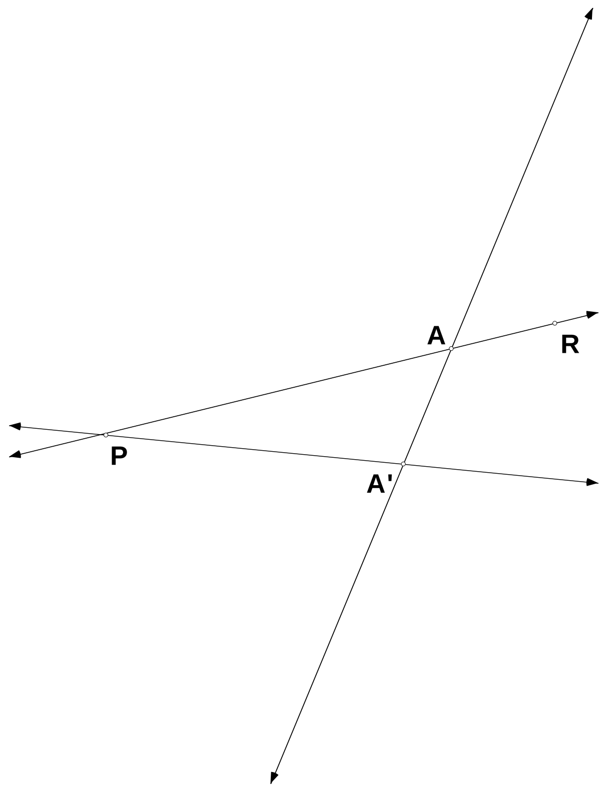

Figure 2.9.3. Proof of Parallelism

We now come to the final axiom of plane geometry:

Axiom PAR. Let L be a line and P a point not on the line L. Then there is exactly one line in the plane $\mathcal{P}$ determined by L and P which contains the point P and is parallel to L.

Theorem 2.9.7. *Let L and L' be parallel lines in a plane $\mathcal{P}$, and let T be a transversal to L and L'. Then corresponding angles for the triple (L, L', T) are congruent.*

Proof. Either all pairs of corresponding angles are congruent or none are. Suppose that no pair of corresponding angles is congruent.

Let B and B' be points on L and L' respectively, which lie on the same side of T. Let Q be a point on T such that $A — A' — Q$. Then angles $\angle QAB$ and $\angle QA'B'$ are corresponding angles, so *not* congruent. By the axiom of angle measure, Axiom AM-2, there is a point B'' on the same side of T as B and B' such that $\angle QA'B'' \cong \angle QAB$. Furthermore B'' does not lie on line L', because then B'' would lie on ray $\overrightarrow{A'B'}$ and $\angle QA'B'$ would equal $\angle QA'B''$, so congruent to $\angle QAB$, contrary to our assumption. So let L'' denote the line $A'B''$.

Then T is a transversal to L and L'', and corresponding angles for the triple (L, L'', T) *are* congruent, so L and L'' are parallel by Theorem 2.9.5.

But then L' and L'' are two distinct lines, both containing point A' and both parallel to L, contradicting Axiom PAR. This contradiction shows that corresponding angles for the triple (L, L', T) are congruent. □

Corollary 2.9.8. *Let L and L' be distinct lines in a plane $\mathcal{P}$, and let T be a transversal to L and L'. Then L is parallel to L' if, and only if, corresponding angles for the triple (L, L', T) are congruent.*

Corollary 2.9.9. *The sum of the measures of the angles of a triangle is 180.*

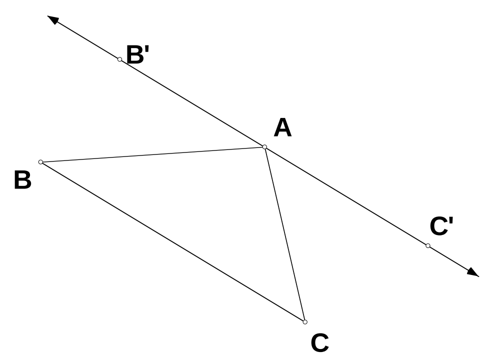

Figure 2.9.4. Sum of Angles in a Triangle

Proof. Consider a triangle $\triangle ABC$ in a plane $\mathcal{P}$. Let L' be the unique line in $\mathcal{P}$ which contains A and which is parallel to $L = \overleftrightarrow{BC}$.

Let B' be a point on L' which is on the same side of $\overleftrightarrow{AC}$ as B. Then angles $\angle B'AB$ and $\angle C$ are alternate interior angles for the triple $(L, L', \overleftrightarrow{AC})$ (two parallel lines and a transversal). Hence $\angle B'AB \cong \angle C$.

Let C' be a point on L' which is on the same side of $\overleftrightarrow{AB}$ as C. Arguing as above, one finds that $\angle C'AC \cong \angle B$.

Using axioms of angle measure, one finds that $m(\angle B'AB) + m(\angle A) + m(\angle C'AC) = 180$. Therefore by the congruences observed in the last two paragraphs, $m(\angle B) + m(\angle A) + m(\angle C) = 180$. □

Exercise 2.9.6. Consider a triangle $\triangle ABC$. The measure of the exterior angle at vertex A is equal to the sum of the measures of the angles ate B and C.

Consider four distinct points A, B, C, D in a plane $\mathcal{P}$. Suppose that no three of the points are colinear, and that the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ intersect only at endpoints of the segments. Then the union of the four segments is called a *quadrilateral*. The segments are called the *sides* of the quadrilateral. Two sides are called adjacent if they intersect (at a common endpoint); two sides which are not adjacent are called opposite.

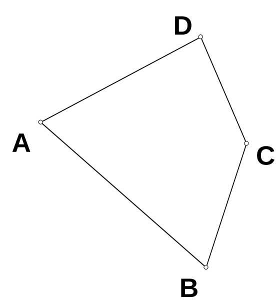

Figure 2.9.5. Quadrilateral

The angles of the quadrilateral are $\angle A = \angle DAB$, $\angle B = \angle ABC$, $\angle C = \angle BCD$, and $\angle D = \angle CDA$.

The points A, B, C, D are called the vertices of the quadrilateral. Two vertices are said to be adjacent if the segment which they determine is a side of the quadrilateral; otherwise the vertices are said to be opposite. Two angles of the quadrilateral are said to be adjacent if their vertices are adjacent vertices; otherwise, they are said to be opposite. The two segments joining opposite vertices are called the diagonals of the quadrilateral.

A quadrilateral is called a *parallelogram* if opposite sides are parallel. It is called a *rectangle* if all four angles are right angles. It is called a *rhombus* if it is a parallelogram with all four sides congruent. It is called a *square* if it is a rectangle with all four sides congruent.

Exercise 2.9.7. Prove: A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

Exercise 2.9.8. Prove: Opposite sides of a parallelogram are congruent. Opposite angles of a parallelogram are congruent. Adjacent angles of a parallelogram are supplementary.

Exercise 2.9.9. Prove: The two diagonals of a parallelogram bisect each other.

Exercise 2.9.10. Prove: Rectangles exist.

Exercise 2.9.11. Prove: Rectangles are parallelograms. Therefore, opposite sides of a rectangle are congruent, and diagonals of a rectangle bisect each other.

Exercise 2.9.12. Prove: The diagonals of a rhombus are perpendicular bisectors of one another. This is in particular true of squares.