Mutation of type D friezes

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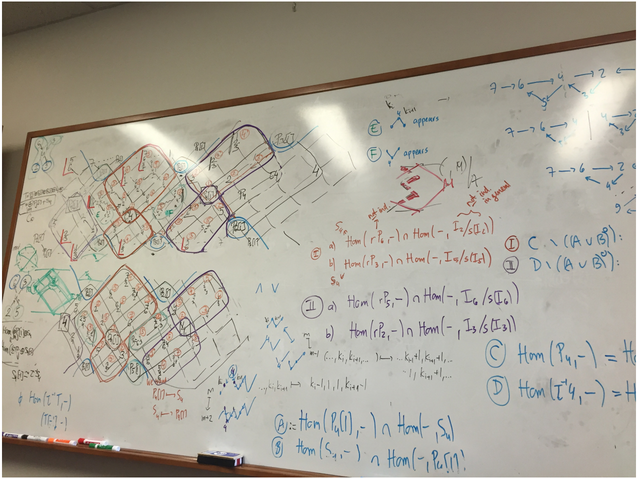
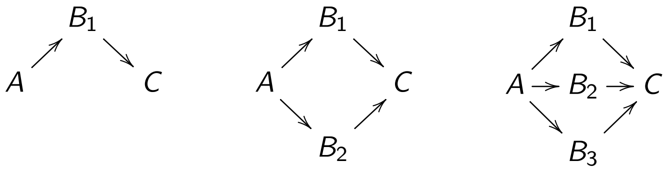


Problem: Define and study mutation of friezes that is compatible with cluster mutation, [Baur-Faber-Graz-S-Todorov] for type A.

Friezes

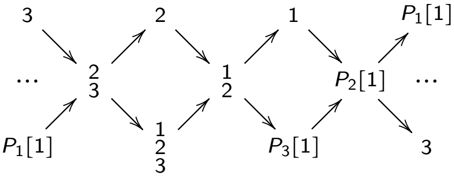
Let B be a cluster-tilted algebra of finite type. A frieze is an assignment of positive integers $F(M)$ for every element M of $\operatorname{ind} B$ and $\operatorname{ind} B[1]$, subject to mesh relations.



$$F(A)F(C) - \prod F(B_i) = 1$$

Frieze of type A

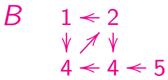
$B = k(1 \to 2 \to 3)$

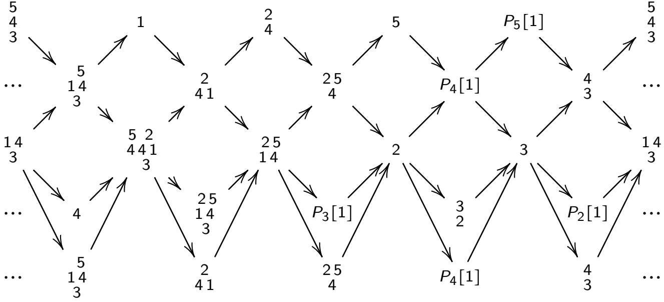


Frieze of type A

	2		2		2		1	
⋯		3		3		1		⋯
	1		4		1		2	

Frieze of type D



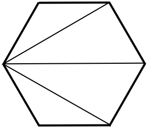


4		2		3		2		1		4
⋯	7		5		5		1		3	⋯
5		17		8		2		2		5
⋯	6		3		3		1		3	⋯
⋯	2		9		1		3		1	⋯

Bijections

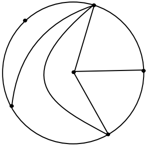
Theorem. [Conway-Coxeter, Baur-Marsh, Caldero-Chapoton, BMRRT, Schiffler, ...]

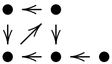
$\left\{ \begin{array}{c} \text{triangulations of} \\ \text{polygons and once-} \\ \text{punctured disks} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{cluster-tilted alg.} \\ \text{of type } A \text{ and } D \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(unitary) friezes} \\ \text{of type } A \text{ and } D \end{array} \right\}$



$\bullet \to \bullet \to \bullet$

frieze of type A





frieze of type D

Bijections

Theorem. [Conway-Coxeter, Baur-Marsh, Caldero-Chapoton, BMRRT, Schiffler, ...]

{ triangulations of polygons and once-punctured disks } $\longleftrightarrow$ { cluster-tilted alg. of type A and D } $\longleftrightarrow$ { (unitary) friezes of type A and D }

Given a cluster-tilted algebra B and $M \in \operatorname{mod} B$

$$F(M) = \sum_{N \subseteq M} \chi(\mathrm{Gr}_{\underline{\dim}\, N} M) \text{ and } F(P_i[1]) = 1$$

In type A we have $F(M) = \sum_{N \subseteq M} 1$

Bijections

Theorem. [Conway-Coxeter, Baur-Marsh, Caldero-Chapoton, BMRRT, Schiffler, ...]

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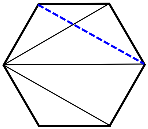
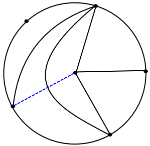
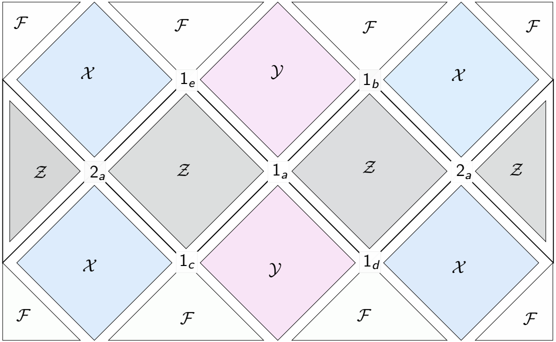
Problem: Define and study mutation of friezes that is compatible with cluster mutation.





Mutation of type A friezes



Theorem. [Baur-Faber-Graz-S-Todorov] Let m be an entry in a frieze of type A and m' the entry at the same place after mutation at arc a. Then $\delta_a(m) = m - m'$ is given by:

If $m \in \mathcal{X}$ then $\delta_a(m) = [\pi_1^+(m) - \pi_2^+(m)]\,[\pi_1^-(m) - \pi_2^-(m)]$
If $m \in \mathcal{Y}$ then $\delta_a(m) = -[\pi_2^+(m) - 2\pi_1^+(m)]\,[\pi_2^-(m) - 2\pi_1^-(m)]$
If $m \in \overline{\mathcal{Z}}$ then $\delta_a(m) = \pi_s^{\downarrow}(m)\pi_p^{\downarrow}(m) + \pi_s^{\uparrow}(m)\pi_p^{\uparrow}(m) - 3\,\pi_p^{\downarrow}(m)\pi_p^{\uparrow}(m)$
If $m \in \mathcal{F}$ then $\delta_a(m) = 0$.

$\pi_*(m)$ are certain projections of m onto the boundary of $\mathcal{Z}$. [Result relies heavily on the representation theory of modules of type A.]

From type D to type A

This approach appears in [Essonana Magnani] to study cluster variables in type D as cluster variables in type A.

Type D

4		2		3		2		1		4		2
···	7		5		5		1		3		7	···
5		17		8		2		2		5		17
···	6		3		3		1		3		2	···
···	2		9		1		3		1		6	···

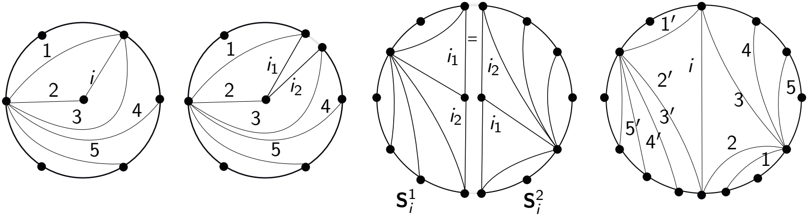
Glued Type D

4		2		3		2		1		4		2
···	7		5		5		1		3		7	···
5		17		8		2		2		5		17
···	12		27		3		3		3		12	···

Next, complete this glued type D pattern to a frieze of type A such that this completion behaves well with mutations. The precise operation is easily seen on the level of surface triangulations.

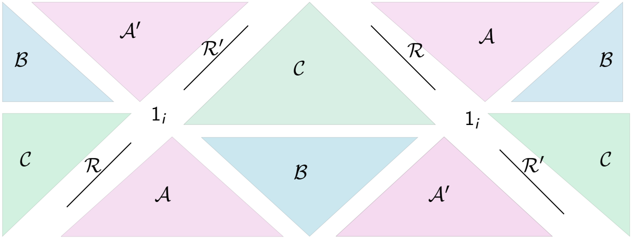
From type D to type A

Let **T** be a triangulation of a once punctured disk, and let i be an arc of **T** attached to the puncture. Then we obtain a new polygon with triangulation by cutting **S** at i and gluing two copies of the cut surface at i as follows.



From type D to type A

The frieze of type A coming from cutting **S** has lots of symmetry $\mathcal{R} = \mathcal{R}'$ correspond to arcs in **S** attached to the puncture, $\mathcal{A} = \mathcal{A}'$, and contains the glued type D as a sub-pattern $\mathcal{A} \cup \mathcal{B}$.

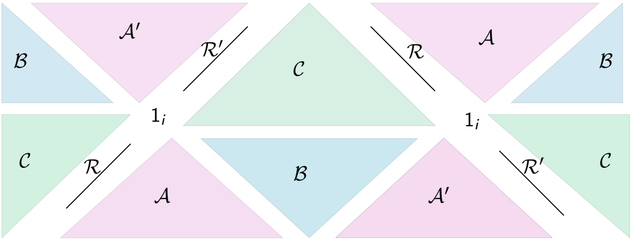


Theorem. [Garcia Elsener - S] Let arc $a \in \mathbf{T}$ such that $a \neq i$. Then mutation at a of the type D frieze is obtained by ungluing the pattern $\mu_a \mu_{a'}(\mathcal{A} \cup \mathcal{B})$ in the corresponding type A frieze.

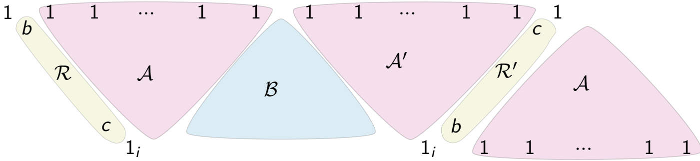
Note: $a \neq i$ is not an obstruction, because we can always choose to cut at a different arc.

Pattern $\mathfrak{G}_{\mathbf{T}}$

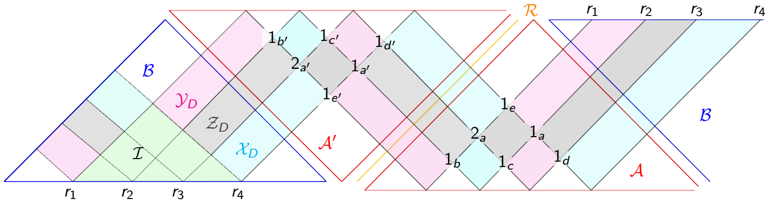
Type A frieze coming from cutting **S** at i



Pattern $\mathfrak{G}_{\mathbf{T}}$: only has entries of type D frieze



Mutation of type D friezes



Theorem. [Garcia Elsener - S] Let m be an entry in $\mathfrak{G}_{\mathbf{T}}$ and $a \neq i$. Then $\delta_a(m) = m - m'$ is given by:

If $m \in \mathcal{X}_D$ then $\delta_a(m) = [\rho_1^+(m) - \rho_2^+(m)]\,[\rho_1^-(m) - \rho_2^-(m)]$
If $m \in \mathcal{Y}_D$ then $\delta_a(m) = -[\rho_2^+(m) - 2\rho_1^+(m)]\,[\rho_2^-(m) - 2\rho_1^-(m)]$
If $m \in \overline{\mathcal{Z}}_D$ then $\delta_a(m) = \rho_s^{\downarrow}(m)\rho_p^{\downarrow}(m) + \rho_s^{\uparrow}(m)\rho_p^{\uparrow}(m) - 3\,\rho_p^{\downarrow}(m)\rho_p^{\uparrow}(m)$
If $m \in \mathcal{F}_D$ then $\delta_a(m) = 0$.
If $m \in \mathcal{I}$ then $m' = \rho_R^+(m)'\rho_A^+(m)' + \rho_R^-(m)'\rho_A^-(m)'$.

$\rho_*(m)$ are certain projections of m onto the boundary of $\mathcal{Z}_D$ or $\mathcal{R}$ or $\mathcal{A}$.

Question: Can we realize this operation of going from type D to type A on the level of the corresponding module categories?

Thank you!