# Algebraic Stability for Arbitrary Orientations of $\mathbb{A}_n$

David Meyer

Smith College

Joint work with Killian Meehan

CGMRT

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# Persistence Modules

A **persistence module** is a representation of a partially ordered set $P$ with values in a category $\mathcal{D}$.

That is, if $\mathcal{D}$ is a category and $P$ is a poset, a persistence module $M$ for $P$ with values in $\mathcal{D}$ assigns

- an object $M(x)$ of $\mathcal{D}$ for each $x \in P$, and
- a morphism $M(x \leq y)$ in $Mor_{\mathcal{D}}(M(x), M(y))$ for each $x, y \in P$ with $x \leq y$,

satisfying

$$M(x \leq z) = M(y \leq z) \circ M(x \leq y) \text{ whenever } x, y, z \in P \text{ with } x \leq y \leq z.$$

# Persistence Modules

**Persistent homology** uses persistence modules to attempt to discern the genuine topological properties of a finite data set.

When $P$ is a finite poset and $\mathcal{D}$ is $K$-mod, persistence modules for $P$ are modules for the poset algebra of $P$.

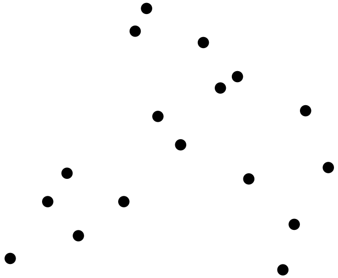
# Introduction/Applications

Persistent homology has been recently used:

- to study atomic configurations (Hiraoka, Nakamura, Hirata)
- to study viral evolution (Chan, Carlsson, Rabadan)
- to analyze neural activity (Giusti, Pastalkova, Curto)
- to filter noise in sensor networks (Baryshnikov, Ghrist)

etc.

# Example (Ambiguous $H_0$)



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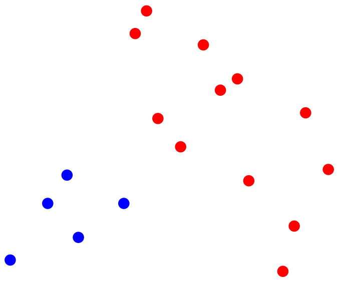

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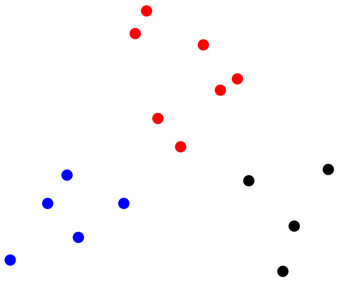
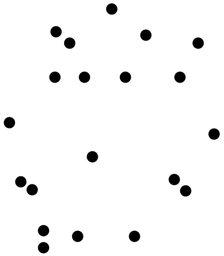


# Another Example (Ambiguous $H_1$)



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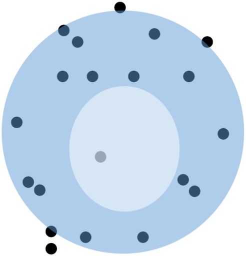

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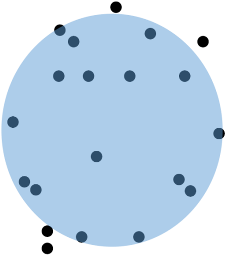


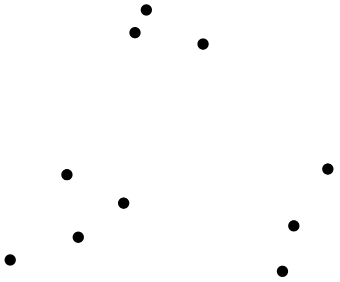
So what do we do?

- Suppose $X$ is a finite data set contained in a metric space with undetermined topological features.
- The data set is associated to its Vietoris-Rips complex $(C_\epsilon)_{\epsilon \geq 0}$
- When $\delta < \epsilon$, $C_\delta \hookrightarrow C_\epsilon$, thus $\epsilon \to C_\epsilon$ is a persistence module.
- We take an appropriate homology, depending on which topological features we wish to distinguish between.

# Summary of Persistent Homology

- As $\epsilon$ increases generators for homology are born and die, as cycles appear and become boundaries.
- One takes the viewpoint that true topological features of the data set can be distinguished from noise by looking for intervals which "persist" for a long period of time.
- Informally, we "keep" an indecomposable summand of $f$ when it corresponds to a wide interval. Conversely, cycles which disappear quickly after their appearance are interpreted as noise and disregarded.
- By passing to the jump discontinuities of the Vietoris-Rips complex, one obtains a representation of equioriented $\mathbb{A}_n$.

# Example



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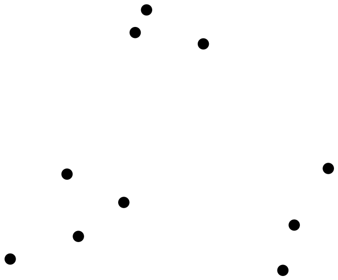

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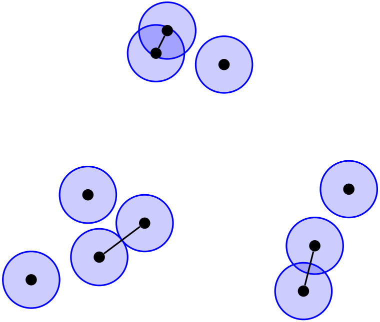

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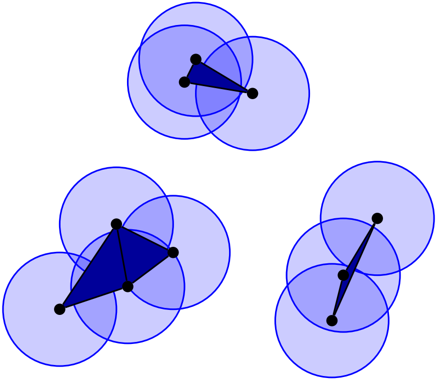

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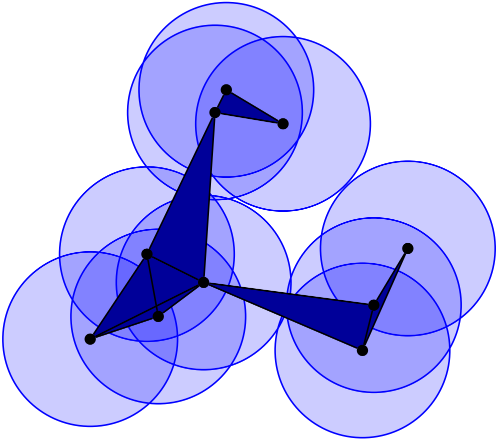
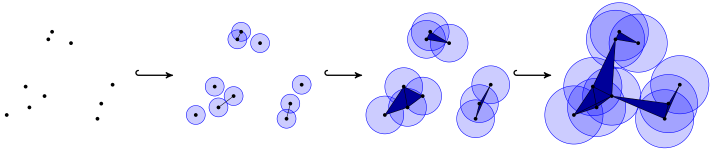


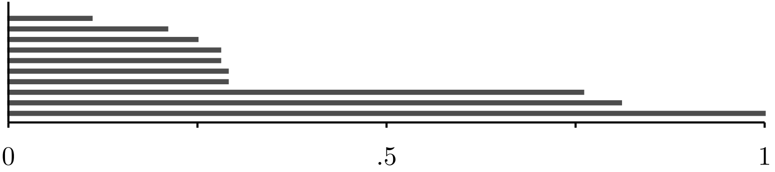
# Example

As $\epsilon$ increases, we obtain an inclusion of simplicial complexes

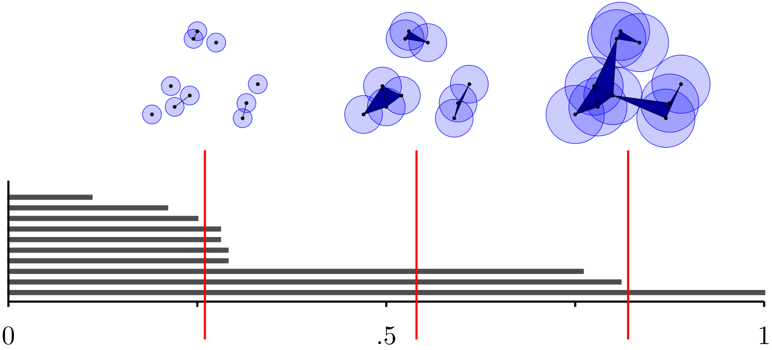


# Example

We take homology



# $H_0$ Example



# Bottleneck Metric

A **bottleneck metric** is a way of defining a metric on the collection of finite multisubsets of a fixed set $\Sigma$.

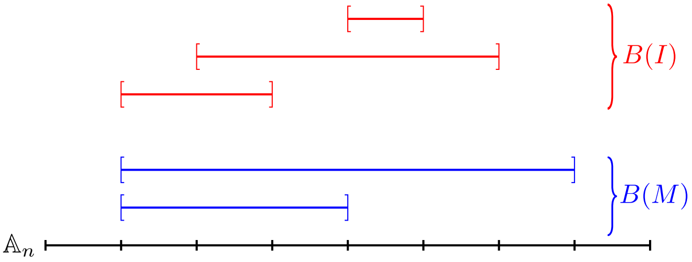
A bottleneck metric comes from

- a metric $d$ on $\Sigma$, and
- a function $W : \Sigma \to (0, \infty)$, satisfying

$$|W(\sigma) - W(\tau)| \leq d(\sigma, \tau), \text{ for all } \sigma, \tau \in \Sigma.$$

Our multisubsets will be the indecomposable summands of a persistence module with their multiplicities.

# Bottleneck Metric Example



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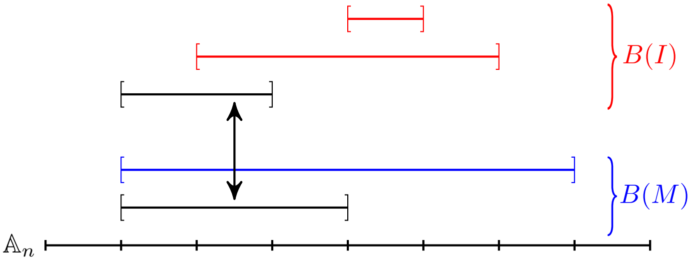

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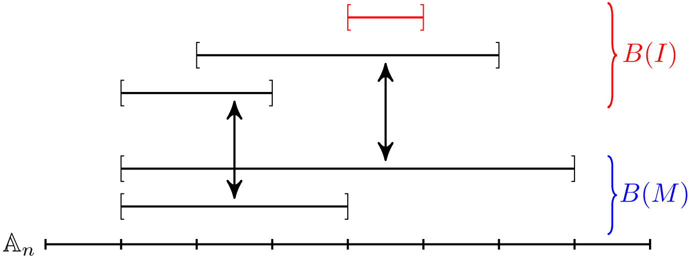

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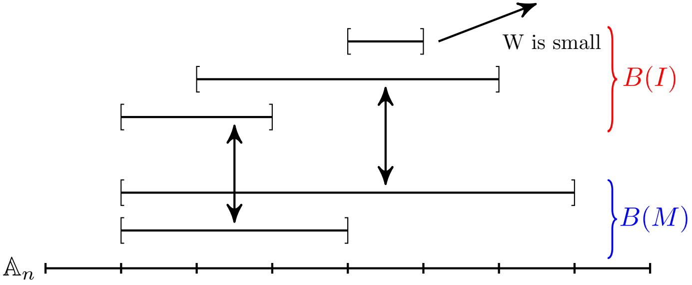


# Interleaving Metrics

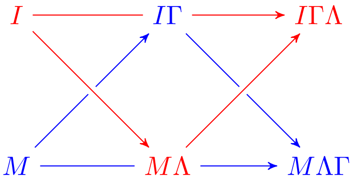
The other metric is an **interleaving metric**. An interleaving metric comes from

- a monoid $\mathcal{T}(P)$ that acts on the category of generalized persistence modules, and
- a metric $d'$ on $P$.

The metric allows us to assign a notion of height to the elements of $\mathcal{T}(P)$.

# Interleaving Metrics

The interleaving distance between two persistence modules $I$ and $M$ is $\inf\{\epsilon : \exists\, \Lambda, \Gamma \in \mathcal{T}(P), h(\Lambda), h(\Gamma) \leq \epsilon\}$, and one obtains the commutative diagram below



# Algebraic Stability

**Theorem (Isometry Theorem)**

*Let $P = (0, \infty)($ or $\mathbb{R})$, $([0, \infty), +) \subseteq \mathcal{T}(P)$. Then the interleaving metric $D$ equals the bottleneck metric $D_B$.*

This suggests the following representation-theoretic analogue of the isometry theorem.

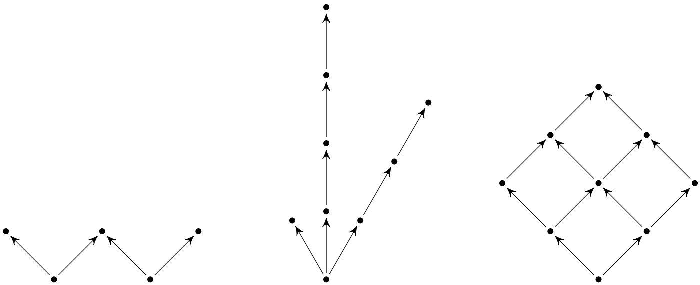
Let $P$ be a finite poset and let $K$ be a field. Choose a full subcategory $\mathcal{C}$ of persistence modules, and let

- $D$ be the interleaving metric restricted to $\mathcal{C}$, and
- $D_B$ be a bottleneck metric on $\mathcal{C}$ which incorporates some algebraic information.

Prove that $Id : (\mathcal{C}, D) \to (\mathcal{C}, D_B)$ is an isometry or a contraction.

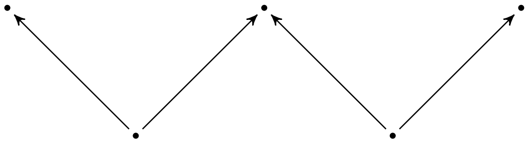
# Metric on $P$

We use a weighted graph metric on the Hasse quiver of the poset.



# Metric on $P$

First, we suspend the poset at infinity.



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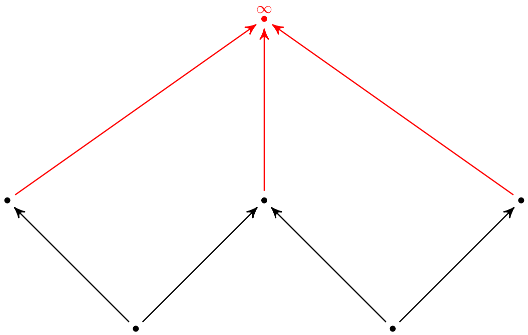
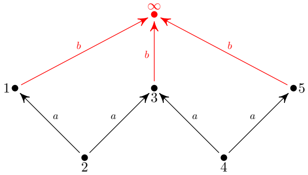
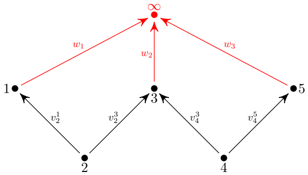


# Metric on $P$

We may use the "democratic" variant




Or an arbitrary choice of weights



# Isometry Theorem 1

## Theorem (Meehan, M.)

*Let $P$ be an n-Vee and let $\mathcal{C}$ be the full subcategory of persistence modules consisting of direct sums of interval modules. Let $(a, b)$ be a democratic choice of weights and let $D$ denote interleaving distance (corresponding to the weight $(a, b)$) restricted to $\mathcal{C}$.*

*Set $W(M) = min\{\epsilon : \mathcal{H}om(M, M\Gamma\Lambda) = 0, \Gamma, \Lambda \in \mathcal{T}(\mathcal{P}), h(\Gamma), h(\Lambda) \leq \epsilon\}$, and let $D_B$ be the bottleneck distance on $\mathcal{C}$ corresponding to the interleaving distance and $W$.*

*Then, the identity is an isometry from $(\mathcal{C}, D) \xrightarrow{Id} (\mathcal{C}, D_B)$.*

# Isometry Theorem 2

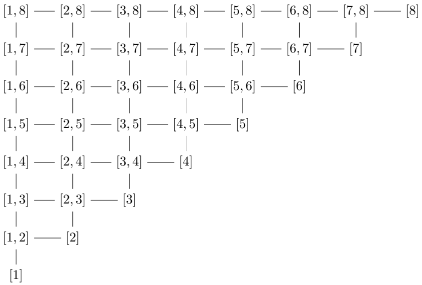
## Theorem (Meehan, M.)

*Let $P$ be equioriented $\mathbb{A}_n$, and let $(a_i, b)$ be any choice of weights. Let $D$ denote interleaving distance, and again*

*set $W(M) = min\{\epsilon : Hom(M, M\Gamma\Lambda) = 0, \Gamma, \Lambda \in \mathcal{T}(\mathcal{P}), h(\Gamma), h(\Lambda) \leq \epsilon\}$. Let $D_B$ be the bottleneck distance corresponding to the interleaving distance and $W$.*

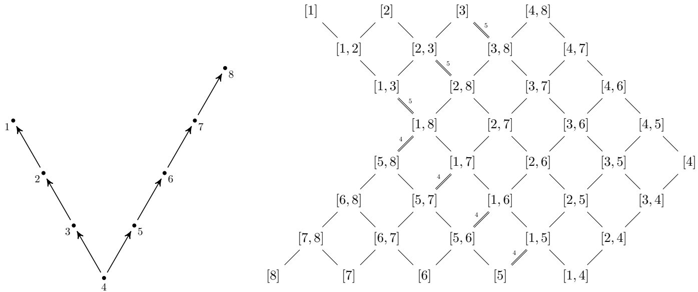
*Then, one obtains a "shifted" isometry theorem.*

# Bottleneck metric on the AR quiver




AR quiver of equioriented $\mathbb{A}_8$ .

# Arbitrary Orientations




A different orientation on $\mathbb{A}_8$ with its AR quiver.

# 3 Metrics

Since

1. the graph metric on the *AR* quiver for $\mathbb{A}_n$ agrees with the classical bottleneck metric, and
2. any orientation on $\mathbb{A}_n$ corresponds to the Hasse quiver of a poset;

we wish to prove a stability theorem for an arbitrary orientation of $\mathbb{A}_n$.

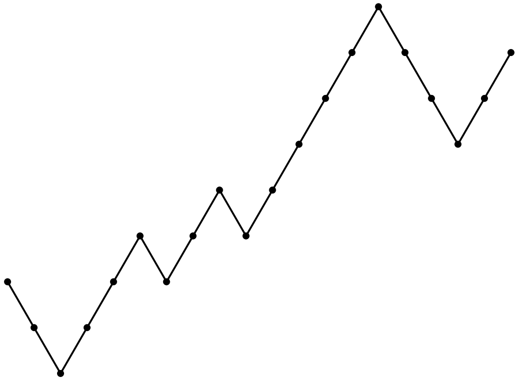
# 3 Metrics

Here are the metrics.

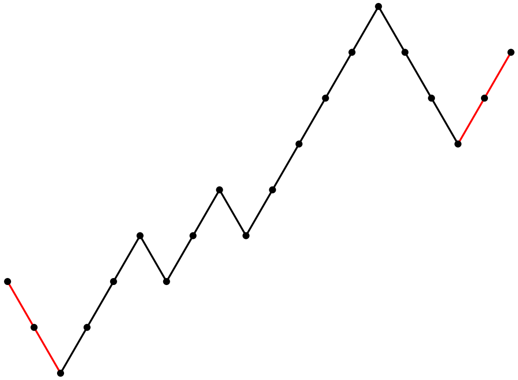
- Bottleneck 1
  $d$=interleaving metric, $W(M) = min\{\epsilon : \mathrm{Hom}(M, M\Gamma\Lambda) = 0$ (same as previous work)
- Interleaving metric (same as previous work)
- Bottleneck 2
  $d$=weighted graph metric on the AR quiver, $W(M)$ is distance to zero (motivated by previous comments)

Goal: Compare the metrics. In particular, find minimal weights $(a, b)$ such that the identity is a contraction from Bottleneck 2 to Bottleneck 1.

# Stability Theorem

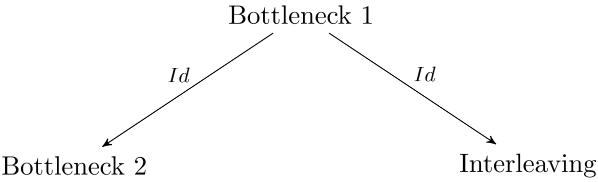


# Stability Theorem



$T$ is the "longest of the shortests sides." Here $T$ equals 2.

# Stability Theorem




$(2, T)$ is the minimal weight such that both arrows are contractions. For many orientations, Bottleneck 1 equals Interleaving.

# THANK YOU!