Upper and Lower Degree Bounds for Generating Invariants

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Geometric Methods in Representation Theory
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Invariant Theory

$K = \mathbb{C}$ base field
G **reductive** algebraic group (e.g., GL_n, semi-simple, finite,...)
V n-dimensional representation of G
$\mathbb{C}[V]$ ring of polynomial functions on V

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Theorem (Hilbert 1890)

$\mathbb{C}[V]^G$ is a finitely generated $\mathbb{C}$-algebra

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Example: SL_2 acts on $V_d = \{a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d\}$
(binary forms of degree d)
$K[V_d] = K[a_0, a_1, \ldots, a_d]$
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Example: G finite, V representation of G

Theorem (E. Noether 1916)

$$\beta_G(V) \leq |G| \textit{ (constant bound if } G \textit{ fixed)}$$

Polynomial Bound for Tori

Example: $T = (\mathbb{C}^\times)^m$ m-dimensional torus
for $t = (t_1, \ldots, t_m) \in T$, $a \in \mathbb{Z}^m$ we write $t^a = t_1^{a_1} \cdots t_m^{a_m}$

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$V = K^n$ representations with weights $\omega_1, \ldots, \omega_n \in \mathbb{Z}^m$
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if T (and m) are fixed, then

$$\beta_T(V) = O(nL^m)$$

where $L = \max\{\|\omega_1\|, \dots, \|\omega_n\|\}$

Polynomial Bounds for Fixed G

V n-dim representation of G
$\mathcal{N} = \{v \in V \mid \forall f \in \mathbb{C}[V]^G \ f(v) = f(0)\}$ null cone

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$$\sigma_G(V) = \min\{d \mid \mathcal{N} \text{ defined by invariants of degree} \leq d\}$$

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Theorem (D. 2001)

$$\beta_G(V) \leq \max\{2, \tfrac{3}{8} n \sigma_G(V)^2\}$$

Polynomial Bounds for Fixed G

$T \subseteq G^0$ max torus of rank r, $\omega_1, \dots, \omega_n$ weights of T acting on V
$L = \max\{\|\omega_1\|, \dots, \|\omega_n\|\}$

Theorem (Kazarnovskii, Popov, Hiss)

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Corollary

$$\beta_G(V) = O(nL^{2m})$$

Non-Constant Symmetric Group

$G = S_n$ acts on $V_n = \mathbb{C}^n$ by permutations
$\mathbb{C}[V_n]^{S_n} = \mathbb{C}[e_1, \ldots, e_n]$, where

$$e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

is k-th elementary symmetric function, so $\beta_{S_n}(V_n) = n$

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For example, for fixed d and $S_n \subseteq S_{n^d}$ we get

$$\beta_{S_n}(\underbrace{V_n \otimes \cdots \otimes V_n}_{d}) = \beta_{S_n}(V_{n^d}) = O(n^{2d})$$

(for $d = 2$ one gets graph invariants)

Matrix Invariants

GL_n acts on $\mathrm{Mat}_{n,n}$ by conjugation

Theorem (Procesi 1976, Razmyslov 1974)

$\mathbb{C}[\mathrm{Mat}_{n,n}^s]^{\mathrm{GL}_n}$ *generated by invariants of the form*

$$(A_1, \ldots, A_s) \mapsto \mathrm{Tr}(A_{i_1} A_{i_2} \cdots A_{i_r})$$

with $r \leq n^2$*, so* $\beta_{\mathrm{GL}_n}(\mathrm{Mat}_{n,n}^s) \leq n^2$

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$\mathrm{SL}_n\times\mathrm{SL}_n$ acts on $\mathrm{Mat}_{n,n}$ by left-right multiplication

Theorem (D.-Makam 2015)

$\mathbb{C}[\mathrm{Mat}_{n,n}^s]^{\mathrm{SL}_n}$ *is generated by invariants of the form*

$$(A_1,\dots,A_s)\mapsto \det(A_1\otimes T_1+\dots+A_s\otimes T_s)$$

with $T_1,\dots,T_s\in\mathrm{Mat}_{d,d}$ *and* $d<n^5$ *and*

$$\beta_{\mathrm{SL}_n\times\mathrm{SL}_n}(\mathrm{Mat}_{n,n}^m)<n^6$$

Non-Constant Torus Action

suppose that $T_n = (\mathbb{C}^\times)^n$ acts on $W_n = \mathbb{C}^{n+1}$ with weights

$$\begin{gathered}(-2, 0, \dots, 0)\\(1, -2, 0, \dots, 0)\\(0, 1, -2, \dots, 0)\\(0, \dots, 0, 1, -2)\\(0, \dots, 0, 0, 1)\end{gathered}$$

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we have

$$\mathbb{C}[W_n]^{T_n} = \mathbb{C}[x_1 x_2^2 x_3^4 \cdots x_{n+1}^{2^n}]$$

and $\beta_{T_n}(W_n) = 2^{n+1} - 1$

Exponential Growth!!

Exponential Lower Bounds for Cubic Forms

Suppose that $G_n = \mathrm{SL}_{3n}$ acts on $V_n = S^3(\mathbb{C}^{3n})$ be the space of cubic forms

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we use the Grosshans principle to reduce the theorem to finding exponential lower bounds for the maximal torus $T_n \subseteq G_n$

we sketch the proof

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V a representation of G, $H \subseteq G$ subgroup
H acts by right multiplication on G: $h \cdot g = gh^{-1}$
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$\mathbb{C}[W \oplus V]^G \twoheadrightarrow (\mathbb{C}[G]^H \otimes \mathbb{C}[V])^G = \mathbb{C}[V]^H$

Kempf-Ness

let $w = (\sum_{i=1}^{n} x_i^2 z_i, \sum_{i=1}^{n} y_i^2 z_i, \sum_{i=1}^{n} \alpha_i x_i y_i z_i) \in W$, where
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the stabilizer of w in $G_n = \mathrm{SL}_{3n}$ is a torus $H_n \subseteq G_n$ (of dim. n)
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from the Corollary we get

$$\beta_{G_n}(V_n^4) = \beta_{G_n}(W \oplus V_n) \geq \beta_{H_n}(V_n) \geq \tfrac{2}{3}(4^n - 1).$$