THE QUANTUM CONTENT OF THE NORMAL SURFACES IN A THREE-MANIFOLD

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Abstract. The formula for the Turaev-Viro invariant of a 3-manifold depends on a complex parameter t. When t is not a root of unity, the formula becomes an infinite sum. This paper analyzes convergence of this sum when t does not lie on the unit circle, in the presence of an efficient triangulation of the three-manifold. The terms of the sum can be indexed by surfaces lying in the three-manifold. The contribution of a surface is largest when the surface is normal and when its genus is the lowest.

1. Introduction

This paper initiates the study of non-perturbative quantum invariants of three-manifolds M away from roots of unity. Turaev and Viro [12] defined invariants of closed 3-manifolds as state sums depending on a complex parameter t. When t is a root of unity this sum is finite. At values of t other than roots of unity the formula for the Turaev-Viro invariant becomes an infinite sum. The partial sums could oscillate wildly, so that even after renormalizing the series does not converge. However, we are able to show that for a special class of spines of some three-manifolds, the oscillation does not occur and there is a limit.

The key is to see the invariant as a sum over surfaces in the manifold. An efficient ideal triangulation [6] of the manifold is one where the only normal spheres and tori are links of the boundary components of the manifold. Given an efficient ideal triangulation we know what the invariant should be. It is a sum over surfaces carried by a spine dual to the efficient triangulation. The surfaces contribute to the sum in a way that fits the modern approach to normal surface theory. The study of normal surfaces [6] has been augmented by looking at surfaces that aren't normal, and coming to an understanding of how a surface fails to be normal. The farther a surface is from being normal, the less it contributes to the sum for the invariant. The higher the genus of a surface the less it contributes. In a very real sense, the Turaev-Viro invariant is a measure of the normal surface theory of M.

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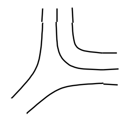

Figure 1. Admissible triple $(2, 3, 3)$.

In section 2 we broach preliminary concepts relating to special functions, and to spinal and normal surfaces in an ideal triangulation of a 3-manifold. This is followed by section 3 that studies properties and limiting behavior of the $6j$-symbols. Section 4 is concerned with estimates of the contributions of spinal surfaces to the state sum. The final section proves the result about the convergence of the infinite state sum.

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2. Preliminaries

2.1. **Special Functions.** The formulas in this section are taken from [7], however we use the variable t instead of A. Throughout this paper t is a real number with $0 < t < 1$. There are several functions of t that we will work with. The first is known as *quantized* n,

$$[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}. \tag{1}$$

The next is just a variation on the first,

$$\Delta_n = (-1)^n \frac{t^{2n+2} - t^{-2n-2}}{t^2 - t^{-2}} = (-1)^n [n+1]. \tag{2}$$

There is quantized factorial, defined recursively by $[0]! = 1$ and

$$[n]! = [n][n-1]!. \tag{3}$$

A triple of nonnegative integers (a, b, c) is admissible if their sum is even and they satisfy every possible triangle inequality. Admissibility is the necessary and sufficient condition for the existence of a Kauffman triad on a, b and c. Suppose that (a, b, c) is admissible. Arrange a points, b points and c points on the sides of a triangle. There exists a system of disjoint proper arcs joining opposite sides of the triangle having those points as their boundary. Figure 1 shows the admissible triple $(2, 3, 3)$. The number of strands running between the family of a points, and the family of b points is $x_1 = \frac{a+b-c}{2}$, between the b points and the c points is $x_2 = \frac{c+b-a}{2}$, and between the a points and the c points is $x_3 = \frac{a+c-b}{2}$. Admissibility is equivalent to the statement that all three functions are nonnegative and integral. We call x_1, x_2 and x_3 the *strand numbers* of the triple (a, b, c).

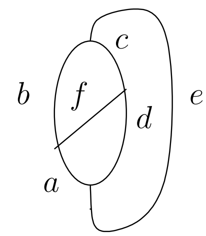

FIGURE 2. Tetrahedral net

You can always add two admissible triples and the result will be admissible, but you cannot always subtract them. However,

Proposition 1. *Suppose that* (a, b, c) *and* (a', b', c') *are admissible and* x_i *and* x'_i *are the strand numbers corresponding to the two triples. If* $x_i \geq x'_i$ *for all* i*, then* $(a - a', b - b', c - c')$ *is admissible.*

Proof. The strand numbers are linear functions of the triples, thus the strand numbers for $(a - a', b - b', c - c')$ are integral and nonnegative. □

If (a, b, c) is an admissible triple, define, [7]

$$\theta(a, b, c) = (-1)^{\frac{a+b+c}{2}} \frac{[\frac{a+b+c}{2} + 1]![\frac{a+b-c}{2}]![\frac{a+c-b}{2}]![\frac{b+c-a}{2}]!}{[a]![b]![c]!}. \tag{4}$$

Suppose that a tetrahedral net has been labeled as in Figure 2 where the letters are nonnegative integers and the triples appearing at each vertex v are admissible. The tetrahedral coefficient [7, 9] is the quantity

$$\mathrm{Tet}\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix} = \tag{5}$$

$$\frac{\prod_v [x_{v,1}]![x_{v,2}]![x_{v,3}]!}{[a]![b]![c]![d]![e]![f]!} \sum_{s=m}^{M} \frac{(-1)^s [s+1]!}{[B_1 - s]![B_2 - s]![B_3 - s]![s - A_1]![s - A_2]![s - A_3]![s - A_4]!},$$

where the B_i are half the sums of the labels over the four cycles, the A_i are half the sums of the labels at each vertex, m is the maximum of the A_i and M is the minimum of the B_i. The $x_{v,i}$ are the strand numbers of the admissible triple at the vertex v.

The *unitary* $6j$ *symbol* [11] is the quantity:

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}_u = \frac{\mathrm{Tet}\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}}{\sqrt{\theta(a, d, e)\theta(b, c, e)\theta(a, b, f)\theta(c, d, f)}}. \tag{6}$$

A consequence of admissibility is that the denominator is a square root of a positive number, so the formula is unambiguous.

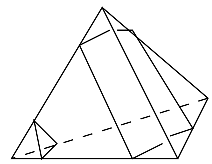

FIGURE 3. Normal surface intersecting a tetrahedron

Letting $q = t^4$, let $(x;q)_n = \prod_{i=1}^{n}(1 - xq^{i-1})$. We need the following fact [5]: The function

$$(x;q)_\infty = \prod_{i=1}^{\infty}(1 - xq^{i-1}) \tag{7}$$

is well defined when $|q| < 1$. In particular, $(q;q)_\infty$ is well defined. Notice that

$$[n] = t^{-2n+2}\frac{1-q^{n-1}}{1-q}, \tag{8}$$

and

$$\Delta_n = (-1)^n t^{-2n}\frac{1-q^n}{1-q}, \tag{9}$$

so that

$$[n]! = t^{-(n-1)n}\frac{(q;q)_n}{(1-q)^n}, \tag{10}$$

and

$$\theta(a,b,c) = (-1)^{\frac{a+b+c}{2}}\frac{t^{-a-b-c}}{1-q}\ \frac{(q;q)_{\frac{a+b+c}{2}+1}(q;q)_{\frac{a+b-c}{2}}(q;q)_{\frac{b+c-a}{2}}(q;q)_{\frac{a+c-b}{2}}}{(q;q)_a(q;q)_b(q;q)_c}. \tag{11}$$

The quantities Δ_n, $\theta(a,b,c)$ and $\mathrm{Tet}\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$ can be understood as the Kauffman brackets of colored graphs [7].

2.2. **Spines and Ideal Triangulations.** An *ideal triangulation* [1] of the compact three-manifold M is a union of tetrahedra joined along faces with their vertices removed so that the result is homeomorphic to the interior of M.

A surface is *normal* with respect to the triangulation [6] if it intersects each tetrahedron in triangles and quadrilaterals (quads) as in Figure 3. Parameterize normal surfaces by their intersection with the edges of the tetrahedra. Arrange these numbers in a 2×3 array, so that each column of the array is the number of points of intersection of the normal surface with two opposite edges of the tetrahedron. More specifically, there is a tetrahedral net dual to the 1-skeleton of the tetrahedron, lying on the boundary of the tetrahedron, as pictured in Figure 4. The intersection of the normal surface with the boundary of the tetrahedron is a family of circles carried by

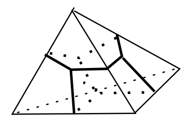

FIGURE 4. Dual tetrahedral net

this net, the number of strands carried by an edge of the net is the intersection number of the normal surface with the edge of the tetrahedron transverse to the particular edge of the net. We form the array of nonnegative integers just as if we were indexing a tetrahedral coefficient, see Figure 2, where the label on the edge is the number of strands carried by that edge. Let C_1, C_2, C_3 be the sums of the columns of the array, named so that $C_1 \geq C_2 \geq C_3$.

Proposition 2. *An array of nonnegative integers corresponds to a normal surface if and only if the integers assigned to the three edges around each face of the tetrahedron form an admissible triple, and* $C_1 = C_2$.

Proof. Think of a face of a tetrahedron as a triangle, and the intersection of the normal surface with the triangle as a system of arcs joining points on the three edges. As we are joining the points on the three sides of the triangle by nonintersecting arcs, the triple around each face must be admissible. The second condition follows from the fact that the intersection of a normal surface with any tetrahedron can only contain triangles and one type of quad. □

You cannot necessarily add normal surfaces, because if two normal surfaces have different quads in the same tetrahedron, their double curve sum may no longer be normal. However, there is always a finite family of normal surfaces so that every normal surface can be written as an integral sum of those surfaces, with nonnegative coefficients.

Letting $\mathbb{N}$ denote the nonnegative integers, a *rational cone* is the solution of a family of linear homogeneous equations with integer coefficients in $\mathbb{N}^k$ for some k. An element of a rational cone is *irreducible* if it cannot be written as the sum of two elements of the cone in a nontrivial way. It is a classical result that a rational cone has only finitely many irreducible elements and they generate the cone additively. The set of irreducibles is called a *Hilbert basis* for the cone.

The normal surfaces form a rational cone. The class of normal surfaces is a subset of a more general class of surfaces, the *spinal surfaces.*

Let

$$X = \{(x, y, z) \in \mathbb{R}^3 | z = 0 \text{ or } (z \geq 0 \text{ and } x = 0) \text{ or } (z \leq 0 \text{ and } y = 0)\}. \tag{12}$$

A subset Y of the 3-manifold M is *modeled* on X if for every $p \in Y$ there is an open neighborhood U of p, open set $V \subset \mathbb{R}^3$ and a homeomorphism $\phi : U \to V$, with $\phi^{-1}(X) = Y \cap U$. There is a decomposition of Y into vertices, (open) edges and

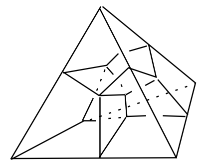

FIGURE 5. Intersection of the spine with a tetrahedron

(open) faces coming from the natural decomposition of X into vertices, edges and faces.

We say $S \subset M$ is a *regular spine* [8, 10] if:

(1) S is modeled on X.
(2) $M - S$ is homeomorphic to $\partial M \times [0, 1)$.
(3) S has at least one vertex.
(4) Every edge of S has a vertex in its closure.
(5) Every face of S is simply connected.

Proposition 3. *Ideal triangulations and regular spines are in one to one correspondence up to isotopy via duality.*

Proof. For each ideal triangulation there is a regular spine. Put a vertex in the center of each tetrahedron. Join the vertices in adjacent tetrahedra by edges, and then form faces of the spine that intersect the edges of the triangulation transversely and are bounded by the edges of the spine. The intersection of the spine with a tetrahedron is pictured in Figure 5. Similarly for each regular spine there is an ideal triangulation, so that its six edges intersect the six faces of the spine coming into the vertex transversely, and each edge intersects exactly one face. □

Given a spine S of M and a simple closed curve $\kappa \subset \partial M$ there is a possibly singular annulus $A_\kappa \subset M$ having κ as one boundary component so that the intersection of A_κ with S is the other boundary component of A_κ. The annulus is constructed by taking the closure of the points lying over κ in the product structure on $M - S$. The singularities of A_κ come from the fact that the map from ∂M to S given by following the lines of the product structure is two to one along faces. Since there is some ambiguity in the product structure we can choose the annulus A_κ so that it is in general position with respect to the spine. This means its boundary misses the vertices of the spine, intersects the edges transversely and its only singular points are transverse double points occurring in the interior of the faces of the spine. If $\mathcal{C}$ is a system of disjoint simple closed curves in ∂M let A_κ, where $\kappa \in C$, be a system of disjoint annuli corresponding to the curves in C that is in general position with respect to the spine S. The union $S(C) = S \cup (\cup_{\kappa \in C} A_\kappa)$ is called *the augmentation of the spine with respect to* C. Except for points on C the augmentation is still modeled on X, so it can be decomposed into vertices, edges and faces just as a spine. If the

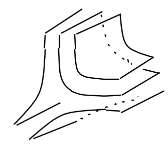

FIGURE 6. Building a spinal surface

spine is regular then the faces of the augmentation are simply connected except for the annular faces with one boundary component a curve in C.

An *admissible coloring* of a spine is an assignment of a nonnegative integer to each face of the spine so that the integers assigned to the three faces meeting along each edge form an admissible triple. Given an admissible coloring of the spine there is a *spinal surface* built as follows. If the face f carries the integer u_f then take u_f parallel copies of f. Along the edges glue the faces together so that they look like the Cartesian product of a triple of arcs at a vertex with an interval. The triple $(2, 3, 3)$ occurring along an edge is shown in Figure 6. So far, the surface constructed intersects the boundary of a small ball at each vertex in a collection of circles arranged along a tetrahedral net. To finish the construction, fill in the surface inside each ball with a disk for each circle in the net.

Topologically, the spinal surfaces are those surfaces that intersect the tetrahedra in disks, so that their intersection with any face of a tetrahedron consists of arcs whose endpoints lie in distinct edges of the face. The spinal surfaces form an additive cone, as the sum of two admissible colorings is an admissible coloring. However, Euler characteristic is not always additive under sum. Clearly, spinal surfaces are a larger class than the normal surfaces associated to the dual triangulation. We can identify the normal surfaces inside the spinal surfaces by looking at the tetrahedral net at each vertex. In specific at each vertex we can define the three column sums of the tetrahedral net and order them so that $C_1 \geq C_2 \geq C_3$.

Remark 1. *By Proposition 2 the surface is normal in the dual ideal triangulation if and only if at each vertex* $C_1 = C_2$.

The spinal surfaces form a rational cone. The proper domain is the Cartesian product of copies of $\mathbb{N}$, one for each strand number. The color on a face is the sum of the two adjacent strand numbers along an edge of the face. The equations defining the cone come from the requirement that the computed color of a face must be the same no matter what edge of the face you compute it along. Thus we have the following:

Fact 1. *There is a set of primitive spinal surfaces* $\{F_i\}$ *so that every spinal surface can be written as a nonnegative sum of the* $\{F_i\}$*'s.*

Suppose now that C is a system of disjoint simple closed curves in ∂M and $S(C)$ is an augmentation of the spine with respect to C. An admissible coloring of $S(C)$ is defined the same way as an admissible coloring of a spine except that the annular faces can only carry the color 1. There is once again a correspondence between admissible colorings and surfaces, but now the surfaces have boundary equal to the union of the curves in C.

In order to understand the Euler characteristic of a surface carried by a spine or an augmented spine we need to understand how many circles there are in a colored tetrahedral net.

Proposition 4. *Suppose that a tetrahedral net has column sums $C_1 \geq C_2 \geq C_3$. The number of circles in the net is* $\gcd(C_1 - C_2, C_1 - C_3)/2 + C_2 + C_3 - C_1$.

Proof. Unless a tetrahedral net is of the form

$$\begin{pmatrix} a & b & a+b \\ a & b & a+b \end{pmatrix} \tag{13}$$

with a and b nonzero then there is always a simple closed curve in the net that is the boundary of one of the faces of the tetrahedron.

Removing a curve that bounds a face does not change $C_1 - C_2$ or $C_1 - C_3$, but $C_2 + C_3 - C_1$ is reduced by one. Remove such curves until there are no more curves that bound faces. The remaining net will be of the form above. If $\gcd(C_1 - C_2, C_1 - C_3) = 2$ then the system consists of a single curve. More generally, the number of components is $\gcd(a, b) = \frac{\gcd(C_1 - C_2, C_1 - C_3)}{2}$. □

From this proof we see that the net is made up of circles that are boundaries of faces along with multiple copies of a single type of circle that appears in a tetrahedral net of type

$$\begin{pmatrix} a & b & a+b \\ a & b & a+b \end{pmatrix} \tag{14}$$

where a and b are relatively prime. Alternatively, the boundary of a simplex with its vertices removed is a four times punctured sphere. Any simple closed curve is either boundary parallel or separates the surface into two pairs of pants. The dearth of disjoint systems of simple closed curves on a pair of pants causes all curves that are not triangles to be parallel. The a and b can be understood in terms of geometric intersection numbers with crosscuts. Name such curves by the pair (a, b) where a and b are relatively prime and $a \leq b$. For each such pair there are six or three different ways (depending on the symmetries of the particular curve type) of labeling the tetrahedron corresponding to the curve of type (a, b). We say that two (a, b) curves are *non-conflicting* if the curves are parallel in a regular neighborhood of the 1-skeleton of the tetrahedral net.

Proposition 5. *Euler characteristic of spinal surfaces is additive when the two surfaces have the same (a, b) types at each vertex and those types are non-conflicting.*

Proof. The surface that corresponds to the sum of the colorings is the disjoint union of the surfaces corresponding to the two colorings. □

The type $(0,1)$ is a quad, the type $(1,1)$ corresponds to an almost normal surface [6]. Further types wind more and more around the tetrahedral net before closing up.

3. $6j$-symbol Details

3.1. Bounding the $6j$-symbols. We begin with a universal bound on the size of the unitary $6j$–symbols in terms of their entries.

Proposition 6. *Let $C_1 \geq C_2 \geq C_3$ be the column sums of the unitary $6j$–symbol $\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}_u$ and assume that $0 < t < 1$. There exists a function $K(t) > 0$ such that*

$$\left| \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}_u \right| \leq K(t) t^{\frac{1}{2}(C_1 - C_2)(C_1 - C_3) + C_1}. \tag{15}$$

Proof. After collecting and canceling terms, $\left| \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}_u \right|$ is equal to

(16)
$$\frac{\sqrt{\prod_v |[x_{v,1}]![x_{v,2}]![x_{v,3}]!|}}{\sqrt{[\frac{a+d+e}{2}+1]![\frac{b+c+e}{2}+1]![\frac{a+b+f}{2}+1]![\frac{c+d+f}{2}+1]!}} \left| \sum_{s=m}^{M} \frac{(-1)^s [s+1]!}{\prod_{i=1}^{3}[B_i - s]! \prod_{j=1}^{4}[s - A_j]!} \right|.$$

Using (10) this is further equal to

$$t^p (1-q)^2 \sqrt{\frac{\prod_v (q;q)_{x_{v,1}} (q;q)_{x_{v,2}} (q;q)_{x_{v,3}}}{(q;q)_{\frac{a+d+e}{2}+1} (q;q)_{\frac{b+c+e}{2}+1} (q;q)_{\frac{a+b+f}{2}+1} (q;q)_{\frac{c+d+f}{2}+1}}} \tag{17}$$

$$\left| \sum_{s=m}^{M} t^{p_s} \frac{1}{1-q} \frac{(-1)^s (q;q)_{s+1}}{\prod_{i=1}^{3} (q;q)_{B_i - s} \prod_{j=1}^{4} (q;q)_{s - A_j}} \right|.$$

Here

(18)
$$p = \frac{1}{2}(-a^2 - b^2 - c^2 - d^2 - e^2 - f^2 + ae + ad + ab + af + be + bc + bf + ce + cd + cf + de + df)$$
$$+ a + b + c + d + e + f$$

and

$$p_s = 6s^2 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \tag{19}$$
$$+ af + ac + ae + fc + fe + ce + be + bd + bf + ed + df + ab + ad + bc + cd$$
$$-2s(1 + 2a + 2b + 2c + 2d + 2e + 2f).$$

After completing the square, p_s is

$$p_s = 6\left(s - \frac{a+b+c+d+e+f+\frac{1}{2}}{3}\right)^2 + \frac{1}{3}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) \tag{20}$$

$$-\frac{1}{3}(af+ae+ac+ad+ab+fe+fc+ce+be+bd+bf+ed+df+bc+cd)-\frac{2}{3}(a+b+c+d+e+f)-\frac{1}{6}.$$

Combining all the factors of $(1-q)$ outside and the powers of t inside the sum, formula (17) can be simplified to

$$(1-q)\sqrt{\frac{\prod_v (q;q)_{x_{v,1}}(q;q)_{x_{v,2}}(q;q)_{x_{v,3}}}{(q;q)_{\frac{a+d+e}{2}+1}(q;q)_{\frac{b+c+e}{2}+1}(q;q)_{\frac{a+b+f}{2}+1}(q;q)_{\frac{c+d+f}{2}+1}}} \left|\sum_{s=m}^{M} t^{p'_s}\frac{(-1)^s(q;q)_{s+1}}{\prod_{i=1}^{3}(q;q)_{B_i-s}\prod_{j=1}^{4}(q;q)_{s-A_j}}\right|. \tag{21}$$

where

$$p'_s = 6\left(s-\frac{a+b+c+d+e+f+\frac{1}{2}}{3}\right)^2-\frac{1}{6}(a^2+b^2+c^2+d^2+e^2+f^2) -\frac{1}{3}(ac+fe+bd)+\frac{1}{6}(ad+ae+de+bc+be+ce+ab+af+bf+cd+cf+df) +\frac{1}{3}(a+b+c+d+e+f)-\frac{1}{6}. \tag{22}$$

The formula (21) is the absolute value of an alternating sum from $s=m$ to $s=M$. Take the quotient whose numerator is the summand at $s+1$ and whose denominator is the summand at s, the result is,

$$(-1)t^{12s-4(a+b+c+d+e+f)+4}\frac{(1-q^{s+2})\prod_{i=1}^{3}(1-q^{B_i-s})}{\prod_{j=1}^{4}(1-q^{s+1-A_j})}. \tag{23}$$

Take the absolute value, with the effect of removing the (-1). In order to see that each one of these quotients is smaller than the last, take the logarithm of the result, giving:

$$(12s-4(a+b+c+d+e+f)+4)\log(t)+\log(1-q^{s+2})+\sum_{i=1}^{3}\log(1-q^{B_i-s})-\sum_{j=1}^{4}\log(1-q^{s+1-A_j}) \tag{24}$$

Apply the Taylor series for $\log(1-x)$ to get

$$(12s-4(a+b+c+d+e+f)+4)\log(t)+\sum_{n=1}^{\infty}\frac{1}{n}\left(-q^{n(s+2)}-\sum_{i=1}^{3}q^{n(B_i-s)}+\sum_{j=1}^{4}q^{n(s+1-A_j)}\right). \tag{25}$$

As $t<1$ the first term gets smaller as s increases. We analyze the sum over n in (25) term by term. As n gets larger, $q^{n(s+2)}$ gets smaller so $-q^{n(s+2)}$ gets larger. However, for each i and j, $-q^{n(B_i-s)}$ and $+q^{n(s+1-A_j)}$ get smaller as s increases. Furthermore the powers of q appearing in any $q^{n(s+1-A_j)}$ are smaller than in $-q^{n(s+2)}$ which means that the amount any one of them is decreasing is greater than the amount that $-q^{n(s+2)}$ is increasing, so each term is getting smaller. Therefore the sum over all n is getting smaller and the quotients are decreasing. Thus, the absolute value of the sum has a unique maximum. Since the sum is alternating we conclude that the absolute value of the summand is less than or equal to the largest term. For any n, using $(q;q)_n \leq 1$ in the numerator and $(q;q)_n \geq (q;q)_\infty$ in the denominator of (21) together with the fact that the power of t is the largest when the exponent is the smallest, the expression in equation (21) is smaller than

$$(1-q)\sqrt{\frac{1}{((q;q)_\infty)^4}}\,t^{\min}\frac{1}{((q;q)_\infty)^7}, \tag{26}$$

where min is the smallest value of p'_s. Analyzing (22) we can see that p'_s is minimal when $s = M = (C_2 + C_3)/2$. Substituting we see that

$$p'_s \geq \frac{(C_1 - C_2)(C_1 - C_3)}{2} + C_1. \tag{27}$$

The final estimate is

$$\left| \begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}_u \right| \leq \frac{1-q}{((q;q)_\infty)^9} \cdot t^{\frac{(C_1-C_2)(C_1-C_3)}{2}+C_1}. \tag{28}$$

□

3.2. **Some important limits.** Let $\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$ be an admissible labeling of the edges of a tetrahedron. For any nonnegative integer k, the labelings $\begin{pmatrix} a+2k & b+2k & e+2k \\ c+2k & d+2k & f+2k \end{pmatrix}$, $\begin{pmatrix} a+2k & b+2k & e \\ c+2k & d+2k & f+2k \end{pmatrix}$, and $\begin{pmatrix} a+2k & b+2k & e \\ c+2k & d+2k & f \end{pmatrix}$ are admissible.

Proposition 7. *Given an admissible labeling* $\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$ *of a tetrahedral net, the sequences*

$$t^{-4k} \begin{Bmatrix} a+2k & b+2k & e+2k \\ c+2k & d+2k & f+2k \end{Bmatrix}_u, \tag{29}$$

$$t^{-4k} \begin{Bmatrix} a+2k & b+2k & e \\ c+2k & d+2k & f+2k \end{Bmatrix}_u, \tag{30}$$

and

$$t^{-4k} \begin{Bmatrix} a+2k & b+2k & e \\ c+2k & d+2k & f \end{Bmatrix}_u \tag{31}$$

are convergent.

We will only prove the first limit exists, the other two are similar. The proof is based on the following elementary lemma.

Lemma 1. *Suppose that $w(k)_n$ is a sequence of sequences so that for each fixed k, the sequence is alternating and converges to zero, and for fixed n the sequence is convergent. Suppose further that there exists N so that, independent of k, if $n \geq N$ then $|w(k)_n| \geq |w(k)_{n+1}|$. The sequence*

$$w(k)_\infty = \sum_n w(k)_n \tag{32}$$

(depending on k) is convergent.

Proof. This is an application of the proof of the alternating series test. □

Proof. (of Proposition 7) Recall Formula (21). The strand numbers increase by 1 each time k increases by 1 so the $(q;q)_{x_{v,i}}$ all converge to $(q;q)_\infty$ as k goes to infinity. Similarly, the functions in the denominator inside the radical all converge to $(q;q)_\infty$. Hence to prove the convergence we must only understand the quantities inside the sum.

Let $M(k)$, $m(k)$, $A_j(k)$, $B_i(k)$ and $p'(k)_s$ be the quantities in (21) associated to

$$\begin{Bmatrix} a+2k & b+2k & e+2k \\ c+2k & d+2k & f+2k \end{Bmatrix}_u, \tag{33}$$

as in the proof of Proposition 6. Let $n = M(k) - s$, and let

$$w(k)_n = t^{p'(k)_s - 4k} \frac{(-1)^s (q;q)_{s+1}}{\prod_{i=1}^3 (q;q)_{B_i(k)-s} \prod_{j=1}^4 (q;q)_{s-A_j(k)}} \tag{34}$$

for $n \leq M(k) - m(k)$, and $w(k)_n = 0$ for $n > M(k) - m(k)$.

As k increases by 1, the $B_i(k)$ increase by 4 and the $A_j(k)$ only increase by 3. So, $M(k) = M(0) + 4k$, $m(k) = m(0) + 3k$, $B_i(k) = B_i(0) + 4k$, and $A_j(k) = A_j(0) + 3k$. When $n = 0$, $s = M(k)$ and

$$p'(k)_{M(k)} = \frac{(C(k)_1 - C(k)_2)(C(k)_1 - C(k)_3)}{2} + C(k)_1, \tag{35}$$

which increases by 4 when k increases by 1, so $t^{-4k+p'(k)_{M(k)}}$ is a constant. We see that $w(k)_0$ is convergent. A similar analysis shows that for fixed n the sequence $w(k)_n$ is convergent. The series is clearly alternating.

We have already seen that for fixed k the sequence $|w(k)_n|$ can have at most one maximum, we just need to see that there is a bound on how big n is at that maximum depending only on t, and $\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$. We do this by looking at $\log |w(k)_n / w(k)_{n+1}|$ and seeing when it becomes nonnegative. When $n > M(0) - m(0) + k$ then $w(k)_n = 0$, so the maximum of $|w(k)_n|$ has already occured. Hence we only need to understand the case when the quotient $w(k)_n / w(k)_{n+1}$ is well defined. To this end we substitute into Formula (24) to get,

$$\log |w(k)_n / w(k)_{n+1}| = (12(M(0) - n) - 4(a+b+c+d+e+f) + 4) \log(t) \tag{36}$$

$$+\log(1-q^{M(0)+4k-n+2})-\sum_{i=1}^{4}\log(1-q^{M(0)+k-n+1-A_j(0)})+\sum_{i=1}^{3}\log(1-q^{(B_i(0)-M(0)+n)}).$$

Choose N large enough so that,

$$(12(M(0)-N)-4(a+b+c+d+e+f)+4)\log(t)> \\ -\log(1-q^{M(0)+4N-N+2})-\sum_{i=1}^{3}\log(1-q^{M(0)-N+2}). \tag{37}$$

Notice that $-\sum_{i=1}^{4}\log(1-q^{M(0)+k-n+1-A_j(0)})>0$. Inequality (37) guarantees that the expression (36) is positive when $k=N$. Increasing k in $-\log(1-q^{M(0)+4k-N+2})$ makes it smaller. Thus, by the argument from the proof of Proposition 6, the sequence $|w_k(n)|$ is monotone decreasing for $n\geq N$.

We have established the criterion for convergence from Lemma 1. □

Let the limit of the sequence (29) from Proposition 7 be denoted by:

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}_\infty = \lim_{k\to\infty} t^{-4k}\begin{Bmatrix} a+2k & b+2k & e+2k \\ c+2k & d+2k & f+2k \end{Bmatrix}_u. \tag{38}$$

Similarly, denote the limits of the sequences (30) and (31) by

$$\begin{Bmatrix} a & b & e_0 \\ c & d & f \end{Bmatrix}_\infty = \lim_{k\to\infty} t^{-4k}\begin{Bmatrix} a+2k & b+2k & e \\ c+2k & d+2k & f+2k \end{Bmatrix}_u, \tag{39}$$

and

$$\begin{Bmatrix} a & b & e_0 \\ c & d & f_0 \end{Bmatrix}_\infty = \lim_{k\to\infty} t^{-4k}\begin{Bmatrix} a+2k & b+2k & e \\ c+2k & d+2k & f \end{Bmatrix}_u. \tag{40}$$

Remark 2.

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}_\infty = (1-q)(q;q)_\infty\sum_{u=0}^{\infty}(-1)^{\frac{C_2+C_3}{2}+u} \\ t^{6u^2+2(2C_1-C_2-C_3+1)u+\frac{(C_1-C_2)(C_1-C_3)}{2}+C_1}\frac{1}{(q;q)_u(q;q)_{u+\frac{C_1-C_2}{2}}(q;q)_{u+\frac{C_1-C_3}{2}}} \tag{41}$$

The limits (39) and (40) are zero unless $a+c=b+d$.

4. Normal and Spinal Surfaces

4.1. **Analysis of the contribution of a surface.** For the remainder of this paper M will be a compact three-manifold with non-empty connected boundary. Although the method works for a more general class of manifolds, this assumption simplifies the arithmetic so that the ideas behind the estimates are in the foreground.

Definition 1. *An ideal triangulation T whose only normal spheres and tori are the link of a vertex is* **efficient**. *An ideal triangulation is* **0-efficient** *if and only if the only embedded, normal 2-spheres are vertex linking.*

The 0-efficient triangulations were studied in [6]. In particular, it is shown there that any triangulation of a closed, orientable irreducible 3-manifold can be modified to a 0-efficient triangulation, or it can be shown that the 3-manifold is one of S^3, $\mathbb{R}P^3$ or $L(3,1)$. It is also shown that any triangulation of a compact, orientable, irreducible and boundary irreducible 3-manifold with non-empty boundary can be modified to a 0-efficient triangulation. In the announced sequel to [6] authors explore the concept of **1-efficient manifolds**. They show that the triangulations of irreducible, atoroidal, closed 3-manifolds can be obtained so that in addition to being 0-efficient, any embedded normal torus is of a very special form or the 3-manifold is S^3, a lens space or a small Seifert fiber space.

Assume that M has an efficient triangulation T. Suppose that S is the spine dual to T. Let F be a surface carried by S. It is induced by an admissible coloring of the spine. Let u_f denote the color assigned to the face f. At each edge e the three faces sharing that edge carry colors a_e, b_e and c_e. At each vertex there is a corresponding coloring of a tetrahedral net, $\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}$. Denote the column sums at vertex v by $C_{1,v} \geq C_{2,v} \geq C_{3,v}$.

We can form the three strand numbers at each edge: $x_{e,1} = \frac{a_e+b_e-c_e}{2}$, $x_{e,2} = \frac{c_e+b_e-a_e}{2}$ and $x_{e,3} = \frac{a_e+c_e-b_e}{2}$. These are in fact linear functionals on the space of spinal surfaces. There is an arbitrariness to the choice of which function is which, so fix this choice along each edge once and for all. Similarly, at each vertex we can form three linear functionals $S_{1,v}$, $S_{2,v}$ and $S_{3,v}$ corresponding to the column sums of the tetrahedral net at the vertex.

Definition 2. *If C is a (possibly empty) set of simple closed curves on the boundary of M, let $\mathcal{S}(C)$ denote the set of spinal surfaces with respect to an augmentation of the spine corresponding to C. For brevity, let $\mathcal{S} = \mathcal{S}(\emptyset)$. A* **sector** *$\mathcal{F}$ is determined by fixing the order of the values of the $S_{i,v}$ at each vertex (that is, deciding which of the $S_{i,v}$'s is the largest column sum, $C_{1,v}$, etc.).*

Specifying these orderings at all vertices breaks the space of spinal surfaces into $6^{\#v}$ sectors. Given any infinite sequence of spinal surfaces we can find a subsequence that lives in one sector, because there are only finitely many sectors.

Proposition 8. *Suppose that the spinal surface F lies in the sector $\mathcal{F}$. Then every connected component of F lies in the same sector.*

Proof. Recall that the intersection of a spinal surface with a tetrahedron consists of triangles along with one family of disks having a particular curve type (a,b). The triangles contribute the same to each column of the corresponding symbol so any restriction on the sector comes from the curve type. Since all components of F are made up of a subset of the components of the intersection of F with each tetrahedron, they lie in any sector that F lies in (and maybe some other sectors too.) $\square$

Definition 3. *A spinal surface F is* k-**peelable** *if k is the maximum non-negative integer such that F can be written as $F = F' + k \cdot \partial M$. Use $\mathcal{S}_k(C)$ to denote the set of all surfaces in $\mathcal{S}(C)$ that are k-peelable. Similarly, use $\mathcal{F}_k(C)$ to denote the k-peelable surfaces in the sector $\mathcal{F}(C)$.*

There is a one-to-one correspondence between $\mathcal{S}_0$ and $\mathcal{S}_k$ for any $k \geq 0$ given by $u_f \to u_f + 2k$ for every f. Furthermore, this correspondence preserves sectors.

Proposition 9. *A spinal surface is in $\mathcal{S}_0$ if and only if some $x_{v,i} = 0$.*

Proof. This follows from Proposition 1 (on being able to subtract admissible triples). □

Consequently, a spinal surface is k-peelable if and only if the minimum over all strand numbers $x_{v,i}$ is equal to k.

Let $Q : \mathcal{S} \to \mathbb{Z}$ be the function that assigns to each surface F,

$$Q(F) = \sum_f -2u_f + \sum_v \frac{1}{2}(C_{1,v} - C_{2,v})(C_{1,v} - C_{3,v}) + C_{1,v}. \tag{42}$$

Proposition 10. **(i)** $-2\chi(F) \leq Q(F)$

(ii) *The function $Q(F)$ is super additive on any sector. That is, for any surfaces F, F' lying in the same sector, if $C_{i,v}$ are the column sums corresponding to F and $C'_{i,v}$ are the column sums corresponding to F' then*

(43)

$$Q(F+F') = Q(F) + Q(F') + (C_{1,v} - C_{2,v})(C'_{1,v} - C'_{3,v})/2 + (C'_{1,v} - C'_{2,v})(C_{1,v} - C_{3,v})/2$$

$$\geq Q(F) + Q(F')$$

(iii) *$Q(F)$ is bounded below on $\mathcal{S}_0$.*

(iv) *The level sets of $Q(F)$ on $\mathcal{S}_0$ are finite.*

(v) *The cardinality of the level sets of Q on $\mathcal{S}_0$ grows at most polynomially in the level.*

Proof. **(i)** The Euler characteristic of the surface F corresponding to the coloring u_f is

$$\sum_f u_f - \sum_e \frac{a_e + b_e + c_e}{2} + \sum_v \gcd(C_{1,v} - C_{2,v}, C_{1,v} - C_{3,v})/2 + C_{2,v} + C_{3,v} - C_{1,v} \tag{44}$$

Because each edge has exactly two ends we can redistribute the sum to eliminate the sum over the edges. This yields,

$$\sum_f u_f + \sum_v \gcd(C_{1,v} - C_{2,v}, C_{1,v} - C_{3,v})/2 + \frac{1}{2}C_{2,v} + \frac{1}{2}C_{3,v} - \frac{3}{2}C_{1,v}. \tag{45}$$

Comparing (45) to the right hand side of the inequality from item **(i)** we see that it is enough to show that for each vertex v,

(46)

$$-\gcd(C_{1,v}-C_{2,v},C_{1,v}-C_{3,v})-C_{2,v}-C_{3,v}+3C_{1,v}\leq\frac{1}{2}(C_{1,v}-C_{2,v})(C_{1,v}-C_{3,v})+C_{1,v}.$$

In the case that $C_{1,v}-C_{2,v}=0$ this reduces to $C_{1,v}\leq C_{1,v}$ thus the proposition is true. Assume that $C_{1,v}-C_{2,v}>0$. The triples at each vertex are admissible so $C_{1,v}-C_{2,v}\leq C_{1,v}-C_{3,v}$ are even and positive. Hence, $\gcd(C_{1,v}-C_{2,v},C_{1,v}-C_{3,v})\geq 2$. Substituting this in (46) and putting everything on the right side, the inequality is equivalent to:

$$\frac{1}{2}(C_{1,v}-C_{2,v}-2)(C_{1,v}-C_{3,v}-2)\geq 0. \tag{47}$$

Since we are assuming $C_{1,v}-C_{2,v}\geq 2$ and $C_{1,v}-C_{3,v}\geq 2$ this is true.

(ii) This is a direct computation from the formula.

In what follows we would like to use this formula, to that end we write it more compactly as follows. Letting F and F' be surfaces in the same sector with $\delta_v=C_{1,v}-C_{2,v}$, $\gamma_v=C_{1,v}-C_{3,v}$ being associated with F and $\delta'_v=C'_{1,v}-C'_{2,v}$, $\gamma'_v=C'_{1,v}-C'_{3,v}$ being associated with F', we have,

$$Q(F+F')=Q(F)+Q(F')+\sum_v\frac{\delta_v\gamma'_v+\delta'_v\gamma_v}{2}. \tag{48}$$

(iii) Since there are only finitely many sectors, if Q is bounded below on each sector, then it is bounded below on $\mathcal{S}_0$. So assume we are working in a particular sector. Suppose that Q is not bounded below. Starting with a surface with $Q<0$ we demonstrate the existence of another surface of a particular form with smaller $Q(F)$. We then bound Q below on surfaces of that form.

Suppose that $Q(F)<0$. Decompose F as a union F_p of components with positive Euler characteristic and a union F_n components with negative Euler characteristic. Since $Q(F)<0$, the surface F_p is nonempty. By superadditivity we have that $Q(F)\geq Q(F_p)+Q(F_n)$. Since $Q(F_n)\geq 0$ this implies that $Q(F_p)\leq Q(F)$. Since F_p is a subsurface of F, by Proposition 8 it is in the same sector. So we can assume that we are working with a surface all of whose components are spheres.

Next assume that F has $\delta_v\geq 4$ for some v. Our estimate that $Q(F)\geq -2\chi(F)$ tells us that if F has a single component then $Q(F)\geq -4$. Using the fact that $\delta_v\geq 4$ for some vertex allows us to improve this to $Q(F)\geq -2$ Assume that F is not connected. We can then write $F=F_1+F_2$ where the F_i are from $\mathcal{S}_0$, and F_2 is connected and has nonempty intersection with a small ball about v. We use $\delta_{v,1}$, $\delta_{v,2}$ to denote the differences between the largest column and second largest column of these two surfaces at the vertex v, and $\gamma_{v,1}$ and $\gamma_{v,2}$ to describe the difference between the largest column and the smallest column. Note, $\delta_v = \delta_{v,1} + \delta_{v,2}$ and $\gamma_v = \gamma_{v,1} + \gamma_{v,2}$. The super-additivity formula gives

$$Q(F) = Q(F_1 + F_2) = Q(F_1) + Q(F_2) + \sum_v \frac{\delta_{v,1}\gamma_{v,2} + \delta_{v,2}\gamma_{v,1}}{2}. \tag{49}$$

Since $\delta_{v,1} + \delta_{v,2} \geq 4$ it follows that $Q(F_1) \leq Q(F)$ and it has smaller δ_v. Replace the surface F with the surface F_1 and continue until all $\delta_v \leq 2$.

Suppose F is a surface in $\mathcal{F}_0$ with all $\delta_v \leq 2$. Since there are no normal spheres in $\mathcal{F}_0$ each sphere making up F has some $\delta_v = 2$. Since δ_v is additive this means that there are no more spheres in F than there are vertices in the spine. Hence Q is bounded below by $-4(\#\text{ vertices})$.

(iv) It is enough to prove that the intersection of any level set with any sector is finite. Suppose that F_i is an infinite sequence of spinal surfaces in a sector with $Q(F_i) = c$. If necessary we can pass to a subsequence so that the strand numbers of the surfaces F_i are monotone increasing. There are two cases.

Case 1 If the $C_{v,1} - C_{v,2} = \delta_v$ stay bounded then we can further refine the sequence so that all these numbers are constant. As the strand numbers are monotone increasing we can subtract the first term of the sequence from every subsequent term to get a new sequence of spinal surfaces which are normal. The values of $Q(F_i)$ are bounded below (by item **(iii)**), hence there is an infinite sequence of surfaces with the same Euler characteristic. Since these surfaces all have some strand number 0, and the triangulation is efficient they can be written as a sum of a finite list of normal surfaces so that none of the surfaces has positive or zero Euler characteristic. This is a contradiction, as their Euler characteristic is increasing.

Case 2 If some $C_{v,1} - C_{v,2} = \delta_v$ is unbounded we refine the sequence so that the δ_v are monotone increasing and the strand numbers are monotone increasing. Let v be a vertex where the δ_v are unbounded, and assume that the first surface in the sequence has $\delta_v > 0$. If not, just start later. Subtracting the first surface from every surface in the sequence the super-additivity formula informs us that this is a sequence of surfaces in $\mathcal{S}_0$ such that Q is not bounded below. This contradicts item **(iii)**.

(v) If V is a finite dimensional free $\mathbb{Z}$-module and v_i is a basis, we can define $N : V \to \mathbb{Z}$ by

$$N(\sum_i c_i v_i) = \sum_i |c_i|. \tag{50}$$

The cardinality of the set of elements in V with $N(v) \leq n$ is less than or equal to a polynomial in n. Fixing a sector $\mathcal{F}$ there is a finite family of surfaces F_i that generate the surfaces in $\mathcal{F}$ as an integer cone. As there are only finitely many surfaces F with $Q(F) \leq 0$, there is an integer K so that for any $\sum_i c_i F_i$, if some $c_i \geq K$ then $Q(\sum_i c_i F_i) > 0$. Let S_j be the set of all surfaces $\sum_i c_i F_i$, so that some c_i is between K and $2K$ and the other c_i are between 0 and $K-1$. It is clear that all but finitely many surfaces in $\mathcal{S}$ can be written as a positive sum of these surfaces. Form a free $\mathbb{Z}$-module with basis v_j corresponding to the S_j and define a map from the nonnegative integer sums of the v_j to $\mathcal{S}$ by sending the v_j to the S_j. This map is onto all but a finite subset of $\mathcal{F}$. Also,

$$N(\sum_i c_j v_j) \leq Q(\sum_j c_j S_j), \tag{51}$$

so the level set $Q(S) = n$ is the image of a subset of V contained inside the set $N(v) \leq n$. Therefore the level sets of Q grow at most polynomially in n.

□

Now suppose that C is a system of simple closed curves on ∂M. We consider colorings of an augmentation of the spine corresponding to C. Let $\chi(f)$ denote the Euler characteristic of the face f. Note that $\chi(f) = 1$ if f is an open disk, and $\chi(f) = 0$ if f is an annulus.

The space of surfaces corresponding to admissible colorings of the augmented spine is much like the space of spinal surfaces, except you can't add two augmented colorings. However, you can add an augmented coloring and any coloring of the original spine. We can divide the space of colorings of the augmented spine into sectors just like we did for spinal surfaces, and we can define k-peelable. Let $\mathcal{F}(C)$ be the surfaces in a sector coming from colorings of an augmentation of the spine, and denote by $\mathcal{F}$ the corresponding sector in space of spinal surfaces associated to the original spine. Use $\mathcal{F}(C)_k$ to denote the k-peelable surfaces in that sector. Define Q from the space of surfaces corresponding to admissible colorings of the augmented spine to the counting numbers by,

$$Q(F) = \sum_f -2\chi(f)u_f + \sum_v \frac{1}{2}(C_{1,v} - C_{2,v})(C_{1,v} - C_{3,v}) + C_{1,v}. \tag{52}$$

Proposition 11. **(i)** $-2x_F \leq Q(F)$

(ii) *The function $Q(F)$ is super additive on sectors. If $C_{i,v}$ are the column sums corresponding to $F \in \mathcal{F}(C)$ and $C'_{i,v}$ are the column sums corresponding to $F' \in \mathcal{F}$ then*

$$Q(F+F') = Q(F)+Q(F')+(C_{1,v}-C_{2,v})(C'_{1,v}-C'_{3,v})/2+(C'_{1,v}-C'_{2,v})(C_{1,v}-C_{3,v})/2 \geq Q(F)+Q(F'). \tag{53}$$

(iii) *$Q(F)$ is bounded below on $\mathcal{S}(C)_k$.*

(iv) *The level sets of $Q(F)$ on $\mathcal{S}(C)_k$ are finite.*

(v) *The cardinality of the level sets of Q on $\mathcal{S}(C)_k$ grows at most polynomially in the level.*

Proof. The proof is an extension of the proof of Proposition 10. The first two parts follow directly from the formula for Q.

The third part we argue as follows. First get the estimate on $\mathcal{S}(C)_0$ by working in sectors. Given a surface $F \in \mathcal{F}(C)_0$ it can be written as a sum of a surface F_1 such that each of its components has nonempty boundary and a surface F_2 each component of which is closed. From Proposition 10 we have a lower bound for $Q(F_2)$, from inequality **(i)** we can bound $Q(F_1)$ below by -2 times the number of components in C. By super-additivity we have bounded Q from below on $\mathcal{F}(C)_0$.

To bound Q below on $\mathcal{S}(C)_k$ use the one-to-one correspondence between surfaces in $\mathcal{S}(C)_k$ and surfaces in $\mathcal{S}(C)_0$ obtained by adding k copies of ∂M. Once again we bound the value of Q on k parallel copies of the boundary using the inequality from item **(i)** and then use the bound on $\mathcal{S}(C)_0$ and super-additivity on sectors.

The proofs of items **(iv)** and **(v)** are completely analogous to the proofs in Proposition 10. □

4.2. **Summing Over k-peelable Surfaces.** Let C be a system of simple closed curves in ∂M, let S be a spine that is dual to an efficient triangulation of M and let $S(C)$ be an augmentation of S with respect to C. Given a coloring F of the augmented spine $S(C)$ let the u_f, a_e, b_e, c_e, a_v, b_v, c_v, d_v, e_v, f_v and $\chi(f)$ be defined as before. Also, let $\chi(e) = 1$ if the edge e has some vertex in its closure and let $\chi(e) = 0$ otherwise ($\chi(e)$ is the Euler characteristic of the edge e).

Definition 4. *The contribution of F is defined to be*

$$E(F) = \frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e, b_e, c_e)^{\chi(e)}}. \tag{54}$$

Notice that faces and edges of the spine contribute to $E(F)$ unless they are annular or belong to the simple closed curves on the boundary respectively. Each vertex is an endpoint of four edges and each edge that counts in the contribution of a surface has two ends. We can thus collect the tetrahedral coefficient at each vertex with the thetas to reparse this product as

$$E(F) = \prod_f \Delta_{u_f}^{\chi(f)} \prod_v \begin{Bmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{Bmatrix}_u. \tag{55}$$

There is a map $S(C)_0 \to S(C)_k$ that adds k copies of the boundary of M (as a union of triangles near the vertex). This map is one to one and onto. If the largest color corresponding to F is N then the largest color corresponding to $F + k\partial M$ is $N + 2k$. We define

$$E_k(F) = E(F + k\partial M). \tag{56}$$

Since $\chi(M) = \#f - \#v$,

$$Q(F + k\partial M) = Q(F) - 4k\chi(M). \tag{57}$$

Using results of Proposition 7 about limits of $6j$ symbols we have,

Proposition 12. *For every surface $F \in \mathcal{S}(C)_0$, the limit*

$$\lim_{k\to\infty} t^{4k\chi(M)} E_k(F) \tag{58}$$

exists. When $C = \emptyset$, it is equal to

$$E_\infty(F) = \prod_f (-1)^{u_f} \frac{t^{-2u_f}}{1-q} \prod_v \begin{Bmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{Bmatrix}_\infty . \tag{59}$$

Proof. Assume first that $C = \emptyset$, thus $\chi(f) = 1$ for all f. Given a surface $F \in \mathcal{S}(C)_0$ and $k > 0$, use (55) together with (8) to express

$$E_k(F) = \prod_f (-1)^{-u_f-2k} t^{-2u_f-4k} \frac{1-q^{u_f+2k}}{1-q} \prod_v t^{4k} t^{-4k} \begin{Bmatrix} a_v+2k & b_v+2k & e_v+2k \\ c_v+2k & d_v+2k & f_v+2k \end{Bmatrix}_u . \tag{60}$$

Since $\chi(M) = \#f - \#v$, equation (60) can be rewritten as

$$t^{-4k\chi(M)} \prod_f (-1)^{u_f} t^{-2u_f} \frac{1-q^{u_f+2k}}{1-q} \prod_v t^{-4k} \begin{Bmatrix} a_v+2k & b_v+2k & e_v+2k \\ c_v+2k & d_v+2k & f_v+2k \end{Bmatrix}_u . \tag{61}$$

By Proposition 7, along with the fact that $\lim_{k\to\infty} \frac{1-q^{u_f+2k}}{1-q} = \frac{1}{1-q}$, limit (58) exists and is given by the formula (59).

In the case when $C \neq \emptyset$ the argument is similar. The product in (60) must be taken over all faces f with $\chi(f) \neq 0$ and for some of the vertices v we need to consider the limit of sequences (30) or (31) instead of the sequence (29) as in equation (61). □

Let $\mathcal{S}(C)_k^N$ be the subset of $\mathcal{S}(C)_k$ where the largest color u_f is less than or equal to N and let $\mathcal{S}(C)_k^{T(N)}$ be the subset of $\mathcal{S}(C)_k$ where the largest u_f is greater than N, the *tail* of the set. Clearly,

$$\mathcal{S}(C)_k = \mathcal{S}(C)_k^N \cup \mathcal{S}(C)_k^{T(N)}. \tag{62}$$

Lemma 2. *For every $\epsilon > 0$ there is N so that for all k,*

$$\sum_{F\in\mathcal{S}(C)_k^{T(N+2k)}} |E(F)| < t^{-4k\chi(M)}\epsilon, \tag{63}$$

where $\chi(M)$ is the Euler characteristic of the manifold M.

Moreover, for every $i \geq 0$, the limit

$$\lim_{k\to\infty} t^{4k\chi(M)} \sum_{F\in\mathcal{S}(C)_k^{k+i}} |E(F)| \tag{64}$$

exists.

Proof. Using (55) along with the estimate from Proposition 6, we see that,

$$|E(F)| \leq D(t, M, C)t^{Q(F)}, \tag{65}$$

where $D(t, M, C)$ is a number that only depends on t, the manifold M and the augmentation of the spine corresponding to C. From Proposition 11 the function $Q(F)$ is bounded below by some $Q_0 \in \mathbb{Z}$, and has finite level sets, so that the level set where Q takes on the value n has its cardinality bounded above by a polynomial $p(n)$. Comparing with

$$\sum_{n \geq Q_0} p(n)t^n, \tag{66}$$

the series

$$\sum_{F \in \mathcal{S}(C)_0} |E(F)| \tag{67}$$

is absolutely summable. This means that for each $\epsilon > 0$ there is N so that

$$\sum_{F \in \mathcal{S}(C)_0^{T(N)}} D(t, M, C)t^{Q(F)} < \epsilon. \tag{68}$$

Using equation (57) we have

$$\sum_{F \in \mathcal{S}(C)_k^{T(N+2k)}} D(t, M, C)t^{Q(F)} < t^{-4k\chi(M)}\epsilon. \tag{69}$$

Combining the above argument with Proposition 12 yields the existence of the limit (64). □

Remark 3. *The first part of this lemma can be restated as follows: for every $\epsilon > 0$ there exists N so that independent of k,*

$$\sum_{F \in \mathcal{S}(C)_0^{T(N)}} t^{4k\chi(\partial M)}|E_k(F)| < \epsilon. \tag{70}$$

Proposition 13. *Let*

$$Z_k(M) = \sum_{F \in \mathcal{S}(C)_k} E(F) = \sum_{F \in \mathcal{S}(C)_0} E_k(F), \tag{71}$$

and

$$|Z|_k(M) = \sum_{F \in \mathcal{S}(C)_k} |E(F)| = \sum_{F \in \mathcal{S}(C)_0} |E_k(F)| = . \tag{72}$$

For each k, $Z_k(M)$ and $|Z|_k(M)$ are well defined. Moreover, the limits $Z_\infty(M) = \lim_{k\to\infty} t^{4k\chi(M)} Z_k(M)$ and $|Z|_\infty(M) = \lim_{k\to\infty} t^{4k\chi(M)}|Z|_k(M)$ exist.

Proof. The well defined part of the proposition follows directly from Lemma 2.

In order to prove convergence of $Z_k(M)$, choose $\epsilon > 0$. There exists N so that for all k

$$\sum_{F\in\mathcal{S}(C)_k^{T(N+2k)}} D(t,M,C)t^{Q(F)} < t^{-4k\chi(M)}\epsilon/4. \tag{73}$$

By Proposition 12 there is a K so that if $k_1, k_2 > K$ then

$$|t^{4k_1\chi(M)}\sum_{F\in\mathcal{S}(C)_{k_1}}^{N} E(F) - t^{4k_2\chi(M)}\sum_{F\in\mathcal{S}(C)_{k_2}}^{N} E(F)| < \epsilon/2. \tag{74}$$

This means that

$$\begin{gathered} |t^{4k_1\chi(M)}Z_{k_1}(M) - t^{4k_2\chi(M)}Z_{k_2}(M)| \leq \\ |t^{4k_1\chi(M)}\sum_{F\in\mathcal{S}(C)_{k_1}}^{N} E(F) - t^{4k_2\chi(M)}\sum_{F\in\mathcal{S}(C)_{k_2}}^{N} E(F)| + \\ |t^{4k_1\chi(M)}\sum_{F\in\mathcal{S}(C)_{k_1}^{T(N+2k_1)}} E(F)| + |t^{4k_2\chi(M)}\sum_{F\in\mathcal{S}(C)_{k_2}^{T(N+2k_2)}} E(F)| \leq \\ \epsilon/2+\epsilon/4+\epsilon/4. \end{gathered} \tag{75}$$

As the sequence is Cauchy it converges. The same proof works for $|Z|_\infty$. □

5. The Invariant Sums

In the section we analyze the sum of contributions of all spinal surfaces in the three-manifold M with an efficient triangulation.

Given any integer $r \geq 3$, all the special functions, Δ_n, $\theta(a,b,c)$, $\mathrm{Tet}\begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$, are well defined for $t = e^{\frac{\pi i}{2r}}$ whenever $a,b,c,d,e,f \leq r-1$ and the condition $a+b+c \leq 2r-4$ is added to the definition of admissibility. Given a system C of disjoint simple closed curves in ∂M and an augmentation $S(C)$ of the spine dual to the triangulation of M, the (finite) sum over all r-admissible colorings of the faces of $S(C)$,

$$\sum_{r\text{-admissible colorings of } S(C)} \frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e,b_e,c_e)^{\chi(e)}}, \tag{76}$$

is a coefficient (corresponding to C) of a vector-valued invariant associated to M by the topological quantum field theory underlying the Turaev-Viro invariant of M at level r. Our idea is the extend the invariant away from the roots of unity. The first major step is to analyze the convergence of the infinite sums like (76), where $t = e^{\frac{\pi i}{2r}}$ is replaced by any $0 < t < 1$ and the colorings are admissible.

Theorem 1. *Let $S(C)^N$ denote the set of admissible colorings of $S(C)$ with all $u_f \leq N$.*

(i) *If the Euler characteristic of M is negative then*

$$\sum_{\text{admissible colorings } u_f \text{ of } S(C)} \frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e, b_e, c_e)^{\chi(e)}} \tag{77}$$

converges absolutely.

(ii) *If $\chi(M) = 0$ then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{S(C)^N} \frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e, b_e, c_e)^{\chi(e)}} \tag{78}$$

exists and is equal to $Z_\infty(M) = \sum_{F\in S_0(C)} E_\infty(F)$ which converges absolutely.

(iii) *If $\chi(M) = 1$ then*

$$\lim_{N\to\infty} t^{8N} \sum_{S(C)^{2N}} \frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e, b_e, c_e)^{\chi(e)}} \tag{79}$$

exists. Given a spinal surface F, let $m(F)$ denote the least even number greater than or equal to the maximal color corresponding to F. The limit (79) is equal to the sum of the absolutely convergent series:

$$\frac{1}{1-q} \sum_{F\in S_0(C)} t^{4m(F)} E_\infty(F). \tag{80}$$

Proof. **(i)** We need to show that the sequence of partial sums of the absolute values of the series (77) converges, that is,

$$\lim_{N\to\infty} \sum_{S(C)^N} \left| \frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e, b_e, c_e)^{\chi(e)}} \right| \tag{81}$$

exists. Notice that

$$\sum_{S(C)^N} \left| \frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e, b_e, c_e)^{\chi(e)}} \right| = \tag{82}$$

$$\sum_k \sum_{F\in\mathcal{S}(C)_k^N} |E(F)| < \sum_k |Z|_k(M).$$

Proposition 13 implies that the series $\sum_k |Z|_k(M)$ converges by comparison with the series $\sum_k t^{-4k\chi(M)}$.

(ii) First, regroup the finite sum in (78) according to k-peelable surfaces. That is, use the fact that $S(C)^N$ is a disjoint union of subsets $S(C)^N_k$ with $k = 0, \dots, N$ (since $\mathcal{S}(C)^N_k$ is empty for $k > N$). Thus,

$$\frac{1}{N}\sum_{S(C)^N}\frac{\prod_f \Delta_{u_f}^{\chi(f)} \prod_v \mathrm{Tet}\begin{pmatrix} a_v & b_v & e_v \\ c_v & d_v & f_v \end{pmatrix}}{\prod_e \theta(a_e, b_e, c_e)^{\chi(e)}} = \frac{1}{N}\sum_{k=0}^{N}\sum_{F\in\mathcal{S}(C)^N_k} E(F). \tag{83}$$

By Proposition 13 we can find K so that for all $k > K$ we have

$$|Z_k(M) - Z_\infty(M)| < \frac{\epsilon}{4}. \tag{84}$$

By Lemma 2 there exists n_1 so that for all k

$$|Z_k(M) - \sum_{F\in\mathcal{S}(C)^{n_1+k}_k} E(F)| < \frac{\epsilon}{4}. \tag{85}$$

Combining these, we get that for all $k > K$, all $n_0 \geq n_1$

$$|Z_\infty(M) - \sum_{F\in\mathcal{S}(C)^{k+n_0}_k} E(F)| < \frac{\epsilon}{2}. \tag{86}$$

Therefore, each of the $N - K - n_1 - 1$ terms of the sum

$$\sum_{k=K+1}^{N-n_1}\sum_{F\in\mathcal{S}(C)^N_k} E(F) \tag{87}$$

is at most $\frac{\epsilon}{2}$ away from $Z_\infty(M)$. Since $\lim_{N\to\infty}\frac{N-K-n_1-1}{N} = 1$ to finish the proof it suffices to show that the first $K+1$ terms and the last n_1 terms inside the outer sum on the right hand side of (83) are bounded regardless of the value of N. For the first $K+1$ terms notice that by (85)

$$|\sum_{k=0}^{K}\sum_{F\in\mathcal{S}(C)^N_k} E(F)| < K(\frac{\epsilon}{4} + B), \tag{88}$$

where $B = \max(|Z_0(M)|, |Z_1(M)|, \dots |Z_K(M)|)$. The fact that the last n_1 inner sums

$$|\sum_{k=N-n_1}^{N}\sum_{F\in\mathcal{S}(C)^N_k} E(F)| \tag{89}$$

are bounded regardless of N follows from the fact that for every i the limit

$$\lim_{k\to\infty} \sum_{F\in\mathcal{S}(C)_k^{k+i}} E(F) \tag{90}$$

exists (see Lemma 2).

(iii) Absolute convergence of the sum (80) follows from the existence of the universal bound on $|E_\infty(F)|$ for $F \in S_0(C)$. Since $\lim_{k\to\infty} t^{4k}E_k(F) = E_\infty(F)$, this in turn follows from a universal bound on $|t^{4k}E_k(F)|$ for $F \in S_0(C)$. By letting $\epsilon = \frac{1}{2}$ in Remark 3 we see that except for finitely many surfaces $F \in S_0(C)$, $t^{4k}E_k(F) < \frac{1}{2}$. Since each of the sequences $t^{4k}E_k(F)$ is convergent for the remaining surfaces, the quantities $|t^{4k}E_k(F)|$ are universally bounded for all surfaces $F \in S_0(C)$.

Our goal is to show that the sequence

$$t^{8N} \sum_{F\in\mathcal{S}(C)^{2N}} E(F) \tag{91}$$

converges to the sum (80). The first step is to rewrite the finite sum in (91) so that it is a sum over 0-peelable surfaces. We get,

$$\sum_{F\in\mathcal{S}_0(C)} \sum_{k=0}^{2N-m(F)} t^{8N}E_k(F). \tag{92}$$

The largest part of this sum is at the end, so we change variables to put the largest part at the beginning. Let $i = 2N - m(F) - k$. Substitution, along with splitting off an appropriate power of t, yields:

$$\sum_{F\in\mathcal{S}_0(C)} \sum_{i=0}^{2N-m(F)} t^{4m(F)+4i}t^{8N-4m(F)-4i}E_{2N-m(F)-i}(F). \tag{93}$$

From Remark 3 there exists K_0 so that, for all $i \geq K_0$,

$$\sum_{F\in\mathcal{S}_0(C)^{T(K_0)}} t^{4i\chi(M)}E_i(F) < \frac{\epsilon(1-q)}{4}, \tag{94}$$

thus

$$\sum_{F\in\mathcal{S}_0(C)^{T(K_0)}} \frac{t^{4m(F)}}{1-q}E_i(F) < \frac{\epsilon}{4}. \tag{95}$$

Estimating based on summing the geometric series $\sum_i t^{4i} = \frac{1}{1-q}$ we can truncate the sum (93) using any $K \geq K_0$ as follows and remain within $\epsilon/4$ of the original sum.

$$\sum_{F\in\mathcal{S}_0(C)^K}\sum_{i=0}^{2N-m(F)} t^{4m(F)+4i}t^{8N-4m(F)-4i}E_{2N-m(F)-i}(F). \tag{96}$$

Since by Proposition 11 the function $Q(F)$ is bounded below on $\mathcal{S}_0(C)$ there exists B so that for all F, N and i

$$t^{8N-4m(F)-4i}E_{2N-m(F)-i}(F) < B. \tag{97}$$

From the elementary theory of the geometric series there exists I so that for all $F \in \mathcal{S}_0(C)^K$,

$$\sum_{i\geq I}^{2N-m(f)} t^{4m(F)+4i}B < \epsilon/4. \tag{98}$$

This means we can truncate the sum (96) again as follows and remain within $\epsilon/4$ of the original sum:

$$\sum_{F\in\mathcal{S}_0(C)^K}\sum_{i=0}^{I} t^{4m(F)+4i}t^{8N-4m(F)-4i}E_{2N-m(F)-i}(F). \tag{99}$$

Using the fact that for any F, and for any fixed i,

$$\lim_{N\to\infty} t^{8N-4m(F)-4i}E_{2N-m(F)-i}(F) = E_\infty(F), \tag{100}$$

together with the fact that the number of terms of the sum (99) is bounded independent of N, we can choose N so large that the sum (99) is within $\epsilon/4$ of

$$\sum_{F\in\mathcal{S}_0(C)^K}\sum_{i=0}^{I} t^{4m(F)+4i}E_\infty(F), \tag{101}$$

leaving us within $\frac{3\epsilon}{4}$ of the original sum (91). Using the absolute convergence of $\sum_{F\in\mathcal{S}_0(C)} t^{4m(F)}E_\infty(F)$, and the fact that the bound B is still valid for $E_\infty(F)$, we can choose I large enough to make this last sum (101) within $\epsilon/4$ of

$$\sum_{F\in\mathcal{S}_0(C)}\sum_{i=0}^{\infty} t^{4m(F)+4i}E_\infty(F). \tag{102}$$

Summing the geometric series yields the final result. □

References

[1] R. Benedetti, C. Petronio, *Branched Standard Spines of Three-Manifolds*, Lecture Notes in Math 1653, Springer-Verlag, Berlin-Heidelberg-New York, 1997.

[2] D. Bullock, C. Frohman, J. Kania-Bartoszynska, *Understanding the Kauffman Bracket Skein Module*, J. Knot Theory Ramifications 8 (1999), no. 3, 265–277.

[3] D. Bullock, C. Frohman, J. Kania-Bartoszynska, *The Yang-Mills measure in the Kauffman bracket skein module*, Comment. Math. Helv. **78** (2003) 1-17

[4] C. Frohman, J. Kania-Bartoszynska, *A Quantum Obstruction to Embedding*, Math. Proc. Camb. Phil. Soc. **131**, (2001), 279-293.

[5] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, 1990.

[6] W. Jaco, H. Rubinstein, *0-Efficient Triangulations of Three-Manifolds*, xxx.lanl.gov/math.GT/0207158.

[7] L. H. Kauffman and S. Lins, *Temperley-Lieb recoupling theory and invariants of* 3*-manifolds*, Ann. of Math. Studies **143**, Princeton University Press, 1994.

[8] S.V. Matveev, *Transformations of Special Spines and the Zeeman Conjecture*, Math. USSR-Izv. **31** (1988), 423-434.

[9] G. Masbaum, P. Vogel, *3-valent graphs and the Kauffman bracket.*, Pacific J. Math. **164** (1994) 361-381.

[10] R. Pergallini, *Standard Moves for Standard Polyhedra and Spines*, Rendicotti Circ. Mat. Palermo **37** suppl. 18 (1988), 391-414.

[11] J. Roberts, *Classical* $6j$*-Symbols and the Tetrahedron*, Geometry and Topology, **3** (1999), pages 21–66.

[12] V. G. Turaev and O. Y. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology, **31** (1992), 866–902.

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