Definition of the definite integral

In class, we introduced a version of the limiting process that defines the definite integral in a different (more general and more standard) way than in the textbook. The purpose of this note is to record this version for you, for your convenience.

Let $f : [a, b] \to \mathbb{R}$ be a function. The definite integral of f over the interval $[a, b]$ is defined by approximation.

To obtain one approximation, divide the interval $[a, b]$ into some number n of subintervals, not necessarily of equal length. To specifiy the division into subintervals, let

$$\mathcal{P} = (x_0 = a < x_1 < x_2 < \cdots < x_n = b)$$

be an increasing sequence of $n+1$ points in $[a, b]$. The corresponding subintervals are

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n].$$

Let Δx_j be the length of the j–th interval, namely $\Delta x_j = (x_j - x_{j-1})$. In addition let

$$x^* = (x_1^*, x_2^*, \ldots, x_n^*)$$

be a sequence of "sampling" points, with x_j^* contained in the j–th subinterval I_j for each j. Corresponding to the choice of the "partition" $\mathcal{P}$ and the "sample" x^* we can form the sum

$$\begin{aligned} S(f, \mathcal{P}, x^*) &= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n \\ &= \sum_{j=1}^{n} f(x_j^*)\Delta x_j. \end{aligned}$$

We saw in class, that if $f(x) \geq 0$ for all x, then this sum is an approximation to the area under the curve $y = f(x)$, above the x–axis, and between the vertical lines $x = a$ and $x = b$.

As an example, take $f(x) = x^2$, $a = 1$, $b = 3$; take 10 subintervals of equal length, and choose x_j^* to be the midpoint of the j–th subinterval. The corresponding sum is the sum of areas of the rectangles shown on the left side of Figure 1 on the next page. On the right side of the figure, we take 50 subintervals of equal length instead, and we can see that we get a better approximation to the area under the curve.

We define the *mesh* of a partition $\mathcal{P}$, denoted $|\mathcal{P}|$ to be the maximum length of the subintervals of $\mathcal{P}$. If the subintervals have equal length, then the mesh is $(b - a)/n$, where n is the number of subintervals.

One can show that if f is a continuous function, or continuous except for finitely many jump discontinuities, then the collection of sums $S(f, \mathcal{P}, x^*)$ approach a limit as the mesh approaches zero (independent of the choice of the sampling points x^*). This limit is called the definite integral of f over the interval $[a, b]$ and denoted

$$\int_a^b f(x)\, dx \quad \text{or, simply} \quad \int_a^b f.$$

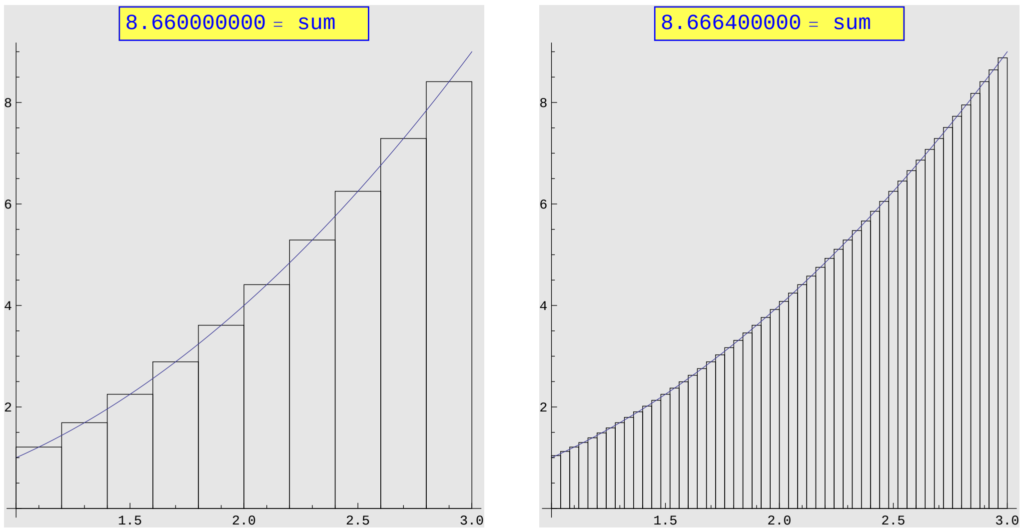

FIGURE 1. Sums approximating area under a curve.

Thus, we have

$$\int_a^b f(x)\ dx = \lim_{|\mathcal{P}|\to 0} S(f, \mathcal{P}, x^*).$$

The meaning of the limit is as follows:

$$\left| \int_a^b f(x)\ dx - S(f, \mathcal{P}, x^*) \right|$$

can be made arbitrarily small by taking a partition $\mathcal{P}$ with sufficiently small mesh, and taking any choice x^* of sample points.

Note: The integral sign $\int$ is actually an elongated "s" and is supposed to suggest a generalized sum.