Tracking the Variety of Interleavings

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Persistence Modules

A **persistence module** is a representation of a partially ordered set P with values in a category $\mathcal{D}$.

That is, if $\mathcal{D}$ is a category and P is a poset, a persistence module M for P with values in $\mathcal{D}$ assigns

- an object $M(x)$ of $\mathcal{D}$ for each $x \in P$, and
- a morphism $M(x \leq y)$ in $Mor_{\mathcal{D}}(M(x), M(y))$ for each $x, y \in P$ with $x \leq y$,

satisfying

$$M(x \leq z) = M(y \leq z) \circ M(x \leq y) \text{ when } x, y, z \in P \text{ with } x \leq y \leq z.$$

Persistence Modules and TDA

Persistent homology uses persistence modules to attempt to discern the genuine topological properties of a finite data set.

When P is a finite poset and $\mathcal{D}$ is K-mod, persistence modules for P are modules for the poset algebra of P.

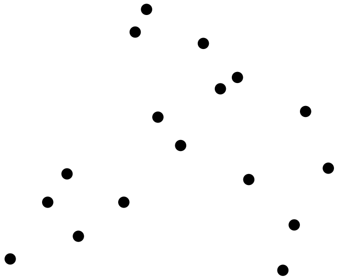
Introduction/Applications

Persistent homology has been recently used:

- to study atomic configurations (Hiraoka, Nakamura, Hirata)
- to study viral evolution (Chan, Carlsson, Rabadan)
- to analyze neural activity (Giusti, Pastalkova, Curto)
- to filter noise in sensor networks (Baryshnikov, Ghrist)

etc.

Example (Ambiguous H_0)



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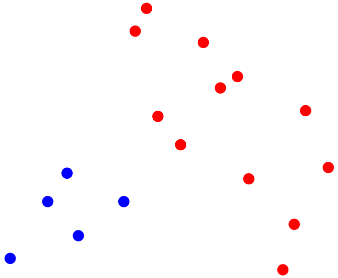

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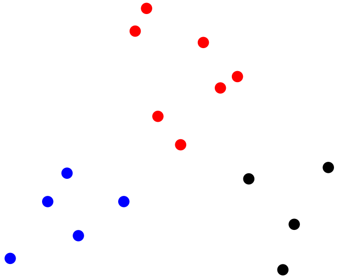
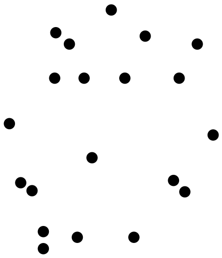


Another Example (Ambiguous H_1)



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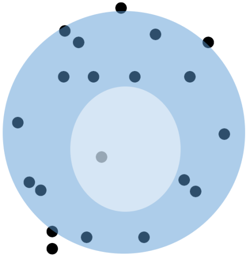

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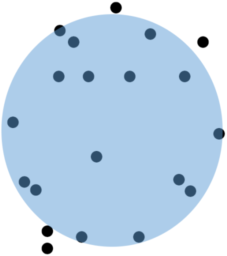


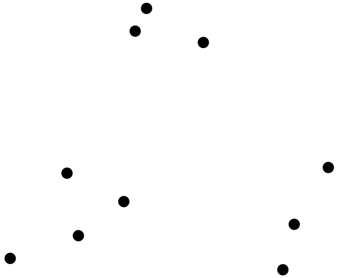
So what do we do?

- Suppose X is a finite data set contained in a metric space with undetermined topological features.
- The data set is associated to its Vietoris-Rips complex $(C_\epsilon)_{\epsilon \geq 0}$
- When $\delta < \epsilon$, $C_\delta \hookrightarrow C_\epsilon$, thus $\epsilon \to C_\epsilon$ is a persistence module.
- We take an appropriate homology, depending on which topological features we wish to distinguish between.

Summary of Persistent Homology

- As ϵ increases generators for homology are born and die, as cycles appear and become boundaries.
- One takes the viewpoint that true topological features of the data set can be distinguished from noise by looking for intervals which "persist" for a long period of time.
- Informally, we "keep" an indecomposable summand when it corresponds to a wide interval. Conversely, cycles which disappear quickly after their appearance are interpreted as noise and disregarded.
- By passing to the jump discontinuities of the Vietoris-Rips complex, one obtains a representation of equioriented $\mathbb{A}_n$.

Example



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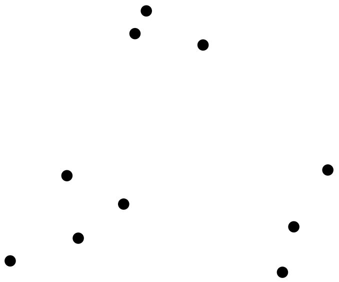

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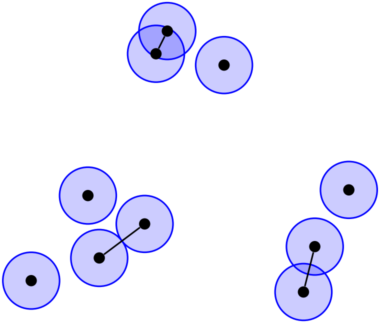

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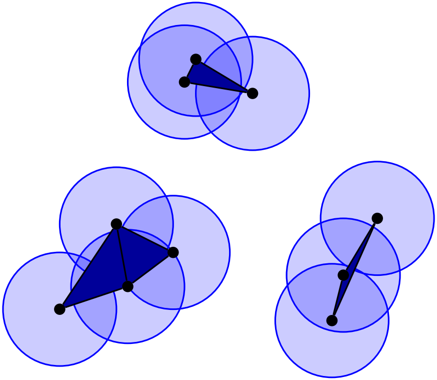

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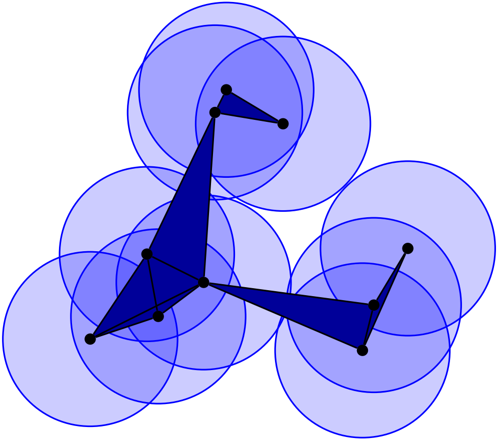
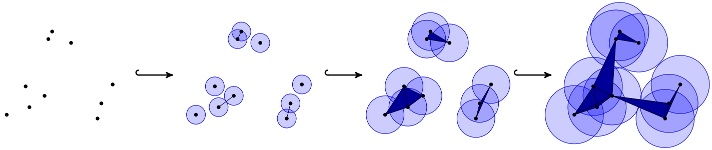


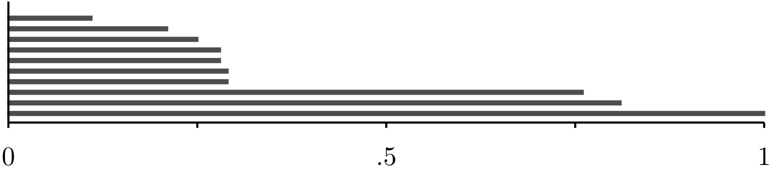
Example

As ϵ increases, we obtain an inclusion of simplicial complexes

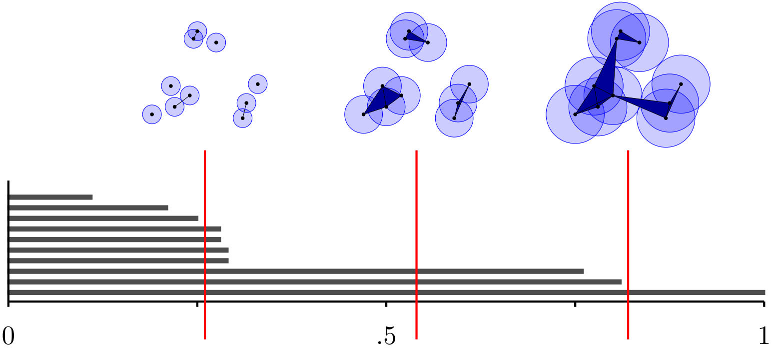


Example

We take homology



H_0 Example



Bottleneck Metric

A **bottleneck metric** is a way of defining a metric on the collection of finite multisubsets of a fixed set Σ.

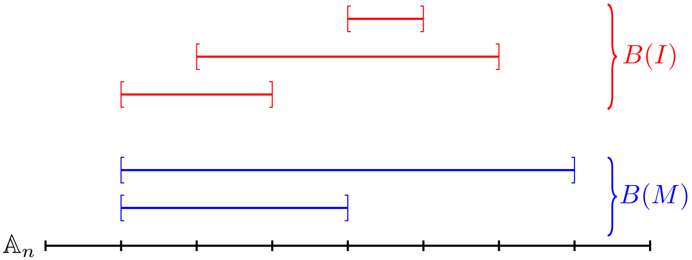
A bottleneck metric comes from

- a metric d on Σ, and
- a function $W : \Sigma \to (0, \infty)$, satisfying

$$|W(\sigma) - W(\tau)| \leq d(\sigma, \tau), \text{ for all } \sigma, \tau \in \Sigma.$$

Our multisubsets will be the indecomposable summands of a persistence module with their multiplicities.

Bottleneck Metric Example



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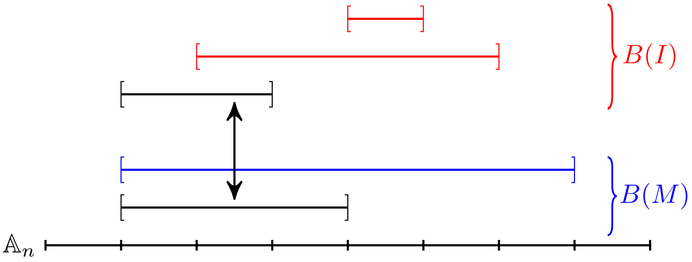

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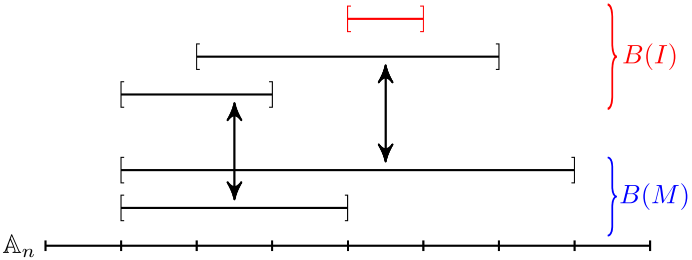

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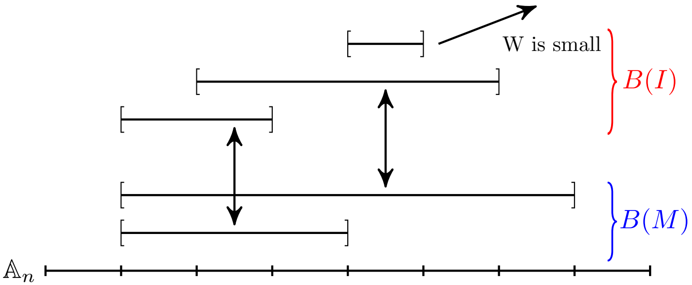


Interleaving Metrics

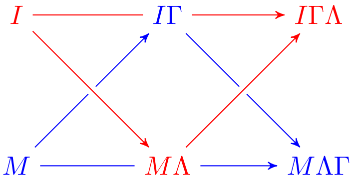
The other metric is an **interleaving metric**. An interleaving metric comes from

- a monoid $\mathcal{T}(P)$ that acts on the category of generalized persistence modules, and
- a metric d' on P.

The metric allows us to assign a notion of height to the elements of $\mathcal{T}(P)$.

Interleaving Metrics

The interleaving distance between two persistence modules I and M is $\inf\{\epsilon : \exists\ \Lambda, \Gamma \in \mathcal{T}(P), h(\Lambda), h(\Gamma) \leq \epsilon\}$, and one obtains the commutative diagram below



Algebraic Stability

Theorem (Isometry Theorem)

Let $P = [0, \infty)($ or $\mathbb{R})$, $([0, \infty), +) \subseteq \mathcal{T}(P)$. Then the interleaving metric D equals the bottleneck metric D_B.

This suggests a representation-theoretic analogue of the isometry theorem.

Let P be a finite poset and let K be a field. Choose a full subcategory $\mathcal{C}$ of persistence modules, and let

- D be the interleaving metric restricted to $\mathcal{C}$, and
- D_B be a bottleneck metric on $\mathcal{C}$ which incorporates some algebraic information.

Prove that $Id : (\mathcal{C}, D) \to (\mathcal{C}, D_B)$ is an isometry or a contraction.

Some Algebraic Stability Theorems

- Isometry Theorem for a class of finite posets which contains finite totally ordered sets (Meehan, M.)
- Isometry Theorems for equioriented $\mathbb{A}_n$ which makes precise the way in which persistence modules for finite totally ordered sets approximize those in data analysis (Meehan, M.)
- Stability Theorems for arbitrary orientations of $\mathbb{A}_n$ which make use of the A-R quiver. (Meehan, M.)

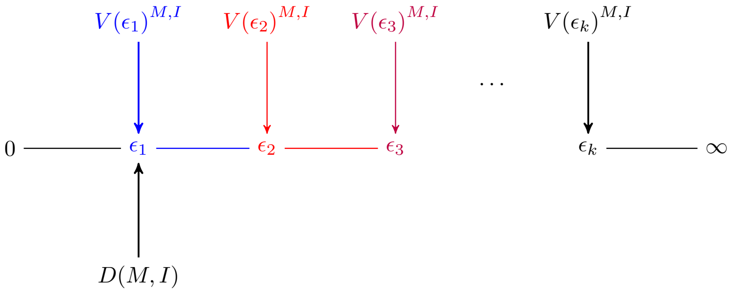
Back to Interleavings

Let $P = \mathbb{R}, M = \bigoplus_{i=1}^{m}[a_i, b_i), I = \bigoplus_{j=1}^{n}[c_j, b_j)$.

For any $\epsilon \geq 0$, the collection of ϵ-interleavings between M and I is a variety $V(\epsilon)^{M,I}$.

For example, $\{(A, A^{-1})|A \in Gl_2(K)\}, \{x, y, z, w|xz = 1, w = 0\}$ are two such varieties.

The Progression of Varieties



Questions

Question: Can the progression of varieties be interpreted in a meaningful way as a persistence module with values in varieties?

Question: Can more information about M and I be extracted from the full progression of varieties than from the interleaving distance?

The two red modules and the two blue modules are both the same distance apart.

Example

Let $M = [a, b), I = [c, d)$. Then,
$D(M, I) = min\left\{ \; max\{|a - c|, |b - d|\}, \;\; max\{\frac{b-a}{2}, \frac{d-c}{2}\}\right\}$.

Theorem (Acharya, Li, M., Noory)

Let $m_2 = max\{|a - c|, |b - d|\}, m_2 = \max\{\frac{b-a}{2}, \frac{d-c}{2}\}$. *Then,*

- $m_1 < m_2 \iff V(\epsilon_1)^{M,I}$ *is a hyperbola* $\iff$ *the full progression is hyperbola - plane - line - point or hyperbola - plane - point*
- $m_1 > m_2 \iff V(\epsilon_1)^{M,I}$ *is a point* $\iff$ *the full progression is point - line - point*
- $m_1 = m_2 \iff V(\epsilon_1)^{M,I}$ *is a line* $\iff$ *the full progression is line - point*

THANK YOU!