Representation varieties of algebras with nodes

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Joint work with Ryan Kinser

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- The action of the base change group

$$GL(\mathbf{d}) = \prod_{z \in Q_0} GL(\mathbf{d}(z))$$

acts on $\mathrm{rep}_Q(\mathbf{d})$ by

$$g \cdot M = (g_{h\alpha} M_\alpha g_{t\alpha}^{-1})_{\alpha \in Q_1},$$

where $g = (g_z)_{z \in Q_0} \in GL(\mathbf{d})$ and $M = (M_\alpha)_{\alpha \in Q_1} \in \mathrm{rep}_Q(\mathbf{d})$.

General questions

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$$\mathrm{rep}_A(\mathbf{d}) = \{M \in \prod_{\alpha \in Q_1} \mathrm{Mat}(\mathbf{d}(h\alpha), \mathbf{d}(t\alpha)) \mid M(r) = 0, \, \forall r \in R\}$$

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- Under the action of $GL(\mathbf{d})$, orbits correspond to isomorphism classes of representations.
- In general $\text{rep}_A(\mathbf{d})$ is not irreducible. We want to study its irreducible components, orbit closures, and their singularities.
- Determine generic decompositions, and moduli space decompositions of semi-stable representations.

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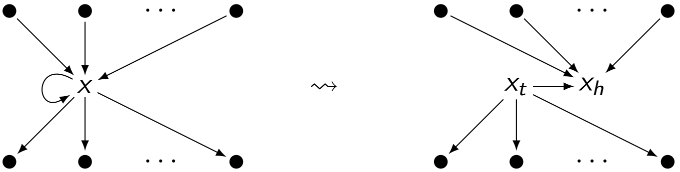

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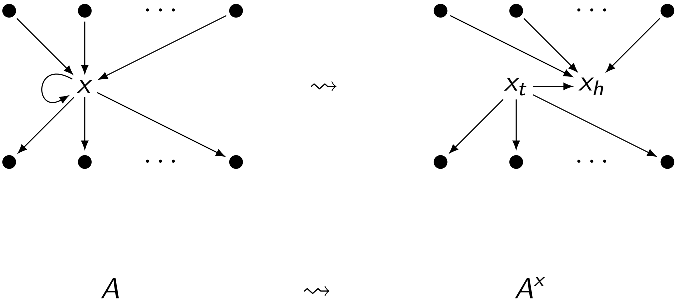

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There is a bijection between the set of isomorphism classes of indecomposable representations of A and the set of isomorphism classes of indecomposable representations of A^x with the simple representation supported at x_h removed.

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Example

Take the following quiver with relation $ab = 0$

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Splitting vertex 2, we get two quivers $1 \to 2_h \qquad 2_t \to 3$; representation varieties for these are affine spaces. However, representation varieties for the original quiver have multiple irreducible components and are singular.

Setup

Assume $x \in Q_0$ is a node of A, and take r with $0 \leq r \leq \mathbf{d}(x)$. We denote by $\mathbf{d}_r^x$ the dimension vector of Q^x obtained by putting $\mathbf{d}^x(x_h) = r$, $\mathbf{d}^x(x_t) = \mathbf{d}(x) - r$, and at the rest of the vertices $\mathbf{d}^x$ coincides with $\mathbf{d}$.

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$$i(M)_\alpha = \begin{cases} M_\alpha & t\alpha \neq x \neq h\alpha \\ \begin{bmatrix} M_\alpha \\ 0 \end{bmatrix} & h\alpha = x \text{ and } t\alpha \neq x, \\ \begin{bmatrix} 0 & M_\alpha \end{bmatrix} & t\alpha = x \text{ and } h\alpha \neq x, \\ \begin{bmatrix} 0 & M_\alpha \\ 0 & 0 \end{bmatrix} & t\alpha = x \text{ and } h\alpha = x. \end{cases}$$

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Let $P_r \leq GL(\mathbf{d}(x))$ be the parabolic subgroup of block upper triangular matrices block size r and $\mathbf{d}(x) - r$. Let $P_r^x(\mathbf{d}) \leq GL(\mathbf{d})$ be the subgroup where the factor $GL(\mathbf{d}(x))$ is replaced by P_r. The variety $\operatorname{rep}_{A^x}(\mathbf{d}_r^x)$ is in fact $P_r^x(\mathbf{d})$-stable subvariety of $\operatorname{rep}_A(\mathbf{d})$, as the unipotent radical of P_r acts trivially on $\operatorname{rep}_{A^x}(\mathbf{d}_r^x)$!

Given subset $S \subset \mathrm{rep}_A(\mathbf{d})$, and a node x, we define the *x-rank* of S to be the number

$$r_x(S) := \max_{M \in S} \left\{ \mathrm{rank} \bigoplus_{h\alpha = x} M_\alpha : \bigoplus_{h\alpha = x} M_{t\alpha} \to M_x \right\}.$$

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Proposition

Let $0 \leq r \leq \mathbf{d}(x)$ and C a $GL(\mathbf{d}_r^x)$-stable irreducible closed subvariety of $\mathrm{rep}_{A^x}(\mathbf{d}_r^x)$ with $r_{x_t}(C) = r$. Then the saturation $GL(\mathbf{d}) \cdot C$ is an irreducible closed subvariety of $\mathrm{rep}_A(\mathbf{d})$, and the following map is a proper birational morphism of $GL(\mathbf{d})$-varieties:

$$\Psi_C : GL(\mathbf{d}) \times_{P_r^x(\mathbf{d})} C \to GL(\mathbf{d}) \cdot C \,, \, (g, M) \mapsto g \cdot M.$$

Main Correspondence

Theorem (Kinser, L. '18)

For each $0 \leq r \leq \mathbf{d}(x)$, the maps below are mutually inverse, inclusion-preserving bijections.

$$\left\{ \begin{array}{c} \textit{irreducible closed} \\ GL(\mathbf{d}_r^x)\textit{-stable subvarieties} \\ \textit{of } \mathrm{rep}_{A^x}(\mathbf{d}_r^x) \textit{ of } x_h\textit{-rank } r \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{irreducible closed} \\ GL(\mathbf{d})\textit{-stable subvarieties} \\ \textit{of } \mathrm{rep}_A(\mathbf{d}) \textit{ of } x\textit{-rank } r \end{array} \right\}$$

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$$C \mapsto GL(\mathbf{d}) \cdot C$$

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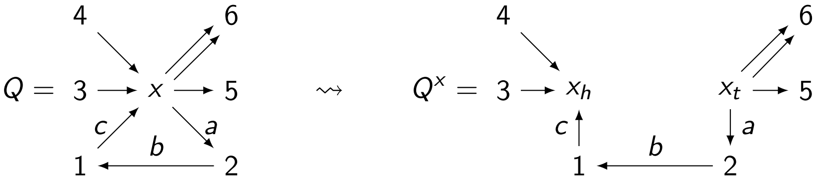
In particular, the irreducible components of representation varieties of A are saturations of irreducible components of representation varieties of A^x.

An Example

Consider the algebra $A = \Bbbk Q/I$, where I is generated by relations declaring that x is a node, along with the relation $abc = 0$.



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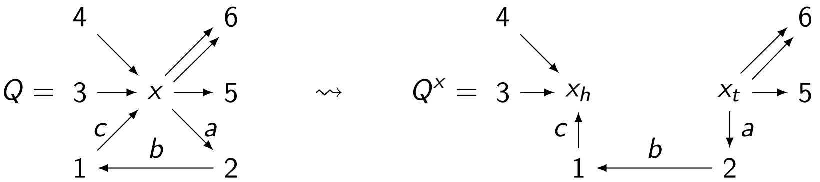

Let $\mathbf{d} = (3, 2, 2, 1, 3, 3, 3)$ (where $\mathbf{d}(x)$ is the last entry). The study of the components of $\mathrm{rep}_A(\mathbf{d})$ reduces to type $\mathbb{A}_4$ quiver with the following dimension vector, for $r = 0, 1, 2, 3$

$$(3 - r) \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} r$$

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- $r = 0$, one component C_0

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- $r = 0$, one component C_0
- $r = 1$, two components:
  $C_1 - (1,1,1,0)^{\oplus 2} \oplus (0,0,1,1)$
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Under saturation, C_1' is contained in C_2 and C_2' is contained in C_3. The irreducible components of $\mathrm{rep}_A(\mathbf{d})$ are given by saturations of C_0, C_1, C_2, C_3.

Radical-square algebras

Theorem (Kinser, L. '18)

Take $A = \Bbbk Q / \operatorname{rad}^2(\Bbbk Q)$ *and a dimension vector* $\mathbf{d}$*. For a dimension vector* $\mathbf{r} \leq \mathbf{d}$*, let* $C_{\mathbf{r}}$ *be the closure of the set of representations* $M \in \operatorname{rep}_A(\mathbf{d})$ *such that* $r_x(M) = \mathbf{r}(x)$*, for all* $x \in Q_0$*. Then* $C_{\mathbf{r}}$ *is irreducible. Furthermore, set* $\mathbf{s} = \mathbf{d} - \mathbf{r}$*, and for* $x \in Q_0$ *let* l_x *be the number of loops at* x *and put*

$$u_x(\mathbf{r}) = \sum_{h\alpha = x} \mathbf{s}(t\alpha) - \mathbf{r}(x), \quad \text{and} \quad v_x(\mathbf{r}) = \sum_{t\alpha = x} \mathbf{r}(h\alpha) - \mathbf{s}(x).$$

Then the irreducible components of $\operatorname{rep}_A(\mathbf{d})$ *are given precisely by the irreducibles* $C_{\mathbf{r}}$ *for which* $\mathbf{r}$ *satisfies the following for all* $x \in Q_0$*:*

$$u_x(\mathbf{r}) \geq 0, \text{ and when } u_x(\mathbf{r}) > l_x \text{ then } v_x(\mathbf{r}) \geq 0.$$

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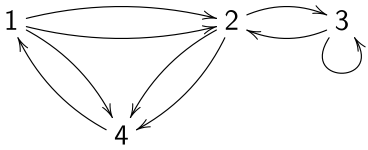
This is complementary to a representation-theoretic algorithm given by [Bleher, Chinburg, Huisgen-Zimmermann '15]

Example

Consider the radical-square algebra A (all compositions zero)



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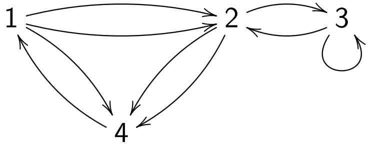

For $\mathbf{d} = (2,2,2,2)$, $\text{rep}_A(\mathbf{d})$ has 13 irreducible components given by the rank sequences:

$$(0,0,1,2),(0,0,2,2),(0,1,1,2),(0,2,0,2),(0,2,1,2),(1,0,1,1),(1,0,2,1)$$

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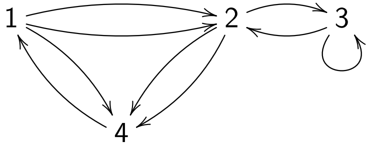

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For $\mathbf{d} = (50,50,50,50)$, we have 60501 irreducible components.

Generic decomposition

Theorem (Kac '80, '82; de la Peña '91; Crawley-Boevey, Schröer '02)

Any irreducible component $C \subseteq \mathrm{rep}_A(\mathbf{d})$ *satisfies a Krull-Schmidt type decomposition*

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Theorem (Kinser, L. '18)

Let $C \subseteq \mathrm{rep}_A(\mathbf{d})$ *be an irreducible component,* $r = r_x(C)$ *and* $C^x = C \cap \mathrm{rep}_{A^x}(\mathbf{d}_r^x)$. *Let* $C^x = \overline{C_1^x \oplus \cdots \oplus C_k^x}$ *be the generic decomposition of the irreducible component* C^x *in* A^x*. Then* $C = \overline{C_1 \oplus \cdots \oplus C_k}$ *is the generic decomposition of* C*, where* $C_i^x = GL(\mathbf{d}) \cdot C$.

Assume $\operatorname{char} \Bbbk = 0$.

Theorem (Kinser, L. '18)

Let C be $GL(\mathbf{d}_r^x)$-stable irreducible closed subvariety of $\operatorname{rep}_{A^x}(\mathbf{d}_r^x)$, for some $0 \leq r \leq \mathbf{d}(x)$. If C is normal (resp. has rational singularities), then the same is true for the variety $GL(\mathbf{d}) \cdot C \subseteq \operatorname{rep}_A(\mathbf{d})$.

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For the proof we use a result of [Kempf '76].

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Corollary

Let A be a finite-dimensional $\Bbbk$-algebra with $\operatorname{rad}^2 A = 0$. Then for any dimension vector $\mathbf{d}$, any irreducible component $C \subseteq \operatorname{rep}_A(\mathbf{d})$ has rational singularities (and is thus also normal, and Cohen-Macaulay).

Example with orbit closures

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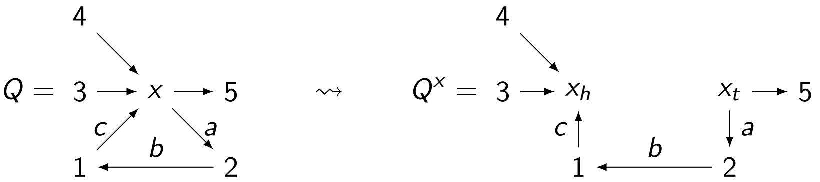

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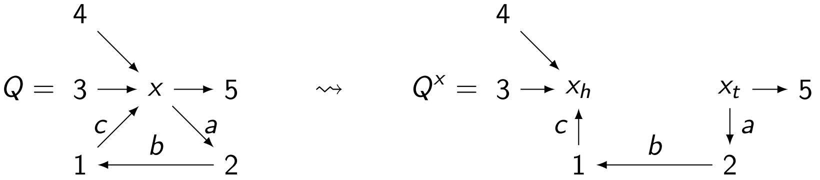
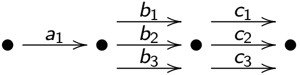


Orbit closures of A^x are orbit closures for a type $\mathbb{D}$ quiver, and thus have rational singularities by [Bobiński-Zwara '02]. Therefore, all orbit closures for A have rational singularities.

Let A be given by the quiver



with relations $a_1 b_1 = b_1 c_1 = b_1 c_2 = b_2 c_1 = b_2 c_2 = b_3 c_3 = 0$.

Let A be given by the quiver $\bullet \xrightarrow{a_1} \bullet \begin{array}{c} \xrightarrow{b_1} \\ \xrightarrow{b_2} \\ \xrightarrow{b_3} \end{array} \bullet \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{c_2} \\ \xrightarrow{c_3} \end{array} \bullet$

with relations $a_1 b_1 = b_1 c_1 = b_1 c_2 = b_2 c_1 = b_2 c_2 = b_3 c_3 = 0$.

A has no nodes, but we can separate relations, and so a representation variety of A can be written as a product of representation varieties of

$$\bullet \xrightarrow{a_1} \bullet \begin{array}{c} \xrightarrow{b_1} \\ \xrightarrow{b_2} \end{array} \bullet \begin{array}{c} \xrightarrow{c_1} \\ \xrightarrow{c_2} \end{array} \bullet \quad \text{and} \quad \bullet \xrightarrow{b_3} \bullet \xrightarrow{c_3} \bullet$$

Representation varieties beyond nodes: example

Let A be given by the quiver $\bullet \xrightarrow{a_1} \bullet \overset{b_1}{\underset{b_3}{\overset{\xrightarrow{b_2}}{\rightrightarrows}}} \bullet \overset{c_1}{\underset{c_3}{\overset{\xrightarrow{c_2}}{\rightrightarrows}}} \bullet$

with relations $a_1 b_1 = b_1 c_1 = b_1 c_2 = b_2 c_1 = b_2 c_2 = b_3 c_3 = 0$.

A has no nodes, but we can separate relations, and so a representation variety of A can be written as a product of representation varieties of

$$\bullet \xrightarrow{a_1} \bullet \underset{b_2}{\overset{b_1}{\rightrightarrows}} \bullet \underset{c_2}{\overset{c_1}{\rightrightarrows}} \bullet \quad \text{and} \quad \bullet \xrightarrow{b_3} \bullet \xrightarrow{c_3} \bullet$$

Both quivers have now nodes. Splitting the node in the former, we obtain the product of an affine space with a representation variety of

$$\bullet \xrightarrow{a_1} \bullet \underset{b_2}{\overset{b_1}{\rightrightarrows}} \bullet$$

Representation varieties beyond nodes: example

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Both quivers have now nodes. Splitting the node in the former, we obtain the product of an affine space with a representation variety of

$$\bullet \xrightarrow{a_1} \bullet \overset{b_1}{\underset{b_2}{\rightrightarrows}} \bullet$$

We can drop b_2, and then split the middle, yielding affine spaces. Hence all representation varieties of A have rational singularities.

Semi-stable representations

Choose a weight $\theta \in \mathbb{Z}Q_0$ with $\theta \cdot \mathbf{d} = 0$. By [King '94], the θ-semistable points of $\text{rep}_A(\mathbf{d})$ are

$$\text{rep}_A(\mathbf{d})^{ss}_\theta = \{M \in \text{rep}_A(\mathbf{d}) \mid \forall N \leq M,\ \theta \cdot \underline{\dim} N \leq 0\}.$$

We have a quotient map $\text{rep}_A(\mathbf{d})^{ss}_\theta \twoheadrightarrow \mathcal{M}(\mathbf{d})^{ss}_\theta$ by GIT.

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We have a quotient map $\text{rep}_A(\mathbf{d})^{ss}_\theta \twoheadrightarrow \mathcal{M}(\mathbf{d})^{ss}_\theta$ by GIT.

Let C be an irreducible component of $\text{rep}_A(\mathbf{d})$ with $C^{ss}_\theta \neq \emptyset$. Consider a collection $\{C_i \subseteq \text{rep}_A(\mathbf{d}_i)\}_{i=1}^k$ of irreducible components, each with a nonempty subset of θ-stable points, $C_i \neq C_j$ for $i \neq j$, and also consider some multiplicities $m_i \in \mathbb{Z}_{>0}$, for $i = 1, \dots, k$. We say that $\{(C_i, m_i)\}_{i=1}^k$ is a *θ-stable decomposition of C* if, for a general representation $M \in C^{ss}_\theta$, its corresponding stable factors are in C_i with multiplicity m_i, and write

$$C = m_1 C_1 + \ldots + m_k C_k.$$

Application to decompositions of moduli spaces

Normality of irreducible components is important also for studying moduli spaces of semi-stable representations.

Theorem (Chindris, Kinser '18)

Let $C \subseteq \operatorname{rep}_A(\mathbf{d})^{ss}_\theta$ be an irreducible component with $C^{ss}_\theta \neq \emptyset$. There exists $C = m_1 C_1 + \ldots + m_k C_k$ a θ-stable decomposition of C where $C_i \subseteq \operatorname{rep}_A(\mathbf{d}_i)$, $1 \leq i \leq k$, are pairwise distinct θ-stable irreducible components. Moreover, if $\mathcal{M}(C)^{ss}_\theta$ is an irreducible component of $\mathcal{M}(\mathbf{d})^{ss}_\theta$, then there is a natural morphism

$$\Psi \colon S^{m_1}(\mathcal{M}(C_1)^{ss}_\theta) \times \ldots \times S^{m_r}(\mathcal{M}(C_k)^{ss}_\theta) \to \mathcal{M}(C)^{ss}_\theta$$

which is finite, and birational. In particular, if C is normal then Ψ is an isomorphism.