Hopf algebra of discrete representation type

Shijie Zhu
(Joint with M. Iovanov, E.Sen, A. Sistko)

GMRC, University of Missouri

November 25, 2019

Notations:

Assume (co)algebras are over an algebraically closed field k.

Notations:

- Assume (co)algebras are over an algebraically closed field k.
- An algebra A is basic if simple A-modules are 1 dimensional over k.

Notations:

Assume (co)algebras are over an algebraically closed field k.

An algebra A is basic if simple A-modules are 1 dimensional over k.

A coalgebra C is pointed if simple C-comodules are 1-dimensional over k.

Notations:

Assume (co)algebras are over an algebraically closed field k.

An algebra A is basic if simple A-modules are 1 dimensional over k.

A coalgebra C is pointed if simple C-comodules are 1-dimensional over k.

An algebra A is finite representation type if there are only finitely many isomorphism classes of indecomposable A-modules.

Notations:

Assume (co)algebras are over an algebraically closed field k.

An algebra A is basic if simple A-modules are 1 dimensional over k.

A coalgebra C is pointed if simple C-comodules are 1-dimensional over k.

An algebra A is finite representation type if there are only finitely many isomorphism classes of indecomposable A-modules.

A coalgebra C is finite (co-)representation type if there are only finitely many isomorphism classes of indecomposable C-comodules.

Background

Representation types of finite dimensional algebras is a fundamental question in representation theory.

> Let G be a finite group. When is the group algebra kG of representation finite type?

Background

Representation types of finite dimensional algebras is a fundamental question in representation theory.

Let G be a finite group. When is the group algebra kG of representation finite type?

When char $k \nmid |G|$, kG is semisimple. Hence it is finite representation type.

Background

Representation types of finite dimensional algebras is a fundamental question in representation theory.

Let G be a finite group. When is the group algebra kG of representation finite type?

When char $k \nmid |G|$, kG is semisimple. Hence it is finite representation type.

When $p =$char $k \mid |G|$, kG is representation finite type if and only if Sylow p subgroups are cyclic.

What about G is a algebraic group? quantum group? Hopf algebras?

What about G is a algebraic group? quantum group? Hopf algebras?

$$rep(G) \cong \mathcal{O}(G) - comod$$

What about G is a algebraic group? quantum group? Hopf algebras?

$$rep(G) \cong \mathcal{O}(G) - comod$$

Problem is equivalent to study comodules over Hopf algebras.

What about G is a algebraic group? quantum group? Hopf algebras?

$$rep(G) \cong \mathcal{O}(G) - comod$$

Problem is equivalent to study comodules over Hopf algebras.

Finite representation type $\to$ Discrete representation type

Definition

Let C be a pointed coalgebra. We say that C is of discrete representation type, if for any finite dimension vector $\underline{d}$, there are only finitely many isoclasses of representations of dimension vector $\underline{d}$.

Our goal is to give a characterization of (possibly infinite dimensional) pointed Hopf algebras of discrete representation type by quivers.

Theorem (Liu-Li [2], 2007)

A finite dimensional basic hopf algebra H over an algebraically closed field k is finite representation type if and only if it is Nakayama. i.e. every indecomposable H-module is uniserial.

Dually, a finite dimensional pointed hopf algebra H over an algebraically closed field k is finite co-representation type if and only if H^* is Nakayama.

Their proof relies on Green and Solberg's covering quiver technique [1].

Path coalgebra: Let $Q = (Q_0, Q_1)$ be a (possibly infinite) quiver. The path coalgebra kQ^c is spanned by all the paths in Q with comultiplication $\Delta(p) = \sum_{p=p_1p_2} p_1 \otimes p_2$; the counit $\epsilon(e_i) = 1$ and $\epsilon(p) = 0$ for $|p| > 0$.

Example: $Q : 3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1$.

$\Delta(\beta\alpha) = \beta\alpha \otimes e_3 + \beta \otimes \alpha + e_1 \otimes \beta\alpha$

Theorem (Gabriel)

A connected basic algebra A is isomorphic to a quiver algebra kQ/I for some admissible ideal I.

Dually,

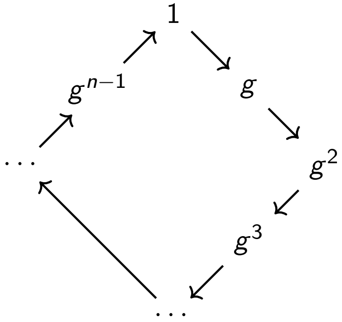
Theorem

A connected pointed coalgebra C is isomorphic to a certain subcoalgebra of a path coalgebra kQ^c.

Here Q is called the Ext-quiver of C.

Given C, how to find Q?
Vertices= group-likes g i.e. $\Delta(g) = g \otimes g$.
Number of arrows $g \to h = \dim_k P(g,h) - 1$, where
$P(g,h) = \{x | \Delta(x) = g \otimes x + x \otimes h\}$ is called the set of g-h skew-primitive elements.

Example: Taft algebra $T_n =< g, x | g^n = 1, x^n = 0, gxg^{-1} = qx >$, where q is a primitive $n-th$ root of unity. The coalgebra structure is given by $\Delta(g) = g \otimes g$, $\Delta(x) = 1 \otimes x + x \otimes g$.
The Ext quiver Q of T_n is



Theorem (Iovanov, Sen, Sistko, Zhu)

If H is a connected pointed Hopf algebras of discrete representation type, then the Ext *quiver of H is one of following:*

(1) *A complete oriented cycle;*

(2) $\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots;$

(3)

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & b^2 & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdots \\
 & & y^2\uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & b & \longrightarrow & ab & \longrightarrow & \cdot & \longrightarrow & \cdots \\
 & & y\uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & 1 & \xrightarrow{x} & a & \xrightarrow{x^2} & a^2 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

(4) The quiver in (3) identifying vertices $a^m = b^n$. (The quiver looks like a tube.)

(4) The quiver in (3) identifying vertices $a^m = b^n$. (The quiver looks like a tube.)

Algebra structures:

(4) The quiver in (3) identifying vertices $a^m = b^n$. (The quiver looks like a tube.)

Algebra structures:

$$ab = ba, a^{-1}xa = -x, b^{-1}xb = -\lambda x, a^{-1}ya = -\lambda^{-1}y, b^{-1}yb = -y;$$

$$x^2 = s(1 - a^2), y^2 = t(1 - b^2), xy + \lambda yx = k(1 - ab).$$

E. Green, and Ø. Solberg, Basic Hopf algebras and quantum groups, Math.Z 229 (1998), 45-76. MR1649318 (2000h:16049).

G. Liu, F. Li, *Pointed Hopf algebras of finite corepresentation type and their classifications*, Proc. Amer. Math. Soc **135** (2007), No.3, 649–657.