Quantum dilogarithm identities: the geometry of quiver representations viewpoint

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Quantum dilogarithm series and pentagon identity

Let $\mathcal{P}_n = \prod_{i=1}^{n} \frac{1}{1-q^i}$

- generating function for $\pi(N; n)$
- Hilbert series (in $q^{1/2}$) of algebra: $\mathbb{R}[c_1, \ldots, c_n]$, $\deg(c_i) = 2i$.
- Poincaré series for $\mathrm{H}^*(B\,\mathrm{GL}(n, \mathbb{C}))$

Definition 1

For a variable z, the **quantum dilogarithm series** in $\mathbb{Q}(q^{1/2})[[z]]$ is

$$\mathbb{E}(z) = 1 + \sum_{n\geqslant 1} \frac{(-z)^n q^{n^2/2}}{\prod_{i=1}^{n}(1-q^i)} = 1 + \sum_{n\geqslant 1} (-z)^n\, q^{n^2/2}\, \mathcal{P}_n.$$

Theorem ($\mathbb{E}$-Pentagon Identity)

In the algebra $\mathbb{Q}(q^{1/2})\langle\langle y_1, y_2\rangle\rangle/(y_1 y_2 - q y_2 y_1)$ *we have*

$$\mathbb{E}(y_1)\,\mathbb{E}(y_2) = \mathbb{E}(y_2)\,\mathbb{E}(-q^{-1/2} y_2 y_1)\,\mathbb{E}(y_1).$$

Pentagon identity, cont.

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(-q^{-1/2}y_2y_1)\mathbb{E}(y_1)$$

Pentagon identity has several interpretations, depending on your tastes.

- Comparing coefficients of $y_1^{\gamma_1}y_2^{\gamma_2}$ on each side gives identities:

$$\mathcal{P}_{\gamma_1}\mathcal{P}_{\gamma_2} = \sum_{(m_{10},m_{01},m_{11})\vdash(\gamma_1,\gamma_2)} q^{m_{10}m_{01}}\,\mathcal{P}_{m_{10}}\mathcal{P}_{m_{01}}\mathcal{P}_{m_{11}},$$

- Combinatorics: Implies Durfee's square/rectangle identities
- Analysis (and number theory and physics): quantum version of the five-term identity for the Rogers dilogarithm
- Geometry: related to *refined DT-invariant* for A_2 quiver; simplest example of "wall-crossing formula"
- Topology: two ways to count Betti numbers of the $GL(\gamma_1,\mathbb{C}) \times GL(\gamma_2,\mathbb{C})$-equivariant cohomology of $V = \mathrm{Hom}(\mathbb{C}^{\gamma_2},\mathbb{C}^{\gamma_1})$—on LHS use that V is contractible, on RHS cut V into orbits

We adopt topological approach:
invent finite stratifications of a quiver's representation space,
and compare Betti numbers via spectral sequence arguments.

Quivers

- Let $Q = (Q_0, Q_1)$ be a quiver with vertex set Q_0 and arrow set Q_1.
- For $a \in Q_1$ let $ta, ha \in Q_0$ respectively denote its head and tail (target and source) vertex.
- For any dimension vector γ we have the representation space

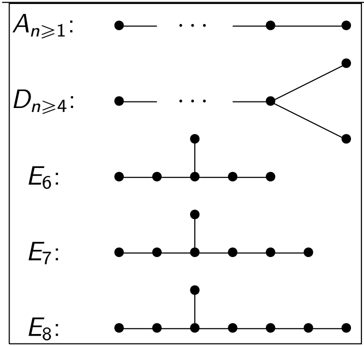
$$\mathrm{Rep}_\gamma = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma(ta)}, \mathbb{C}^{\gamma(ha)})$$

  with action of $\mathbf{G}_\gamma = \prod_{i \in Q_0} \mathrm{GL}(\mathbb{C}^{\gamma(i)})$ by base-change at each vertex.
- Let λ denote the form (extend linearly to all dimension vectors)

$$\lambda(e_i, e_j) = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

Summary of representation theory of Dynkin quivers

Dynkin quivers $\iff$ orientations of ADE Dynkin diagrams



Root Systems

- **simples**: $\Delta = \{\alpha_i : i \in Q_0\}$
- **positives**: $\Phi = \{\beta_j\}$
- For each $\beta \in \Phi$ there are unique positive integers d_α^β such that

$$\beta = \sum_{\alpha \in \Delta} d_\alpha^\beta \, \alpha$$

Theorem (Gabriel's Theorem)

For any γ*:* $\left\{ \begin{array}{c} \mathbf{G}_\gamma\textit{-orbits} \\ \textit{in } \mathrm{Rep}_\gamma \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \textbf{Kostant Partitions} \\ (m_\beta)_{\beta \in \Phi} \in \mathbb{N}^\Phi : \\ \sum_{\beta \in \Phi} m_\beta \, \beta = \gamma \end{array} \right\}$

Quantum algebra of Q

Let $q^{1/2}$ be an indeterminate and q denote its square. The **quantum algebra** $\mathbb{A}_Q$ of the quiver is the $\mathbb{Q}(q^{1/2})$-algebra

- spanned as vector space by symbols y_γ, one for each dimension vector γ
- subject to the relation

$$y_{\gamma_1+\gamma_2} = -q^{-\frac{1}{2}\lambda(\gamma_1,\gamma_2)} y_{\gamma_1} y_{\gamma_2}.$$

- The elements y_{e_i} form a set of algebraic generators.
- Let $\hat{\mathbb{A}}_Q$ denote the **completed quantum algebra** (allow power series in y-variables)

Example: A_2

Consider the quiver $1 \leftarrow 2$. Set $y_i = y_{e_i}$. Then

$$y_1\, y_2 = q\, y_2\, y_1 \qquad\qquad y_{e_1+e_2} = -q^{-1/2} y_2\, y_1$$

Thus the Pentagon Identity says that

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(y_{e_1+e_2})\mathbb{E}(y_1).$$

- The left-hand side reflects an ordering of the **simple** roots of A_2;

- the right-hand side reflects an ordering for the **positive** roots of A_2.

Reineke's EPI generalization

- For each $i \in Q_0$, $\exists \alpha_i$ simple root, identified with dimension vector e_i.
- Since each positive root has unique decomposition

$$\beta = \sum_{i \in Q_0} d^{\beta}_{\alpha_i} \alpha_i,$$

the positive root β is also identified with a dimension vector

Theorem (Reineke (2010), Rimányi (2013))

For Dynkin quivers Q there exist orderings on the simple and positive roots such that

$$\prod_{\alpha \text{ simple}}^{\curvearrowright} \mathbb{E}(y_\alpha) = \prod_{\beta \text{ positive}}^{\curvearrowright} \mathbb{E}(y_\beta).$$

where "$\curvearrowright$" indicates the products are taken in the specified orders.

- Actually, the common value of both sides is denoted $\mathbb{E}_Q$ and called the **refined DT invariant** of the quiver (Keller, 2010).

General acyclic factorizations

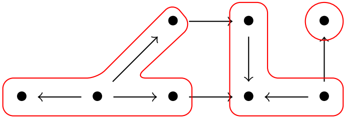
Theorem (A. (2018))

For Q acyclic and any admissible Dynkin subquiver partition $(Q_1, \ldots, Q_r)$ we have a factorization of the DT-invariant

$$\mathbb{E}_Q = \left(\overrightarrow{\prod_{\sigma \in \Phi(Q_1)}} \mathbb{E}(y_\sigma) \right) \cdots \left(\overrightarrow{\prod_{\tau \in \Phi(Q_r)}} \mathbb{E}(y_\tau) \right).$$

- Suppose Q is Dynkin. When $r = 1$ we obtain the "positive root side" of the Reineke identity.
- When $r = |Q_0|$, and hence each Q_i = "the singleton vertex i", we obtain the "simple root side" of the Reineke identity.

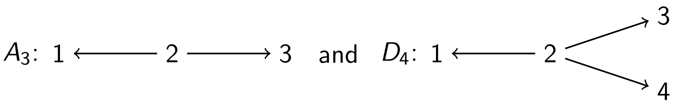
General acyclic quivers



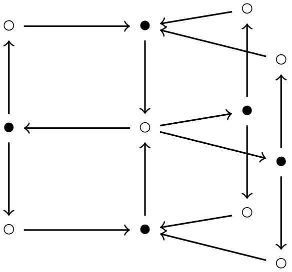
- Acyclic quiver with **admissible Dynkin subquiver partition** (circled in red)
- **Admissible** means that when each circled subquiver is shrunk to a vertex, the resulting quiver is still acyclic.
- This is exactly the condition which allows the roots corresponding to the circled diagrams to be totally ordered
- The ordering $\Phi(A_1) \prec \Phi(A_3) \prec \Phi(D_4)$ is determined by a $\lambda \leqslant 0$ condition.
- Corresponding factorization of $\mathbb{E}_Q$ has $1 + 6 + 12 = 19$ terms:

$$\mathbb{E}_Q = \mathbb{E}(y_{A_1}) \cdot \left(\overrightarrow{\prod_{\sigma \in \Phi(A_3)}} \mathbb{E}(y_\sigma) \right) \cdot \left(\overrightarrow{\prod_{\tau \in \Phi(D_4)}} \mathbb{E}(y_\tau) \right).$$

Square products



- Take alternating orientations of two quivers as above
- Form $A_3 \square D_4$ with grid of vertices $A_3 \times D_4$ and reverse arrows in the full sub-quivers $\{i\} \times D_4$ and $A_3 \times \{j\}$ whenever i is sink in A_3 and j is source in D_4



Square products: a result

Theorem (A.–Rimányi (2016))

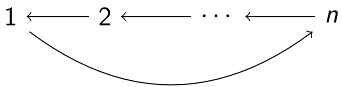
For the square product $A_n \square A_m$ we have the identity

$$\overrightarrow{\prod_{(i,\phi)\in\Delta(A_n)\times\Phi(A_m)}} \mathbb{E}(y_{(i,\phi)}) = \overrightarrow{\prod_{(\psi,j)\in\Phi(A_n)\times\Delta(A_m)}} \mathbb{E}(y_{(\psi,j)})$$

for prescribed orders on the root sets above.

- Keller, via cluster algebras/categories
  - Find a **maximal green sequence** of quiver mutations
  - From this, perform an algorithm and each side is implicitly defined by the end point of this algorithm
  - The result must be the DT-invariant $\mathbb{E}_{Q,W}$
- A.-Rimányi, via topology/geometry
  - For each γ, stratify Rep_γ (finitely many strata)
  - Need theory of **quivers with potential** (here, sum of all the cycles)
  - Spectral sequence in **rapid-decay cohomology** relates Betti numbers

More wild examples: n-cycles



- Quiver with potential $W = -a_1 a_2 \cdots a_n$ (a_i has head i)
- Intersects square product case: 4-cycle is $A_2 \square A_2$

Theorem (A. (2018))

Let $n \geqslant 3$, $1 \leqslant \ell < n$, and $j = n - \ell$. In the completed quantum algebra $\hat{\mathbb{A}}_{\Gamma_n}$, we have the following quantum dilogarithm identity

$$\overrightarrow{\prod_{\phi \in \Phi(A_n) \setminus \{\beta_0\}}} \mathbb{E}(y_\phi) = \overrightarrow{\prod_{\psi \in \Phi(A_\ell) \times \Phi(A_j)}} \mathbb{E}(y_\psi)$$

for specified orders on the root sets.

n-cycles, upshot

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- Able to conjecture MGSs which achieve each side via Keller's algorithm by looking at **Auslander–Reiten** graphs.
- The MGS which achieves the left has length $|\Phi(A_n)| - 1 = \frac{1}{2}n(n+1) - 1$.
- Conjecture that this is the *maximal length* of an MGS
- Gives upper bound for No Gap Conjecture (Brüstle, Dupont, Perotin, 2014)

Thank you