Sets with an action of a category

Peter Webb

University of Minnesota

November 17, 2018

Outline:

What should the Burnside ring of a category be?

A solution

Biset functors

Cohomology as a biset functor

History and Goals

- Extend the Burnside ring of a group to categories. For groups, induction theorems and idempotent formulas control representations and provide decompositions of group cohomology. Extend this to categories of interest such as the orbit category.
- Extend biset functors from groups to categories. These have given an approach to computing group cohomology, and provide a context for fundamental constructions, including the rational character ring and the (torsion-free part of) the Dade group.

The orbit category of a group: objects are finite G-sets, morphisms equivariant maps.
The Burnside ring $b(G)$ is the Grothendieck of the orbit category with respect to disjoint unions.

Sets with an action of a category

$\mathcal{C}$ is a finite category.
A $\mathcal{C}$-set is a functor $\Omega : \mathcal{C} \to$ some category; where 'some category' might be
$\mathcal{F} =$ finite sets,
$\mathcal{FI} =$ finite sets with injective maps,
$\mathrm{Span}\mathcal{FI} =$ the span category of finite sets with injections $= \mathcal{FI}\#$,
or:
something else!
The Burnside ring of $\mathcal{C}$ is $b(\mathcal{C}) =$ Grothendieck group finite $\mathcal{C}$-sets with relations $\Theta = \Omega + \Psi$ if $\Theta \cong \Omega \sqcup \Psi$ as $\mathcal{C}$-sets.
The product of $\mathcal{C}$-sets is defined pointwise:
$(\Omega \cdot \Psi)(x) := \Omega(x) \times \Psi(x)$.

Example: the poset $\mathcal{C} = x < y$

The $\sqcup$-indecomposable $\mathcal{C}$-sets have the form

$$\Omega_n := \{1, \ldots, n\} \to \{*\}, \quad n \geq 0.$$

We see that a finite category may have infinitely many non-isomorphic 'transitive' sets.

$$b_{\mathcal{F}}(\mathcal{C}) = \mathbb{Z}\{\Omega_0, \Omega_1, \Omega_2, \ldots\} \cong \mathbb{Z}\mathbb{N}^{\times}_{\geq 0}$$

What is wrong with this?
The ring is too complicated. It is not even finitely generated.

Issues

- If $\mathcal{C} = G$ is a finite group it does not matter which of the target categories we choose. They all give the 'same' notion of a G-set and $b(G)$. For categories the target category does make a difference. Which should we take?
- The Burnside ring of a small category is already overly complicated.

Conclusion: what definition of Burnside ring should we take?

More on the poset $\mathcal{C} = x < y$

On the other hand

$$b_{\mathcal{FI}}(\mathcal{C}) = \mathbb{Z}\Omega_0 \oplus \mathbb{Z}\Omega_1.$$

has rank 2, and is not a complicated ring.
Furthermore

$$\begin{aligned} b_{\mathcal{F}}(\mathcal{C}) &= \mathbb{Z}\{\Omega_0, \Omega_1, \Omega_2, \ldots\} \cong \mathbb{Z}\mathbb{N}^{\times}_{\geq 0} \\ &= b_{\mathcal{FI}}(\mathcal{C}) \oplus J \end{aligned}$$

where $J = \mathbb{Z}\{\Omega_2 - \Omega_1, \Omega_3 - \Omega_1, \ldots\}$ is an ideal in $b_{\mathcal{F}}(\mathcal{C})$.

For what it is worth, $b_{\mathrm{Span}\mathcal{FI}}(\mathcal{C})$ has rank 3, with basis in bijection with the positive roots of the A_2 root system.

What should the Burnside ring of a category be?

We have several choices for what the target category of a $\mathcal{C}$-set should be, giving different Burnside rings. The Burnside ring should

- extend the notion for finite groups,
- give a reasonable answer that is not too complicated for easy categories,
- be a home for invariants such as the Lefschetz invariant of a $\mathcal{C}$-space,
- have a good connection with representation theory,
- be a projective biset functor,
- have some symmetry properties, so that $b(\mathcal{C}) \cong b(\mathcal{C}^{\mathrm{op}})$, for instance.

All of $\mathcal{F}, \mathcal{FI}$ and $\mathrm{Span}\mathcal{FI}$, as well as other constructions, fail some of these criteria.

Conclusions

What may happen with $\mathcal{C}$-sets is more complicated that what happens when $\mathcal{C}$ is a group.

We may allow more than one candidate for the Burnside ring and explore the relationships between the different possibilities.

$\mathcal{C}$-sets with values in $\mathcal{FI}$

The relationship we saw before between $b_{\mathcal{F}}(\mathcal{C})$ and $b_{\mathcal{FI}}(\mathcal{C})$ holds in general.

Theorem

Each $\mathcal{C}$-set $\Omega \in \mathcal{F}^{\mathcal{C}}$ has a unique largest quotient with values in $\mathcal{FI}$. The natural map $\eta_\Omega : \Omega \to \Omega^{\mathcal{FI}}$ preserves products of $\mathcal{C}$-sets.

Corollary

There are ring homomorphisms $b_{\mathcal{FI}}(\mathcal{C}) \xrightarrow{i_*} b_{\mathcal{F}}(\mathcal{C}) \xrightarrow{j_*} b_{\mathcal{FI}}(\mathcal{C})$ so that $j_* i_* = 1$. Thus $b_{\mathcal{F}}(\mathcal{C}) = b_{\mathcal{FI}}(\mathcal{C}) \oplus J$ where $J = \ker J_*$ is an ideal of $b_{\mathcal{F}}(\mathcal{C})$.

The marks homomorphism

There is a ring homomorphism $m : b(\mathcal{C}) \to \mathbb{Z}^{\mathrm{cc}(\mathcal{C})}$ that is the usual marks homomorphism in case $\mathcal{C}$ is a group.

$\mathrm{cc}(\mathcal{C}) :=$ set of conjugacy classes of non-empty, connected subcategories of $\mathcal{C}$.

Subcategories $\mathcal{D}_1, \mathcal{D}_2$ are conjugate if $\mathcal{D}_1 = \eta\mathcal{D}_2$ where $\eta : \mathcal{C} \to \mathcal{C}$ is a self-equivalence naturally isomorphic to the identity.

The coordinate value $m_{\mathcal{D}}(\Omega)$ is the number of indecomposable constant $\mathcal{D}$-sets in $\Omega \downarrow^{\mathcal{C}}_{\mathcal{D}}$. In case $\mathcal{C}$ is a group this definition gives the usual marks homomorphism.

A semisimple subalgebra of the Burnside ring

A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is an ideal if whenever $x \in \mathcal{D}$ and there is a homomorphism $x \to y$ then $y \in \mathcal{D}$. Ideals form a lattice $\mathcal{L}$ under union and intersection.

For each ideal $\mathcal{D}$ there is an idempotent element $e_{\mathcal{D}} \in b_{\mathcal{FI}}(\mathcal{C})$ that is a constant single point on $\mathcal{D}$ and empty at other objects.

Theorem

The span in $b_{\mathcal{FI}}(\mathcal{C})$ of the $e_{\mathcal{D}}$ is a subalgebra isomorphic to the Möbius algebra of $\mathcal{L}$. It is mapped by the marks homomorphism isomorphically to $\mathbb{Z}^{\mathcal{L}}$.

Bisets for categories

These were introduced in
J. Bénabou, *Les distributeurs,* 1973.
and appear also
Marta Bunge, *Categories of Set-Valued Functors,* University of Pennsylvania, 1966.

Given categories $\mathcal{C}$ and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{D})$-biset is a $\mathcal{C} \times \mathcal{D}^{\mathrm{op}}$-set ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$.

Composition of bisets: given a $(\mathcal{C}, \mathcal{D})$-biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ and a $(\mathcal{D}, \mathcal{E})$-biset ${}_{\mathcal{D}}\Psi_{\mathcal{E}}$ there is a $(\mathcal{C}, \mathcal{E})$-biset $\Omega \circ \Psi$ given by

$$\Omega \circ \Psi(x, z) = \bigsqcup_{y \in \mathcal{D}} \Omega(x, y) \times \Psi(y, z) / \sim$$

where $\sim$ is the equivalence relation generated by $(u\beta, v) \sim (u, \beta v)$ whenever $u \in \Omega(x, y_1)$, $v \in \Psi(y_2, z)$ and $\beta : y_2 \to y_1$ in $\mathcal{D}$.

Biset functors

Proposition

The operation $\circ$ is associative. For each category $\mathcal{C}$ there is an identity biset ${}_{\mathcal{C}}\mathcal{C}_{\mathcal{C}}$.

Let $A(\mathcal{C}, \mathcal{D})$ be the Grothendieck group of finite $(\mathcal{C}, \mathcal{D})$-bisets with respect to $\sqcup$, thus extending the notion of the *double Burnside ring* for groups.

The biset category $\mathbb{B}$ has as objects all finite categories, with homomorphisms $\mathrm{Hom}_{\mathbb{B}}(\mathcal{C}, \mathcal{D}) = A(\mathcal{D}, \mathcal{C})$.

A biset functor is a linear functor $\mathbb{B} \to \mathbb{Z}$-mod.

This extends the usual notion of biset functors defined on groups.

The usual formalities for such functor categories apply: every simple functor on $\mathbb{B}$ restricts to groups either giving a simple functor or zero, and every simple functor on groups extends uniquely to a simple functor defined on categories.

Bisets free on each side

A $\mathcal{C}$-set Ω is representable if $\Omega \cong \bigsqcup \mathrm{Hom}(x_i, -)$, for certain objects $x_i \in \mathcal{C}$.
A biset ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ is biadjoint over a ring R if the left and right adjoints of the functor $R\Omega \otimes_{R\mathcal{D}} -$ coincide.

Theorem

If ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ and ${}_{\mathcal{D}}\Psi_{\mathcal{E}}$ are bisets that are representable on restriction to each side, or are biadjoint, then so is ${}_{\mathcal{C}}\Omega \circ \Psi_{\mathcal{E}}$.

Let $\mathbb{B}_{1,1}^{\mathrm{biadjoint}}$ be the subcategory of $\mathbb{B}$ obtained by using only biadjoint bisets that are representable on each side.

The notation is suggested by the fact that, for categories that are groups, all stabilizers of elements in such bisets, on the left and on the right, are 1. In this case such bisets are automatically biadjoint.

Cohomology as a biset functor

The cohomology ring of a category $\mathcal{C}$ is $H^*(\mathcal{C}) := \mathrm{Ext}^*_{\mathbb{Z}\mathcal{C}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}})$, where $\underline{\mathbb{Z}}$ is the constant functor.

Theorem

$\mathcal{C} \mapsto H^*(\mathcal{C})$ has the structure of a functor on $\mathbb{B}^{\mathrm{biadjoint}}_{1,1}$.

This provides a solution to the problem of finding a 'corestriction' map in the cohomology of categories, related to an approach of Carlson, Peng and Wheeler (1998).

Biset functors on $\mathbb{B}^{\mathrm{biadjoint}}_{1,1}$ were called global Mackey functors in Webb, 'Two classifications ...', J. Pure Appl. Alg. (1993).
They are easier to work with than general biset functors.
A formula was given in Webb (1993) for the simple global Mackey functors and an application given to computing group cohomology.
The theory is not yet so advanced in the context of categories.