A MATRIX MODEL FOR QUANTUM SL_2

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Abstract. We describe a topological ribbon Hopf algebra whose elements are sequences of matrices. The algebra is a quantum version of $U(sl_2)$.

For each nonzero $t \in \mathbb{C}$ that is not a root of unity, we give a quantum analog $\overline{\mathcal{A}}_t$ of $U(sl_2)$. The underlying algebra of the model is $\prod_{n=1}^{\infty} M_n(\mathbb{C})$. Consequently, the algebra structure, which comes from matrix multiplication, is independent of the variable t.

Define $\mathcal{A}_t$ to be the unital Hopf algebra on X, Y, K, K^{-1}, with relations:

$$KX = t^2 XK, \quad KY = t^{-2} YK, \tag{1}$$

$$XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}}, \quad KK^{-1} = 1. \tag{2}$$

The comultiplication is the algebra morphism given by:

$$\Delta(X) = X \otimes K + K^{-1} \otimes X, \quad \Delta(Y) = Y \otimes K + K^{-1} \otimes Y,$$

$$\Delta(K) = K \otimes K.$$

The antipode is the antimorphism given by $S(X) = -t^2 X$, $S(Y) = -t^{-2} Y$, $S(K) = K^{-1}$, and the counit is the morphism given by $\epsilon(X) = \epsilon(Y) = 0$, and $\epsilon(K) = 1$.

The standard representations $\underline{m}$, where m is a nonnegative integer, of $\mathcal{A}_t$ have basis e_i, where i runs in integer steps from $-m/2$ to $m/2$. Hence as a vector space $\underline{m}$ has dimension $m+1$. Recall that

$$[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}},$$

and $[n]! = [n][n-1]\ldots[1]$.

The action of $\mathcal{A}_t$ is given by

$$\begin{aligned} X \cdot e_i &= [m/2 + i + 1] e_{i+1} \quad \text{but} \quad X \cdot e_{m/2} = 0, \\ Y \cdot e_i &= [m/2 - i + 1] e_{i-1} \quad \text{but} \quad Y \cdot e_{-m/2} = 0, \\ K \cdot e_i &= t^{2i} e_i. \end{aligned}$$

The representation $\underline{m}$ can be seen as a homomorphism

$$\rho_m : A_t \to M_{m+1}(\mathbb{C}).$$

Lemma 1. *The homomorphisms $\rho_m : A_t \to M_{m+1}(\mathbb{C})$ are onto.*

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Proof. Using the ordered basis, $\{e_{-m/2}, \dots e_{m/2}\}$, $\rho_m(X)$ is the matrix that is zero except on the first subdiagonal, where the entries going from the top to the bottom are $1, [2], [3], \dots, [m]$. Similarly, the matrix $\rho_m(Y)$ is zero except on the first superdiagonal, where starting from the bottom and going up the entries are $1, [2], [3], \dots, [m]$. The image of X^nY^p is a matrix with zero entries except on a particular super- or sub-diagonal , whose distance from the diagonal is $|n-p|$. Starting from the top, the first $\min\{p, n\}$ entries of that diagonal are zero, and the subsequent entries are all nonzero. Thus there exist linear combinations of the matrices $\rho_m(X^nY^p)$, with $p \leq n$, corresponding to each of the elementary matrices whose only nonzero entry lies on the $n-p$ subdiagonal, or on the diagonal. We are using the pattern of zero and nonzero entries on the $n-p$ subdiagonal to see this. By a similar analysis of $\rho_m(Y^pX^n)$ we see that all elementary matrices where the nonzero entry lies on a superdiagonal can be written as a linear combination of the $\rho_m(Y^pX^n)$. Since $M_{m+1}(\mathbb{C})$ is spanned by the elementary matrices, this finishes the proof. ☐

Define the linear functionals ${}^mc^i_j : A_t \to \mathbb{C}$ by letting ${}^mc^i_j(Z)$ be the ij-th coefficient of the matrix $\rho_m(Z)$. Let ${}_qSL_2$ be the stable subalgebra of the Hopf algebra dual A_t^o generated by linear functionals ${}^mc^i_j$.

Proposition 1. *The linear functionals ${}^mc^i_j$ form a basis for the algebra ${}_qSL_2$.*

Proof. Since

$$\underline{m} \otimes \underline{n} = \bigoplus_{q=|m-n|}^{m+n} \underline{q},$$

the linear functionals ${}^mc^i_j$ span the algebra ${}_qSL_2$. We need to show that they are also linearly independent. The quantum Casimir is given by

$$C = \frac{(tK - t^{-1}K^{-1})^2}{(t^2 - t^{-2})^2} + YX \in \mathcal{A}_t. \tag{3}$$

Since C is central in $\mathcal{A}_t$, it acts as scalar multiplication in any irreducible representation. In fact, it acts on $\underline{m}$ as $\lambda_m = \frac{(t^{m+1}-t^{-m-1})^2}{(t^2-t^{-2})^2}$. Let

$$C_{m,n} = \frac{C - \lambda_n}{\lambda_m - \lambda_n}. \tag{4}$$

Notice that $C_{m,n}$ is zero under ρ_n and is sent to the identity in ρ_m. The product

$$D_{m,N} = \prod_{p=1, p\neq m}^{N} C_{m,n} \tag{5}$$

is an element of $\mathcal{A}_t$ that is sent to 0 in all of the representations from $\underline{1}$ to $\underline{N}$, except $\underline{m}$ where it is sent to the identity matrix.

If some linear combination $\sum \alpha_{i,j,n}{}^nc^i_j$ is equal to zero, it means that for all $Z \in \mathcal{A}_t$,

$$\sum_{i,j,n} \alpha_{i,j,n}{}^nc^i_j(Z) = 0.$$

Let N be the largest n for which $\alpha_{i,j,n} \neq 0$. For each m such that $\alpha_{i,j,m} \neq 0$, apply the functional $\sum_{i,j,n} \alpha_{i,j,n} {}^n c^i_j$ to $D_{m,N}Z$. Since

$$\sum_{i,j,n} \alpha_{i,j,n} {}^n c^i_j (D_{m,N}Z) = 0,$$

it follows that for all $Z \in \mathcal{A}_t$, and fixed m,

$$\sum_{i,j} \alpha_{i,j,m} {}^m c^i_j (Z) = 0.$$

Finally, from lemma 1 the homomorphisms ρ_m are surjective, so the independence of the ${}^m c^i_j$ follows from the independence of the matrix coefficients on $M_{m+1}(\mathbb{C})$. Therefore all the $\alpha_{i,j,m} = 0$. □

The product of any two matrix coefficients can be written as a linear combination of matrix coefficients

$$ {}^m c^i_j \cdot {}^n c^k_l = \sum_{u,v,p} \gamma^{i,j,m,k,l,n}_{u,v,p}(t) \; {}^p c^u_v \tag{6}$$

Since the functionals ${}^p c^u_v$ are linearly independent, the coefficients $\gamma^{i,j,m,k,l,n}_{u,v,p}(t)$ are uniquely defined. The $\gamma^{i,j,m,k,l,n}_{u,v,p}(t)$ are versions of the Clebsch-Gordan coefficients. Notice that $|m-n| \leq p \leq m+n$, consequently for each tuple (i,j,m,k,l,n) there are only finitely many (u,v,p) with $\gamma^{i,j,m,k,l,n}_{u,v,p}(t) \neq 0$.

A similar computation can be performed with the analogously defined ${}^m c^i_j$ associated to $Sl_2(\mathbb{C})$. The limit as t approaches 1 of the coefficients $\gamma^{i,j,m,k,l,n}_{u,v,p}(t)$ gives the corresponding quantities for $Sl_2(\mathbb{C})$.

Let

$$\overline{\mathcal{A}_t} = M_1(\mathbb{C}) \times M_2(\mathbb{C}) \times M_3(\mathbb{C}) \times \dots$$

be the Cartesian product of all the matrix algebras over $\mathbb{C}$ given the product topology.

Proposition 2. *The homomorphism*

$$\Theta : \mathcal{A}_t \to \overline{\mathcal{A}_t} \tag{7}$$

given by $\Theta(Z) = (\rho_0(Z), \rho_1(Z), \rho_2(Z), \dots)$ *is injective and its image is dense in* $\overline{\mathcal{A}_t}$.

Proof. The fact that the ρ_m are onto and the existence of the elements $C_{m,n}$ defined by equation (4) can be used to prove that the image of Θ is dense in $\overline{\mathcal{A}_t}$.

A version of the Poincaré-Birkhoff-Witt theorem says that the monomials $K^m X^n Y^p$ form a basis for $\mathcal{A}_t$ as a vector space. Using the relation $XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}}$, this can be replaced by the basis $Z_{m,n,p}$, with $Z_{m,n,p} = K^m X^n Y^p$ for $n \geq p$ and $Z_{m,n,p} = K^m Y^p X^n$, when $n < p$. In order to prove that the map Θ is injective, consider an element $\sum \alpha_i Z_{m_i,n_i,p_i} \in \mathcal{A}_t$. It is our goal to show that if $\Theta(\sum \alpha_i Z_{m_i,n_i,p_i}) = 0$ then all α_i are zero.

In any representation the image of $Z_{m,n,p}$ is a matrix that is zero off of the super (or sub)-diagonal corresponding to $n-p$. Thus it suffices to consider the sums where

$n_i - p_i$ is a constant, as long as we only work with the parts of the matrices in the image that lie on the super- or sub- diagonal corresponding to that constant.

Assume that $n_i \geq p_i$, The argument is similar when $n_i < p_i$. Suppose that, for $k \geq 0$, the image under Θ of

$$\sum \alpha_i K^{m_i} X^{p_i+k} Y^{p_i}, \tag{8}$$

on the kth subdiagonal is zero. The map Θ takes $K^m X^{p+k} Y^p$ to a sequence of matrices such that the first p entries along the k-th subdiagonal are zero. Let p be the minimum of the p_i appearing in (8). The $(p+1)$-st entry of each k-th subdiagonal of each matrix in the sequence $\Theta(\sum \alpha_i K^{m_i} X^{p_i+k} Y^{p_i})$ is the image under Θ of the collection of terms in (8) with $p_i = p$. All the other terms are mapped to matrices with a zero there. Thus it is enough to show that whenever all the $(p+1)$-st entries on the k-th subdiagonal in each entry of $\Theta(\sum \alpha_i K^{m_i} X^{p+k} Y^p)$ are zero, then all α_i are zero.

Assume that all the $(p+1)$-st entries on the k-th subdiagonal of $\Theta(\sum \alpha_i Z_{m_i,p+k,p})$ are zero. Make a sequence consisting of the $(p+1)$-st entries of the k-th diagonal of the image of $Z_{0,p+k,p}$. This sequence is:

$$(0, 0, \ldots, [p+k]! \prod_{r=1}^{p} [k+r], [p+k]! \prod_{r=1}^{p} [k+r+1], \ldots),$$

where the first nonzero entry corresponds to the representation ρ_{p+k+1}. Hence, the sequence corresponding to $Z_{m_i,p+k,p}$ is

$$(0, 0, \ldots, t^{m_i(p+k)} [p+k]! \prod_{r=1}^{p} [k+r], t^{m_i(p+k-1)} [p+k]! \prod_{r=1}^{p} [k+r+1], \ldots).$$

Supposing that we have J terms in our sum, we can truncate these sequences to get a $J \times J$ matrix, so that the coefficients α_i as a column vector, must be in the kernel of that matrix. Notice that the coefficient of the power of t in each column is the same product of quantized integers. Hence its determinant is a product of quantized integers times the determinant of the matrix,

$$\begin{pmatrix} t^{m_1(p+k)} & t^{m_1(p+k-1)} & \cdots \\ t^{m_2(p+k)} & t^{m_2(p+k-1)} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}.$$

Factoring out a large power of t from each row we get the Vandermonde determinant,

$$\begin{vmatrix} 1 & t^{-m_1} & t^{-2m_1} & \cdots \\ 1 & t^{-m_2} & t^{-2m_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix},$$

which is nonzero as long as the t^{m_i} are not equal to one another. Since t was chosen specifically not to be a root of unity, all the α_i must be zero. ☐

The topology induced on $\overline{\mathcal{A}}_t$ by its image under Θ is the weak topology from ${}_qSL_2$. That is a sequence Z_n is Cauchy if for every $\phi \in {}_qSL_2$, $\phi(Z_n)$ is a Cauchy sequence of complex numbers. Hence $\overline{\mathcal{A}_t}$ is the completion of $\mathcal{A}_t$ by equivalence classes of Cauchy sequences in the weak topology from ${}_qSL_2$.

Let $e_{i,j}(m) \in \overline{\mathcal{A}_t}$ be the sequence of matrices that is the zero matrix in every entry, except the $m+1$-st, where it is the elementary matrix that is all zeroes except for a 1 in the ij-th entry. Notice that the $e_{i,j}(m)$ are dual to the ${}^m c^i_j$ in the sense that ${}^m c^i_j(e_{k,l}(p))$ is zero unless the indices are identical, in which case it is one. Also notice that any $A \in \overline{\mathcal{A}_t}$ can be written uniquely as $\sum_{i,j,m} \alpha_{i,j,m} e_{i,j}(m)$. The infinite sum makes sense!

Proposition 3. *The algebra $\overline{\mathcal{A}_t}$ has a structure of a topological ribbon Hopf algebra.*

Proof. We need to define comultiplication on $\overline{\mathcal{A}_t}$. Every element of $\overline{\mathcal{A}_t} \otimes \overline{\mathcal{A}_t}$ can be written as an infinite sum,

$$\sum_{i,j,m,k,l,n} \tau_{i,j,m,k,l,n} e_{i,j}(m) \otimes e_{k,l}(n) \tag{9}$$

so that no $e_{i,j}(m) \otimes e_{k,l}(n)$ is repeated. There are infinite sums of this form that cannot be decomposed as a finite sum of tensors of elements of $\overline{\mathcal{A}_t}$. We topologize $\overline{\mathcal{A}_t} \otimes \overline{\mathcal{A}_t}$ by saying that a sequence W_n is Cauchy if and only if for every ${}^m c^i_j \otimes {}^n c^k_l$ the sequence $\left({}^m c^i_j \otimes {}^n c^k_l\right)(W_n)$ is Cauchy. Let $\overline{\mathcal{A}_t \otimes \mathcal{A}_t}$ be the completion of $\overline{\mathcal{A}_t} \otimes \overline{\mathcal{A}_t}$ by equivalence classes of Cauchy sequences. Notice that every sum of the type like in equation (9) yields an equivalence class of Cauchy sequences in $\overline{\mathcal{A}_t} \otimes \overline{\mathcal{A}_t}$ by truncating to get a sequence of partial sums. Conversely, if $Z_n \in \overline{\mathcal{A}_t} \otimes \overline{\mathcal{A}_t}$ is Cauchy, by applying the ${}^m c^i_j \otimes {}^n c^k_l$ to the sequence, and taking the limit we get the coefficients of a unique expression of the type (9), and two Cauchy sequences are equivalent if and only if they give rise to the same expression. Hence we can identify $\overline{\mathcal{A}_t \otimes \mathcal{A}_t}$ with the set of expressions like in equation (9).

In order to define the comultiplication on $\overline{\mathcal{A}_t}$ with values in $\overline{\mathcal{A}_t \otimes \mathcal{A}_t}$, take the adjoint of multiplication on ${}_qSL_2$. Use $< \, , \, >$ to denote evaluation of elements of $\overline{\mathcal{A}_t}$ on ${}_qSL_2$, and extend this to evaluating elements of ${}_qSL_2 \otimes {}_qSL_2$ on elements of $\overline{\mathcal{A}_t \otimes \mathcal{A}_t}$ pairwise. Then,

$$< {}^m c^i_j \otimes {}^n c^k_l, \Delta(e_{u,v}(q)) > = < {}^m c^i_j \cdot {}^n c^k_l, e_{u,v}(q) > = \gamma^{u,v,q}_{i,j,m,k,l,n}.$$

Therefore,

$$\Delta(e_{u,v}(q)) = \sum_{i,j,m,k,l,n} \gamma^{u,v,q}_{i,j,m,k,l,n} e_{i,j}(m) \otimes e_{k,l}(n).$$

The sum makes sense for an arbitrary element of $\overline{\mathcal{A}_t}$ as there are only finitely many nonzero $\gamma^{u,v,q}_{i,j,m,k,l,n}$ for any $e_{i,j}(m) \otimes e_{k,l}(n)$. So one can sum

$$\Delta(\sum_{i,j,m} \alpha_{u,v,p} e_{u,v}(q)) = \sum \alpha_{i,j,m} \Delta(e_{i,j}(m)) =$$

$$\sum \alpha_{i,j,m} \gamma^{u,v,q}_{i,j,m,k,l,n}(t) e_{i,j}(m) \otimes e_{k,l}(n).$$

Comultiplication is continuous since its composition with every ${}^m c^i_j \otimes {}^n c^k_l$ is continuous.

Let $q = t^4$. The standard formula for the universal R-matrix [1] in the Jimbo-Drinfeld model of $U_h(sl_2)$ is

$$R = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{[n]} q^{-n(n+1)/2} t^{H \otimes H + nH \otimes 1 - 1 \otimes nH} (X^n \otimes Y^n). \tag{10}$$

Recall that the standard Drinfeld-Jimbo model [1] of $U_h(sl_2)$ is generated by X, Y, H. If we let $K = t^H$ then the relations (1), (2) for $\mathcal{A}_t$ can be derived from the relations for the Drinfeld-Jimbo model. Consequently, interpret H as the traditional image of H under the standard irreducible representations of $U(sl_2)$. That is, H is the sequence of matrices,

$$(1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \ldots).$$

Taking t raised to this sequence gives the sequence $\Theta(K)$, where Θ is defined in equation (7). Interpret X and Y as the sequences of matrices coming from the standard representations of $\mathcal{A}_t$, i.e., $\Theta(X)$ and $\Theta(Y)$. The resulting expression (10) makes sense as an element of $\overline{\mathcal{A}}_t \otimes \overline{\mathcal{A}}_t$ since in any particular irreducible representation only finitely many terms are nonzero. Thus the R matrix is well defined as an element of $\overline{\mathcal{A}}_t \otimes \overline{\mathcal{A}}_t$, and has the desired properties. $\square$

References

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