Mathematics 121 Review for Final (Field Theory portion)
May, 2005

Definitions you should know:

1. Conditions on elements in a field extension: algebraic, transcendental, separable.
2. Conditions on a field extension: finite, algebraic, normal, separable, Galois.
3. Splitting field of a polynomial.

Theorems you should be able to prove:

1. Multiplicativity of dimensions of field extensions.
2. A field extension is finite if, and only if, it is finitely generated and algebraic.
3. The set of algebraic elements in a field extension is a subfield.
4. If $K \subseteq M \subseteq L$, and $K \subseteq M$ and $M \subseteq L$ are algebraic extensions, then $K \subseteq L$ is an algebraic extension.
5. Let p be an irreducible polynomial over a field K. There exists a field extension in which p has a root. Moreover, if α is a root of p in some field extension, then $K(\alpha) = K[\alpha] \cong K[x]/(p)$, and $\dim_K(K(\alpha)) = \deg(p)$.
6. Existence and uniqueness, up to isomorphism, of splitting fields.
7. Let $f(x) \in K[x]$ be an irreducible polynomial. Let L be a splitting field of $f(x)$. Then $\mathrm{Aut}_K(L)$ acts on the roots of $f(x)$ in L, and, moreover, the action is faithful. If $p(x)$ is an irreducible factor of $f(x)$, then the set of roots of $p(x)$ is invariant under the action of $\mathrm{Aut}_K(L)$, and the action on the roots of $p(x)$ is transitive.
8. Let $f(x) \in K[x]$ be an irreducible polynomial. Let L be a splitting field of $f(x)$. Then $\mathrm{Aut}_K(L)$ is finite.
9. Artin's lemma: linear independence of any (finite) set of field automorphisms.

Be able to state:

1. Three equivalent conditions for a field extension to be Galois.
2. The fundamental theorem of Galois theory.

Be able to compute: Field operations, including inverses, in $K(\alpha)$ where α is a root of a given irreducible polynomial $p(x) \in K[x]$.