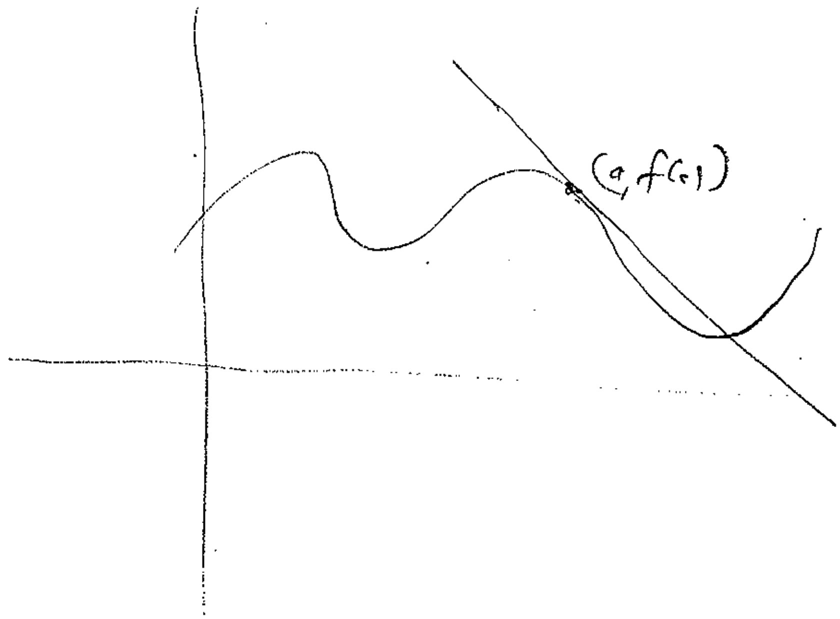
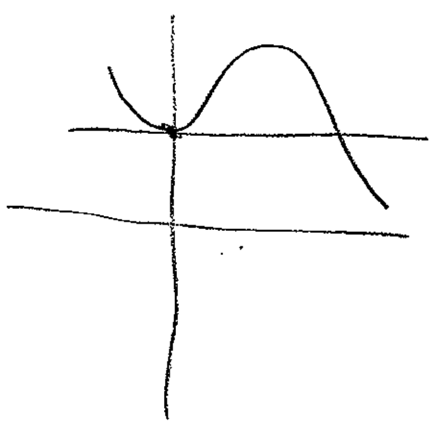
What do we mean by the tangent line to a curve $y = f(x)$ at a point $(a, f(a))$



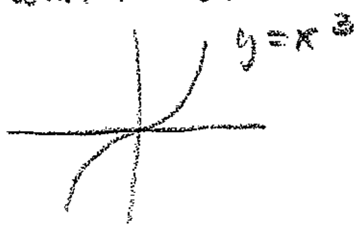
Non answers: Not the line which only touches the curve at $(a, f(a))$

e.g



Not the line which touches but does not cross the curve

e.g



tangent line is $y = 0$

A tentative definition: We are looking for a line which best approximates the curve near $(a, f(a))$.
(It's hopeless to ask for a line to approximate the curve well far away from $(a, f(a))$.)

It's not clear what this actually means. We have to find out.

Computer example: Take $y = f(x) = 2x - x^2 + x^3$

$a = 1.6 \qquad f(a) \doteq 4.736$

Look at part of graph in small rectangle around $(a, f(a))$, namely $y = f(x)$ for $a - h \le x \le a + h$ with h small ($h = 1$, $h = .1$, $h = .01$, $h = .001$) but magnified to "full size"

Find: for h small enough, the graph cannot be distinguished from a straight line.

Question: What is that straight line?

We know it goes thru $(a, f(a)) \cong (1.6, 4.736)$

To specify the line we need its slope

When h is small enough so that the graph looks like a line, the line it looks like has slope approximately the slope of the segment from $(a, f(a))$ to $(a+h, f(a+h))$, namely

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

When we take $h = .1, .01, .001$, etc these slopes seem to approach the value $\boxed{6.48}$

So the line we are seeking is the line thru $(a, f(a)) \cong (1.6, 4.736)$ with slope 6.48

namely
$$\frac{y - 4.736}{x - 1.6} = 6.48 \quad \text{or}$$

$$y = 6.48x - 5.632$$

Notation

Think of a function $y = f(x)$ and a point $(a, f(a))$ on its graph.

We are interested in x near a, so $|x-a|$ is small

Let $h = x-a$ so $|h|$ is small (but h could be pos. or neg.)

Dictionary

x	$a+h$
$x-a$	h
$f(x)$	$f(a+h)$
$\frac{f(x)-f(a)}{x-a}$	$\frac{f(a+h)-f(a)}{h}$

Last time we took $f(x) = 2x - x^2 + x^3$

$(a, f(a)) = (1.6, 4.736)$

We asked what is the tangent line to graph of $y = f(x)$ at $(a, f(a))$?

We found if magnified a small part of the graph corresponding around $(a, f(a))$, it looks like a straight line & we could find the approximate slope by considering $(a, f(a))$ & $(a+h, f(a+h))$ for h small

slope $= \frac{f(a+h) - f(a)}{h}$. Experimentally, we found if h is small enough, slope $\cong 6.48$

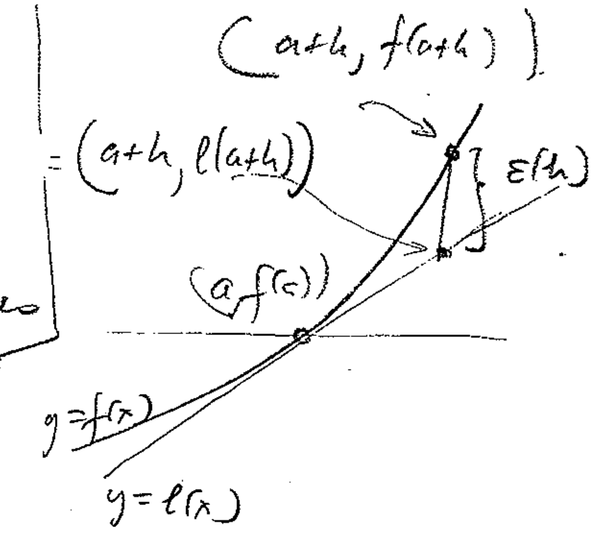
The tangent line is the line thru $(a, f(a))$ with this slope, namely

$$y = l(x) = 4.736 + (6.48)(x - 1.6)$$

This linear function $l(x)$ approximates the actual function $f(x)$ near $x = a$. How close is the approximation?

Lets write $x = a+h$ and

$$\varepsilon(h) = f(a+h) - l(a+h)$$



Lets look at a table of values

h	$\varepsilon(h)$	$\frac{\varepsilon(h)}{h}$

for h small. See computer demo.

We find as gets small so does $\varepsilon(h)$. But, even more, $\frac{\varepsilon(h)}{h}$. Remember $\frac{\varepsilon(h)}{h}$ is

$\varepsilon(h)$ multiplied by the HUGE number $\left(\frac{1}{h}\right)$.

We have been skirting around the idea of limit

namely (1) $\frac{f(a+h) - f(a)}{h}$ approaches a limiting value

$m = 6.48$ as h approaches 0.

(2) $\frac{\varepsilon(h)}{h} = \frac{f(a+h) - l(a+h)}{h}$ approaches

zero as h approaches 0.

Defn Say $\lim_{x \to b} g(x) = L$ if

we can make $|g(x) - L|$ as small as we like by taking $|x-b|$ small enough. (but $(x-b) \neq 0$).

We have 2 examples so far: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 6.48$

(f, a as in example above)

and $\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$

(Both of these facts are only established experimentally.)

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Theorem Let $y = f(x)$ be a function and $(a, f(a))$ a point on the graph.

TFAE

(1) There is a good linear approximation to $y = f(x)$ near $x = a$, in the following sense: There is a number m such that the linear function

$$\boxed{y = l(x) = f(a) + m(x-a)}$$

satisfies

$$\lim_{h\to 0} \frac{f(a+h) - l(a+h)}{h} = 0$$

(2) $$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = m.$$

If this holds we say (a) f is differentiable at $x=a$

(b) The derivative of f at $x=a$ is m

notation $\left.\frac{df}{dx}\right|_{x=a} = m$ or $f'(a) = m$

(c) The line $y = l(x) = f(a) + f'(a)(x-a)$

is the tangent line to the graph of $y=f(x)$

at $x=a$.

Proof. note $l(a+h) = f(a) + m(a+h-a)$

$= f(a) + mh$

Thus $\frac{1}{h}\left(f(a+h) - l(a+h)\right)$

$= \frac{1}{h}\left(f(a+h) - f(a) - mh\right)$

$= \frac{f(a+h) - f(a)}{h} - m$

Hence

$$\lim_{h\to 0} \frac{f(a+h) - l(a+h)}{h} = 0 \iff$$

$$\lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{h} - m \right) = 0 \iff$$

$$\lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{h} \right) = m$$

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Restatement:

A function $y=f(x)$ is differentiable at $x=a$ if

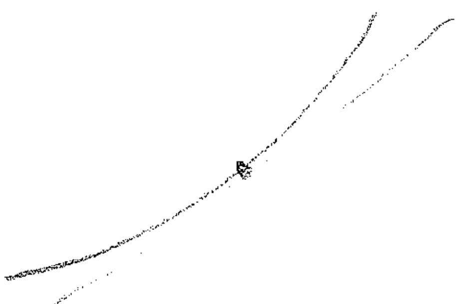
$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists. Let's call the limit

$f'(a)$ ($f'(a)$ is a number)

(Example if $f(x) = 2x - x^2 + x^3$ then $f'(1.6) = 6.48$)

In this case the line $\boxed{y = l(x) = f(a) + f'(a)(x-a)}$

is the tangent line to the graph of $y=f(x)$ at $(a, f(a))$



For x near a, $f(x) - \left(f(a) + f'(a)(x-a)\right)$

is small. In fact $\lim_{x\to a} \frac{f(x) - \left(f(a) - f'(a)(x-a)\right)}{x-a} = 0$

$$\left(= \lim_{h\to 0} \frac{f(a+h) - \left(f(a) - f'(a)(h)\right)}{h} \right.$$

Example $f(x) = x^2$, a arbitrary

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^2 - a^2}{h} =$$

$$\frac{a^2 + 2ah + h^2 - a^2}{h} = \frac{2ah + h^2}{h} = 2a + h$$

$$\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right) = \lim_{h \to 0}(2a+h) = 2a$$

So f is differentiable at arbitrary a and $\boxed{f'(a) = 2a}$

That means f' is itself a function

$f'(a) = 2a$

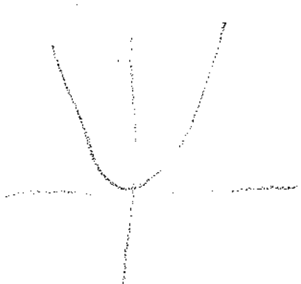
$f'(3) = 6$

$f'(b) = 2b$

$f'(x) = 2x$

Example Find the tangent line to $y = x^2$ at $x = 3$

$f(3) = 9$



$f'(3) = 6$

Tangent line is $y = f(3) + f'(3)(x-3)$

$= 9 + 6(x-3)$