Further Exercises

Let N and H be groups. An *extension of N by H* is a group E along with a monomorphism $i\colon N \to E$ and an epimorphism $\pi\colon E \to H$ such that $i(N) = \ker \pi$ (so that N imbeds in E as a normal subgroup, with the quotient group being isomorphic with H). We shall usually refer to an extension (E, i, π) simply by the group E; however, the nature of the maps i and π are important in distinguishing between extensions. We identify N with its image under i, and H with the quotient of E by N. As an example, let $\varphi\colon H \to \mathrm{Aut}(N)$ be a homomorphism; then the semidirect product $N \rtimes_\varphi H$ is an extension of N by H in an obvious way, taking i to be the inclusion map sending $n \in N$ to $(n, 1)$ and π to be the projection map sending (n, h) to h.

11. We say that an extension E of N by H is a *split extension* if there is a homomorphism $t\colon H \to E$ (called a *splitting map* for the extension) such that $\pi \circ t$ is the identity map on H, in which case $t(H)$ will be a transversal for N in E. Show that E is a split extension iff it is a semidirect product of N by H.
12. (cont.) Let Q be the quaternion group of order 8. (We can consider Q as the set $\{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication given by the rules $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$.) Show that Q can be realized as a non-trivial extension in four ways—thrice as an extension of $\mathbf{Z}_4$ by $\mathbf{Z}_2$, and once as an extension of $\mathbf{Z}_2$ by $\mathbf{Z}_2 \times \mathbf{Z}_2$—but that none of these extensions is split. (In other words, Q cannot be written non-trivially as a semidirect product.)

If E is an extension of N by H, then we cannot expect to find a homomorphism $t\colon H \to E$ such that $t(H)$ will be a transversal for N in E, for if such a t existed then E would be split. However, since $H \cong E/N$, we can always find a set map $t\colon H \to E$ whose image is a transversal for N; such a map is called a *section* of the extension. Moreover, we can always choose t so that $t(1) = 1$, in which case we say that t is *normalized*. (We use normalized sections instead of arbitrary sections in order to keep the notational complexity to a minimum.)

13. (cont.) Let t be a normalized section of an extension E. Let $\Psi\colon E \to \mathrm{Aut}(E)$ be the homomorphism sending an element of E to the corresponding inner automorphism of E. We shall, for $x \in E$, regard $\Psi(x)$ as being an automorphism of N, which is possible since $N \trianglelefteq E$. Define set maps $f\colon H \times H \to N$ and $\varphi\colon H \to \mathrm{Aut}(N)$ by
$$f(\alpha, \beta) = t(\alpha)t(\beta)t(\alpha\beta)^{-1},$$
$$\varphi(\alpha) = \Psi(t(\alpha)).$$
We call (f, φ) the *factor pair* arising from t. Show that (f, φ) has

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the following properties:

1 $f(\alpha, 1) = f(1, \alpha) = 1$ for every $\alpha \in H$, and $\varphi(1)$ is the identity in $\mathrm{Aut}(N)$.

2 $\varphi(\alpha)\varphi(\beta) = \Psi(f(\alpha,\beta))\varphi(\alpha\beta)$ for $\alpha, \beta \in H$.

3 $f(\alpha,\beta)f(\alpha\beta,\gamma) = \varphi(\alpha)(f(\beta,\gamma))f(\alpha,\beta\gamma)$ for $\alpha,\beta,\gamma \in H$.

14. (cont.) Just as we were able to externalize the notion of semidirect product, so should we be able to externalize the notion of extension; that is, given groups N and H and appropriate additional data, we should be able to construct an extension of N by H. Using Exercise 13 as a guide, formulate such an external construction and prove that it works.

We shall return to these ideas in the further exercises to Section 9.

3. Group Actions

Let G be an arbitrary group. A (*left*) *action* of G on a set X is a map from $G \times X$ to X, with the image of (g, x) being denoted by gx, which satisfies the following conditions:

- $1x = x$ for every $x \in X$.
- $(g_1g_2)x = g_1(g_2x)$ for every $g_1, g_2 \in G$ and $x \in X$.

(Right actions are defined analogously and are used in lieu of left actions by many authors; however, in this book virtually all actions considered will be left actions.) If we have an action of G on X, then we say that G *acts* on X or that X is a *G-set*. If X is a G-set, then X is also an H-set for any $H \leqslant G$, as the action of G on X restricts to give an action of H on X.

For example, let $H \leqslant G$ and consider the coset space G/H. We have an obvious map from $G \times G/H$ to G/H, namely the left multiplication map sending (g, xH) to gxH. This is easily seen to be a left action of G on G/H. Whenever we refer to a coset space G/H as being a G-set, it is this action of G on G/H that we have in mind.

We now provide an alternate perspective on group actions.

PROPOSITION 1. There is a natural bijective correspondence between the set of actions of G on a set X and the set of homomorphisms from G to Σ_X.

Exercises

Throughout these exercises, p denotes a prime.

1. Show that if P is a non-cyclic finite p-group, then P has a normal subgroup N such that $P/N \cong \mathbf{Z_p} \times \mathbf{Z_p}$.
2. Let P be a group of order p^n. Show that P has a normal subgroup N_a of order p^a for every $0 \leq a \leq n$, and that these subgroups can be chosen so that N_a is contained in N_b whenever $a \leq b$.
3. Let $G = \mathrm{GL}(n,p)$, and let P be a Sylow p-subgroup of G. What is the order of $Z(P)$? What is the order of $Z(P/Z(P))$? If we let $Z_2(P) \leqslant P$ be such that $Z_2(P)/Z(P) = Z(P/Z(P))$ and continue in this way, what happens?
4. Let U be the subgroup of $\mathrm{GL}(n,p)$ consisting of the upper unitriangular matrices, and let Q be the subgroup of U consisting of all matrices whose (i,j)-entry is zero whenever $1 < i < j < n$. Determine $Z(Q)$, and show that $Q/Z(Q)$ is abelian.
5. Show that subgroups and quotient groups of finite nilpotent groups are nilpotent, and that direct products of finite nilpotent groups are nilpotent.
6. Let U be the subgroup of $\mathrm{GL}(3,p)$ consisting of the upper unitriangular matrices. Show that if p is an odd prime, then U is a non-abelian group of order p^3 having no elements of order p^2. If $p = 2$, with which group of order 8 is U isomorphic?

Further Exercises

7. Show that a finite group G has a largest nilpotent normal subgroup, in the sense that it contains all nilpotent normal subgroups of G. (This subgroup is called the *Fitting subgroup* of G.)

The intersection of all maximal subgroups of a finite group G is called the *Frattini subgroup* of G and is denoted by $\Phi(G)$.

8. Show that $\Phi(G)$ is a nilpotent normal subgroup of G.
9. Show that $g \in \Phi(G)$ iff whenever $G = \langle S \rangle$ and $g \in S$, then $G = \langle S - \{g\} \rangle$.
10. Show that if P is a finite p-group, then $P/\Phi(P)$ is an elementary abelian p-group.

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Further Exercises

These exercises are a continuation of the further exercises to Section 2. Let N and H be groups. A *factor pair* of N by H is a pair (f, φ) of set maps $f: H \times H \to N$ and $\varphi: H \to \mathrm{Aut}(N)$ satisfying properties **1**, **2**, and **3** listed on page 27. Let $\mathcal{E}$ be the set of extensions of N by H, and let $\mathcal{F}$ be the set of factor pairs of N by H. In what follows, we will always use "extension" to mean an element of $\mathcal{E}$, and "factor pair" to mean an element of $\mathcal{F}$.

3. Let (f, φ) be a factor pair, and define

$$(x, \alpha) \cdot (y, \beta) = (x\varphi(\alpha)(y)f(\alpha, \beta), \alpha\beta)$$

for $(x, \alpha), (y, \beta) \in N \times H$. Show that this gives a group structure on $N \times H$; call this group $E_{f,\varphi}$. Show further that $(E_{f,\varphi}, i, \pi)$ is an extension, where $i(x) = (x, 1)$ and $\pi(x, \alpha) = \alpha$, and that (f, φ) is the factor pair arising from some normalized section of $E_{f,\varphi}$. (Observe that this construction generalizes the notion of external semidirect product.)

We have seen in Exercise 2.13 that an extension gives rise to a factor pair via a choice of normalized section, and we have just given an explicit construction of an extension from a given factor pair. We view these processes as giving maps between $\mathcal{E}$ and $\mathcal{F}$, and we now investigate the relationship between these maps. We must first consider the relation between factor pairs arising from different normalized sections of the same extension.

4. (cont.) Suppose that t and u are normalized sections of an extension E, and let (f, φ) and (g, ρ) be the factor pairs arising from t and u, respectively. Let $c: H \to N$ be the set map such that $u(\alpha) = c(\alpha)t(\alpha)$ for every $\alpha \in H$. Show that the following properties hold:

   4 $\rho(\alpha) = \Psi(c(\alpha))\varphi(\alpha)$ for $\alpha \in H$, where $\Psi(c(\alpha))$ is the inner automorphism of N coresponding to $c(\alpha)$.
   5 $g(\alpha, \beta) = c(\alpha)\varphi(\alpha)(c(\beta))f(\alpha, \beta)c(\alpha\beta)^{-1}$ for $\alpha, \beta \in H$.

The above exercise motivates the following definition: We say that two factor pairs (f, φ) and (g, ρ) are *equivalent* if there is a map $c: N \to H$ such that properties **4** and **5** hold. (Verify that this is an equivalence relation on $\mathcal{F}$.) Let $\overline{\mathcal{F}}$ denote the set of equivalence classes of factor pairs; we will use $[f, \varphi]$ to denote the class of the factor pair (f, φ). We have a well-defined map from $\mathcal{E}$ to $\overline{\mathcal{F}}$ which sends an extension to the class of a factor pair arising from any normalized section. We must now consider what happens when we pass from $\overline{\mathcal{F}}$ back to $\mathcal{E}$ via the construction in Exercise 3. Here we will need to recall the exact definition of an extension.

5. (cont.) Let (f,φ) and (g,ρ) be factor pairs, with $[f,\varphi] = [g,\rho]$. Let $i\colon N \to E_{f,\varphi}$ and $j\colon N \to E_{g,\rho}$ be the natural inclusions (of N into the underlying set $N \times H$), and let $\pi\colon E_{f,\varphi} \to H$ and $\tau\colon E_{g,\rho} \to H$ be the natural projections (of the underlying set $N \times H$ onto H). Show that there is an isomorphism $\xi\colon E_{f,\varphi} \to E_{g,\rho}$ such that $\xi \circ i = j$ and $\tau \circ \xi = \pi$.

Motivated by the above exercise, we say that two extensions (E,i,π) and (F,j,τ) are *equivalent* if there is an isomorphism $\xi\colon E \to F$ such that $\xi \circ i = j$ and $\tau \circ \xi = \pi$. (Verify that this gives an equivalence relation on $\mathcal{E}$.) We let $\overline{\mathcal{E}}$ denote the set of equivalence classes of extensions, and we let $[E]$ denote the class of an extension E. In this context, Exercise 5 asserts that there is a well-defined map from $\overline{\mathcal{F}}$ to $\overline{\mathcal{E}}$, sending $[f,\varphi]$ to $[E_{f,\varphi}]$.

6. (cont.) If p is an odd prime, show that $\mathbf{Z_{p^2}}$ can be realized in $p-1$ nonequivalent ways as an extension of $\mathbf{Z_p}$ by $\mathbf{Z_p}$.
7. (cont.) We have already obtained a map from $\mathcal{E}$ to $\overline{\mathcal{F}}$, sending an extension E to the class of the factor pair arising from any normalized section of E. Show that this map induces a map from $\overline{\mathcal{E}}$ to $\overline{\mathcal{F}}$.
8. (cont.) Show that the map from $\overline{\mathcal{E}}$ to $\overline{\mathcal{F}}$ obtained in Exercise 7 is inverse to the map from $\overline{\mathcal{F}}$ to $\overline{\mathcal{E}}$ sending $[f,\varphi]$ to $[E_{f,\varphi}]$. Conclude that there is a bijective correspondence between the set of equivalence classes of extensions and the set of equivalence classes of factor pairs.

Exercise 8 implies that in order to study extensions up to equivalence, it suffices to study equivalence classes of factor pairs. The next two exercises give a slight refinement of the correspondence just obtained.

9. (cont.) Let $\eta\colon \mathrm{Aut}(N) \to \mathrm{Out}(N)$ be the natural map. Show that if (f,φ) and (g,ρ) are any two factor pairs arising from an extension E, then $\eta \circ \varphi = \eta \circ \rho$, and this map from H to $\mathrm{Out}(N)$ (which we denote by ψ_E) is a homomorphism. Conclude that there is a well-defined map from $\mathcal{E}$ to the set of homomorphisms from H to $\mathrm{Out}(N)$, sending E to ψ_E.
10. (cont.) Let $\psi\colon H \to \mathrm{Out}(N)$ be a given homomorphism. Show that there is a bijective correspondence between the set of classes $[E]$ of $\overline{\mathcal{E}}$ for which $\psi_E = \psi$ and the set of classes $[f,\varphi]$ of $\overline{\mathcal{F}}$ for which $\eta \circ \varphi = \psi$, where $\eta\colon \mathrm{Aut}(N) \to \mathrm{Out}(N)$ is the natural map.

We now consider the case where the group N is abelian; we write A instead of N, and we will use additive notation for A. Observe that $\mathrm{Out}(A) = \mathrm{Aut}(A)$. We fix a homomorphism $\varphi\colon H \to \mathrm{Aut}(A)$, and we write xa in lieu of $\varphi(x)(a)$ for $x \in H$ and $a \in A$. We would like to study those equivalence classes $[E]$ of extensions of N by H for which $\psi_E = \varphi$; we say that such extensions *respect* the action of H on A. By Exercise 10, it suffices to study equivalence classes $[f, \varphi]$ of factor pairs of A by H. We suppress φ in our notation, so that we are studying functions $f\colon H \times H \to A$ such that $f(x, 1) = f(1, x) = 0$ for all $x \in H$ and which in addition satisfy

$$f(x,y) + f(xy,z) = xf(y,z) + f(x,yz)$$

for all $x, y, z \in H$, with two such functions f and g being equivalent if there is a map $c\colon H \to A$ such that $c(1) = 0$ and

$$g(x,y) = f(x,y) + c(x) + xc(y) - c(xy)$$

for all $x, y \in H$. As discussed in the section, the set $H^2(H, A)$ of equivalence classes of such functions forms an abelian group that is called the second cohomology group of (H, A). It now follows from Exercise 10 that there is a bijective correpondence between the group $H^2(H, A)$ and the set of equivalence classes of extensions of A by H which respect the action of H on A. In particular, if $H^2(H, A) = 0$, then every extension of A by H is split.

11. (cont.) Suppose that H and A are both finite. Show that the order of each element of $H^2(H, A)$ divides both $|H|$ and the exponent of A. (This implies that $H^2(G/A, A) = 0$ when A is an abelian normal Hall subgroup of a finite group G, which we established in proving the Schur-Zassenhaus theorem.)