Generalized Littlewood-Richardson coefficients for branching rules of $\mathrm{GL}(n)$

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Background and motivation: Horn's conjecture

In 1912, H. Weyl asked the following question:

Weyl's eigenvalue problem: Letting $\lambda(i)$ denote a weakly decreasing sequence of n real numbers,

$$\lambda(i): \quad \lambda_1(i) \geq \lambda_2(i) \geq \ldots \geq \lambda_n(i),$$

describe the triples $(\lambda(1), \lambda(2), \lambda(3))$ for which there exist $n \times n$ Hermitian matrices $H(1)$, $H(2), H(3)$ with eigenvalues $\lambda(1)$, $\lambda(2)$, $\lambda(3)$, respectively, such that

$$H(2) = H(1) + H(3).$$

Littlewood-Richardson coefficients

A weakly decreasing sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a ***partition*** if $\lambda_i \in \mathbb{Z}_{\geq 0}$ for all i. In this case, we say it has at most n nonzero parts. Over $\mathbb{C}$,

$$\{\text{weakly decreasing sequences of } n \text{ integers } \lambda = (\lambda_1, \ldots, \lambda_n)\}$$
$$\updownarrow$$
$$\{\text{irreducible rational representations of } \mathrm{GL}(V), \text{ denoted } S^\lambda V\}$$

Definition

Given any three weakly decreasing sequences of n integers λ, μ, ν, the ***Littlewood-Richardson coefficient*** $c^\nu_{\lambda,\mu}$ is defined to be the multiplicity of $S^\nu V$ in $S^\lambda V \otimes S^\mu V$, *i.e.*,

$$c^\nu_{\lambda,\mu} = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}(V)}(S^\nu V, S^\lambda V \otimes S^\mu V).$$

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Theorem (Horn's conjecture (1962))

Let $\lambda(i) = (\lambda_1(i), \ldots, \lambda_n(i))$, $i \in \{1,2,3\}$, be weakly decreasing sequences of n real numbers. Then the following are equivalent:

1. *there exist $n \times n$ complex Hermitian matrices $H(i)$ with eigenvalues $\lambda(i)$ such that*
$$H(2) = H(1) + H(3);$$
2. *the numbers $\lambda_j(i)$ satisfy*
$$|\lambda(2)| = |\lambda(1)| + |\lambda(3)|$$
together with
$$\sum_{j \in I_2} \lambda_j(2) \leq \sum_{j \in I_1} \lambda_j(1) + \sum_{j \in I_3} \lambda_j(3)$$
for every triple (I_1, I_2, I_3) of subsets of $\{1, \ldots, n\}$ of the same cardinality $r < n$ and $c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} \neq 0$;
3. *if $\lambda_j(i)$ is an integer for each $1 \leq j \leq n$, $i \in \{1,2,3\}$, (1) and (2) are equivalent to $c^{\lambda(2)}_{\lambda(1),\lambda(3)} \neq 0$.*

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Horn's conj. (cont.)

A. Klyachko (1998) proved the equivalence of (1) and (2) and noted the connection between Horn's conjecture and Littlewood-Richardson coefficients.

P. Belkale (2001) showed that all inequalities for which $c^{\lambda(I_2)}_{\lambda(I_1),\lambda(I_3)} > 1$ are redundant.

The remaining inequalities would be irredundant by a theorem of Klyachko provided the saturation of Littlewood-Richardson coefficients.

Horn's conj. (cont.)

Theorem (Saturation conjecture)

For weakly decreasing sequences of n integers λ, μ, ν, $c^{N\nu}_{N\lambda,N\mu} \neq 0$ for some positive integer N if and only if $c^{\nu}_{\lambda,\mu} \neq 0$.

This was first proven by A. Knutson and T. Tao (1999) using combinatorial gadgets called honeycombs and hive models.

It was proven again in the context of quiver theory by H. Derksen and J. Weyman (2000).

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Generalized Littlewood-Richardson coefficients

For an m-tuple of weakly decreasing sequences of n integers $\underline{\lambda} = (\lambda(1), \ldots, \lambda(m))$, $\lambda(i) = (\lambda_1(i), \ldots, \lambda_n(i))$,

1. $f_1(\underline{\lambda}) := \sum c^{\lambda(1)}_{\alpha(1),\alpha(2)} c^{\lambda(2)}_{\alpha(2),\alpha(3)} \cdots c^{\lambda(m-1)}_{\alpha(m-1),\alpha(m)} c^{\lambda(m)}_{\alpha(m),\alpha(1)}$, $m \geq 4$ and even;
2. $f_2(\underline{\lambda}) := \sum c^{\alpha(1)}_{\lambda(1),\lambda(2)} c^{\lambda(3)}_{\alpha(1),\alpha(2)} \cdots c^{\lambda(m-2)}_{\alpha(m-4),\alpha(m-3)} c^{\alpha(m-3)}_{\lambda(m-1),\lambda(m)}$, $m \geq 4$;
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Motivation

f_1 describes the coefficients arising from the branching rule for the diagonal embedding $\mathrm{GL}(n) \subseteq \mathrm{GL}(n) \times \mathrm{GL}(n)$ in the case $m = 6$.

f_2 describes the branching rule for the direct sum embedding $\mathrm{GL}(n) \times \mathrm{GL}(n') \subseteq \mathrm{GL}(n + n')$ when $m = 6$.

The multiplicity f_3 describes the tensor product multiplicities for extremal weight crystals of type $A_{+\infty}$ when $m = 6$. This generalized multiplicity is described by C. Chindris, and is found to have connections with long exact sequences of finite, abelian p-groups, parabolic affine Kazhdan-Lusztig polynomials, and decomposition numbers for q-Schur algebras.

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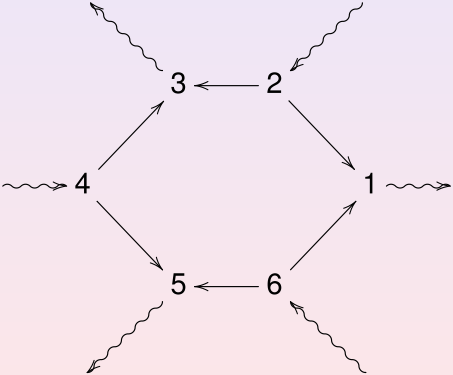
Sun quiver

$$f_1(\underline{\lambda}) := \sum_{\alpha(i)} c^{\lambda(1)}_{\alpha(1),\alpha(2)} c^{\lambda(2)}_{\alpha(2),\alpha(3)} c^{\lambda(3)}_{\alpha(3),\alpha(4)} c^{\lambda(4)}_{\alpha(4),\alpha(5)} c^{\lambda(5)}_{\alpha(5),\alpha(6)} c^{\lambda(6)}_{\alpha(6),\alpha(1)}$$



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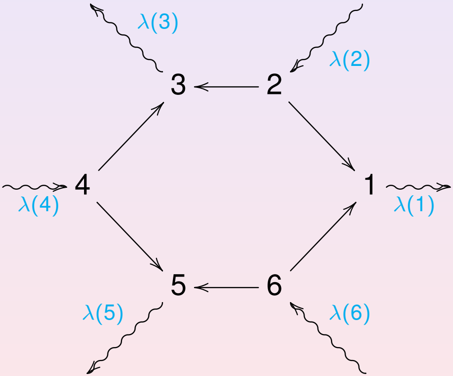

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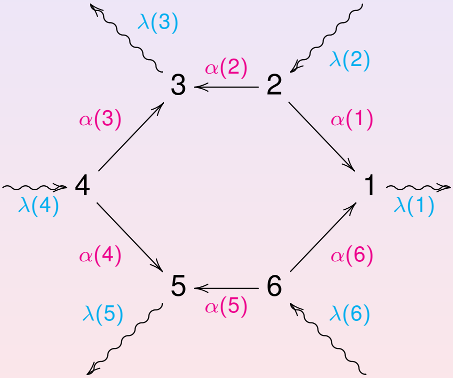

$$\beta(i,j) = j,\ 1 \le i \le 6,\ 1 \le j \le n$$

Saturation property

Lemma

Let $\lambda(1), \ldots, \lambda(m)$, $m \geq 4$ and even, be weakly decreasing sequences of n integers. Then for every integer $r \geq 1$, we have

$$f_1(r\lambda(1), \ldots, r\lambda(m)) = \sum_{\alpha(i)} c^{r\lambda(1)}_{\alpha(1),\alpha(2)} c^{r\lambda(2)}_{\alpha(2),\alpha(3)} \cdots c^{r\lambda(m)}_{\alpha(m),\alpha(1)}$$

$$= \dim \mathrm{SI}(Q, \beta)_{r\sigma_1},$$

where

$$\sigma_1(j,i) = \begin{cases} (-1)^i(\lambda(i)_j - \lambda(i)_{j+1}) & 1 \leq i \leq m,\ 1 \leq j \leq n-1 \\ (-1)^i \lambda(i)_n & 1 \leq i \leq m,\ j = n. \end{cases}$$

Saturation property (cont.)

Theorem (C.) (Saturation property)

Let $\lambda(1), \ldots, \lambda(m)$ be weakly decreasing sequences of n integers for $m \geq 4$ and even. For every integer $r \geq 1$,

$$f_1(r\lambda(1), \ldots, r\lambda(m)) \neq 0 \iff f_1(\lambda(1), \ldots, \lambda(m)) \neq 0.$$

Horn-type inequalities

Let $\beta_1 \leq \beta$ be a dimension vector which is weakly increasing with jumps of at most one along each of the flags.

Define the jump sets

$$I_i = \{l \mid \beta_1(l,i) > \beta_1(l-1,i),\ 1 \leq l \leq n\}$$

Conversely, each tuple $I = (I_1, \ldots, I_m)$ of subsets of $\{1, \ldots, n\}$ defines a dimension vector β_I because if

$$I_i = \{z_1(i) < \cdots < z_r(i)\},$$

then $\beta_I(j,i) = j - 1$ for all $z_{k-1}(i) \leq j < z_k(i)$ for all $1 \leq k \leq r+1$.

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Horn-type inequalities (cont.)

For a subset $I = \{z_1 < \ldots < z_r\} \subseteq \{1, \ldots, n\}$, define the partition

$$\lambda(I) = (z_r - r, \ldots, z_1 - 1).$$

Define

$$\lambda_1(I_i) = \begin{cases} \lambda'(I_i) & i \text{ even} \\ \lambda'(I_i) - ((|I_i| - |I_{i-1}| - |I_{i+1}|)^{n-|I_i|}) & i \text{ odd.} \end{cases}$$

$\lambda_1(I_i)$ is a weakly decreasing sequence of integers for each i.

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Proposition (C.) (Horn-type inequalities)

Let $\lambda(1), \ldots, \lambda(m)$ be weakly decreasing sequences of n reals, $m \geq 4$ and even. The following are equivalent for the sun quiver Q:

1. $\dim \mathrm{SI}(\mathrm{Q}, \beta)_\sigma \neq 0$;
2. the numbers $\lambda(i)_j$ satisfy

$$\sum_{i \text{ even}} |\lambda(i)| = \sum_{i \text{ odd}} |\lambda(i)|$$

and

$$\sum_{j \in I_i} \sum_{i \text{ even}} \lambda(i)_j \leq \sum_{j \in I_i} \sum_{i \text{ odd}} \lambda(i)_j$$

for every tuple $(I_1, \ldots, I_m)$ for which $|I_i| < n$ for some i, the $\underline{\lambda}_1(I_i)$ are partitions, $1 \leq i \leq m$, and

$$f_1(\underline{\lambda}_1(I_1), \ldots, \underline{\lambda}_1(I_m)) \neq 0;$$

In particular, this provides a recursive procedure for finding all nonzero generalized Littlewood-Richardson coefficients of this type.

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Example

For $n = 2$ and $m = 6$ for the sun quiver Q, $\dim \mathrm{SI}(Q, \beta)_\sigma \neq 0$ if and only if the defining partitions $\lambda(1), \ldots, \lambda(6)$ satisfy

$$|\lambda(1)| + |\lambda(3)| + |\lambda(5)| = |\lambda(2)| + |\lambda(4)| + |\lambda(6)|,$$

and

$$\begin{array}{ll}
\lambda(2)_1 \leq \lambda(1)_1 + \lambda(3)_1 & \lambda(2)_2 \leq \lambda(1)_1 + \lambda(3)_2 \\
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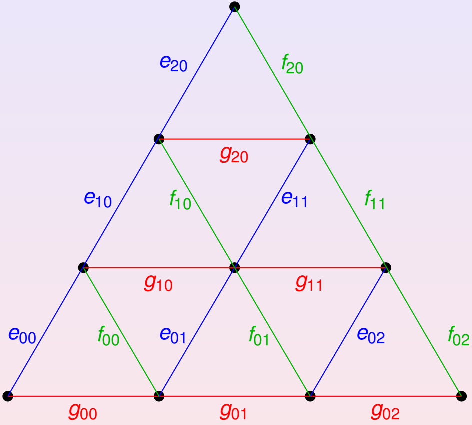
\end{array}$$

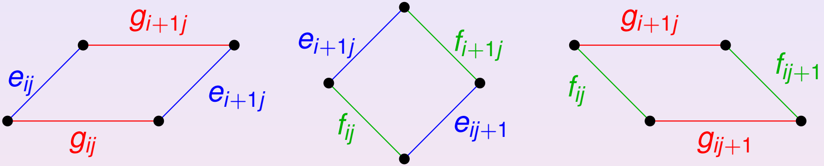
along with the inequalities obtained by permutations of the flags that respect the symmetries of the sun quiver. Moreover, this is a minimal list.

Hive models



$$e_{ij} + f_{ij} = g_{ij} \qquad e_{i\,j+1} + f_{ij} = g_{i+1\,j}$$

Rhombus inequalities



$$e_{ij} \geq e_{ij+1}, \quad g_{ij} \geq g_{i+1j} \qquad\qquad f_{i+1j} \geq f_{ij}, \quad e_{ij+1} \geq e_{i+1j}$$

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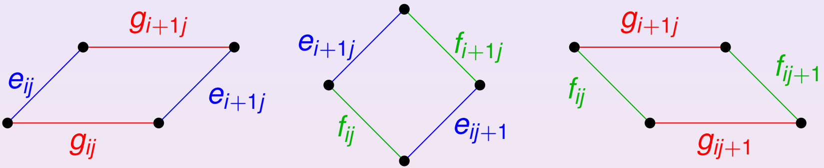

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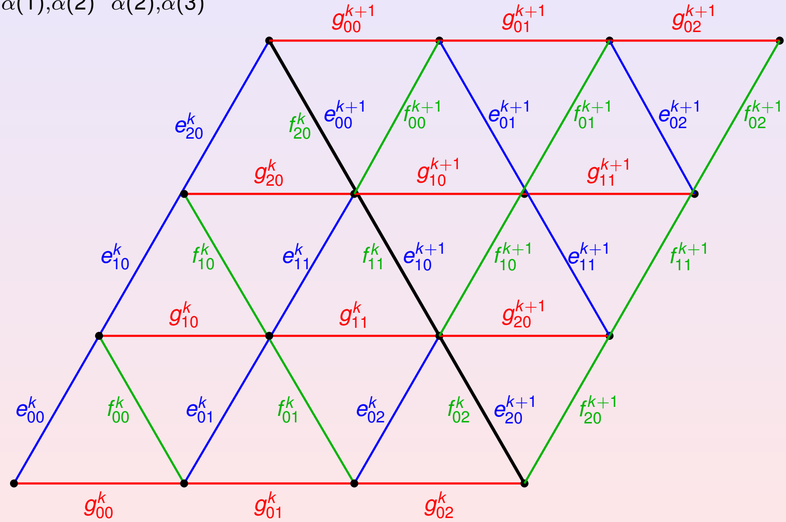
LR hives (cont.)

Theorem (Knutson, Tao (1999))

The Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ is the number of integer LR hives with boundary labels determined by λ, μ, and ν.

Gluing LR hives

$$c^{\lambda(1)}_{\alpha(1),\alpha(2)}\, c^{\lambda(2)}_{\alpha(2),\alpha(3)}$$

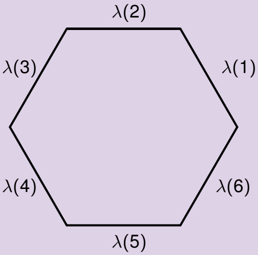


Theorem (C.)

For partitions $\lambda(1),\dots,\lambda(m)$, $m \geq 4$ and even, of no more than n parts, the generalized Littlewood-Richardson coefficient

$$\sum c^{\lambda(1)}_{\alpha(1),\alpha(2)} c^{\lambda(2)}_{\alpha(2),\alpha(3)} \cdots c^{\lambda(m-1)}_{\alpha(m-1),\alpha(m)} c^{\lambda(m)}_{\alpha(m),\alpha(1)}$$

is equal to the number of integer (m,n)-LR sun hives with external boundary labels determined by the $\lambda(i)$ in cyclic orientation (so that the edge labeled $\lambda(r)$ is between the edges labeled $\lambda(r+1)$ and $\lambda(r-1)$). For instance, the boundary labels of a $(6,n)-$LR sun hive are



Geometric complexity theory

K. Mulmuley and M. Sohoni introduced geometric complexity theory for the purpose of approaching fundamental problems in complexity theory (such as P vs. NP) through algebraic geometry and representation theory.

A. Knutson and T. Tao (1999) and J. De Loera and T. McAllister (2006) independently showed that whether $c^{\nu}_{\lambda,\mu} > 0$ could be computed in polynomial time. Mululey and Sohoni (2005) showed positivity could be computed in strongly polynomial time.

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Making a linear program

For each $1 \leq r \leq m$, the rhombus inequalities and boundary conditions may be written as a linear program $A_r\mathbf{x_r} \leq \mathbf{b_r}$, where

- A_r is a matrix with entries $0, 1, -1$,
- $\mathbf{x_r}$ is the vector of interior edges e^r_{ij}, f^r_{ij}, g^r_{ij},
- the entries of $\mathbf{b_r}$ are homogeneous, linear forms in the entries of $\lambda(r)$, and are thus integral when $\lambda(r)$ is a partition.

Combining each of these produces a linear program $A\mathbf{x} \leq \mathbf{b}$.

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Complexity of generalized LR coefficients

Theorem (C.)

For partitions $\lambda(1), \ldots, \lambda(m)$, $m \geq 4$ and even, determining whether

$$f_1(\lambda(1), \ldots, \lambda(m)) = \sum c^{\lambda(1)}_{\alpha(1),\alpha(2)} \cdots c^{\lambda(m-1)}_{\alpha(m-1),\alpha(m)} c^{\lambda(m)}_{\alpha(m),\alpha(1)}$$

is positive can be decided in strongly polynomial time.

Sketch of proof

Proving the multiplicity is nonzero is equivalent to showing that the polytope contains an integer (m, n)-LR sun hive.

If the polytope is nonempty, it has a vertex $\mathbf{v}$, A must be of full rank, and necessarily $A\mathbf{v} = \mathbf{b}$. Because the entries of A and $\mathbf{b}$ are integers, the entries of $\mathbf{v}$ are rational.

Scaling the polytope produces an integral vertex. Hence, the scaled polytope has an integral (m, n)-LR sun hive, which shows the scaled multiplicity is nonzero.

By the saturation property, this proves the unscaled multiplicity is nonzero.

Determining whether the polytope is nonempty can be determined in polynomial time using techniques from linear programming.

$A\mathbf{x} \leq \mathbf{b}$ is combinatorial, so this can in fact be checked in strongly polynomial time (É. Tardos, 1986).

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Thank you!