Auslander's formula in dualizing variaties

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November 19, 2017

The Auslander's formula

Theorem (Auslander)

Let Λ be an artin algebra.
$(\Lambda-\mathrm{mod})-\mathrm{mod}$*: the category of finitely presented (contravariant) functors,*
$(\Lambda-\mathrm{mod})-\mathrm{mod}_0$*: the category of finitely presented functors vanishing on projective modules.*
Then

$$\frac{(\Lambda-\mathrm{mod})-\mathrm{mod}}{(\Lambda-\mathrm{mod})-\mathrm{mod}_0} \cong \Lambda-\mathrm{mod}$$

Remark

$$
\begin{aligned}
(\Lambda - \text{mod}) - \text{mod}_0 &= \{F \mid (-, X) \stackrel{(-,f)}{\to} (-, Y) \to F \to 0 \\
&\quad \textit{for some epimorphism } f : X \to Y\} \\
&\cong (\underline{\Lambda - \text{mod}}) - \text{mod}
\end{aligned}
$$

Let $\mathcal{A}$ be an additive category with pseudo-kernel. i.e. for any morphism $f : A \to B$, there is a morphism $g : K \to A$ such that

$$\mathrm{Hom}(-, K) \stackrel{\mathrm{Hom}(-,g)}{\to} \mathrm{Hom}(-, A) \stackrel{\mathrm{Hom}(-,f)}{\to} \mathrm{Hom}(-, B)$$

is exact.

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Proposition

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An additive category $\mathcal{A}$ has pseudo-kernel if and only if $\mathcal{A}-\mathrm{mod}$ is abelian.

If $\mathcal{A}$ has pseudo-kernel, then any contravariantly finite subcategory $\mathcal{X}$ has pseudo-kernel.

Let $\mathcal{X}$ be a contravariantly finite subccategory of $\mathcal{A}$; $\mathcal{A}-\text{mod}$ be the category of finitely presented functors on $\mathcal{A}$;
$\mathcal{T}_{\mathcal{X}} = \{F \in \mathcal{A}-\text{mod} \,|\, (-, X) \to F \to 0 \text{ for some } X \in \mathcal{X}\}$;
$\mathcal{F}_{\mathcal{X}} = \{F \in \mathcal{A}-\text{mod} \,|\, F(X) = 0 \text{ for all } X \in \mathcal{X}\}$.

Definition

$(\mathcal{F}, \mathcal{T})$ is a torsion theory in abelian category $\mathcal{C}$, if
(1) $\mathcal{T}^{\perp} = \mathcal{F}$ and ${}^{\perp}\mathcal{F} = \mathcal{T}$.
(2) For any $M \in \mathcal{C}$, there is an exact sequence
$0 \to tM \to M \to rM \to 0$, *where $tM \in \mathcal{T}$ and $rM \in \mathcal{F}$.*

Theorem (Gentle, Todorov)

$(\mathcal{F}_{\mathcal{X}}, \mathcal{T}_{\mathcal{X}})$ is a torsion theory.

Let $F \in \mathcal{A}-\text{mod}$.

$$(-, A) \longrightarrow (-, B) \longrightarrow (-, C) \longrightarrow F \longrightarrow 0$$

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$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
(-,A) & \longrightarrow & (-,E) & \longrightarrow & (-,X_C) & \longrightarrow & tF & \longrightarrow & 0 \\
\| & & \downarrow & (p.b.) & \downarrow{\scriptstyle (-,f_C)} & & \downarrow & & \\
(-,A) & \longrightarrow & (-,B) & \longrightarrow & (-,C) & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

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(-,A) & \longrightarrow & (-,B) & \longrightarrow & (-,C) & \longrightarrow & F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
(-,E) & \longrightarrow & (-,B \oplus X_C) & \longrightarrow & (-,C) & \longrightarrow & rF & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}
$$

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Define a functor $e : \mathcal{X}-\text{mod} \to \mathcal{A}-\text{mod}$:
If $F \in \mathcal{X}$ has a presentation

$$(\mathcal{X}, X_1) \stackrel{(\mathcal{X},f)}{\to} (\mathcal{X}, X_0) \to F \to 0,$$

define eF by

$$(\mathcal{A}, X_1) \stackrel{(\mathcal{A},f)}{\to} (\mathcal{A}, X_0) \to eF \to 0.$$

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Theorem

For any $F \in \mathcal{A}-\text{mod}$, $\text{res}_{\mathcal{X}} F \in \mathcal{X}-\text{mod}$.

Let $F \in \mathcal{A}-\text{mod}$.

$$
\begin{array}{ccccccccc}
(-,K) & \longrightarrow & (-,X_E) & \longrightarrow & (-,X_C) & \longrightarrow & e\,\mathrm{res}_{\mathcal{X}}\,F & \longrightarrow & 0 \\
\big\downarrow & (p.b.) & \big\downarrow (-,f_E) & & \big\| & & \big\downarrow & & \\
(-,A) & \longrightarrow & (-,E) & \longrightarrow & (-,X_C) & \longrightarrow & tF & \longrightarrow & 0 \\
\big\| & & \big\downarrow & (p.b.) & \big\downarrow (-,f_C) & & \big\downarrow & & \\
(-,A) & \longrightarrow & (-,B) & \longrightarrow & (-,C) & \longrightarrow & F & \longrightarrow & 0
\end{array}
$$

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- $r : \mathcal{A}-\text{mod} \to \mathcal{F}_{\mathcal{X}}$ is a left adjoint of the inclusion $i : \mathcal{F}_{\mathcal{X}} \to \mathcal{A}-\text{mod}$.

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- $\text{res}_{\mathcal{X}} : \mathcal{A}-\text{mod} \to \mathcal{X}-\text{mod}$ is a right adjoint of $e : \mathcal{X}-\text{mod} \to \mathcal{A}-\text{mod}$.

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- $r : \mathcal{A}-\text{mod} \to \mathcal{F}_{\mathcal{X}}$ is a left adjoint of the inclusion $i : \mathcal{F}_{\mathcal{X}} \to \mathcal{A}-\text{mod}$.
- $\text{res}_{\mathcal{X}} : \mathcal{A}-\text{mod} \to \mathcal{X}-\text{mod}$ is a right adjoint of $e : \mathcal{X}-\text{mod} \to \mathcal{A}-\text{mod}$.

Proposition

There is an exact sequence of categories

$$\mathcal{O} \longrightarrow \mathcal{F}_{\mathcal{X}} \xrightarrow{i} \mathcal{A}-\text{mod} \xrightarrow{\text{res}_{\mathcal{X}}} \mathcal{X}-\text{mod} \longrightarrow \mathcal{O}$$

where i is the inclusion functor with $r \dashv i$ and $e \dashv \text{res}_{\mathcal{X}}$.

Question: when does the functor $\mathrm{res}_{\mathcal{X}}$ has a right adjoint?

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Theorem (Asadollahi,J., Hafezi, R., Keshavarz, M.H, 2017)

When $\mathcal{A}$ is a contravariantly finite subcategory of $\Lambda-\mathrm{mod}$ for some artin algebra Λ containing all the projective Λ modules and $\mathcal{X}$ is the category of projective Λ modules, there is a recollement

$$\mathcal{F}_{\mathcal{X}} \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i}}} \mathcal{A}-\mathrm{mod} \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{res}_{\mathcal{X}}}}} \mathcal{X}-\mathrm{mod},$$

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$$\mathcal{F}_{\mathcal{X}} \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i}}} \mathcal{A}-\text{mod} \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\text{res}_{\mathcal{X}}}}} \mathcal{X}-\text{mod},$$

Notice in this situation, $\mathcal{F}_{\mathcal{X}} \cong \mathcal{A}-\text{mod}_0$ and $\mathcal{X}-\text{mod} \cong \Lambda-\text{mod}$.

Definition

Let $\mathbf{k}$ be a commutative artin ring with radical $\mathbf{r}$ and $E(\mathbf{k}/\mathbf{r})$ be the injective envelope of the $\mathbf{k}$ module $\mathbf{k}/r$. Denote by $D = \mathrm{Hom}_{\mathbf{k}}(-, E(\mathbf{k}/r))$ the duality.
Then a Hom*-finite additive $\mathbf{k}$ category $\mathcal{C}$ is called a dualizing $\mathbf{k}$-variety if there is an equivalence*

$$\begin{array}{ccc} \mathcal{C}-\mathrm{mod} & \to & \mathcal{C}^{op}-\mathrm{mod} \\ F & \mapsto & DF. \end{array}$$

For example, $\Lambda-\mathrm{mod}$ is a dualizing variety. Any functorially finite subcategory of a dualizing variety is again a dualizing variety.

Theorem (Ogawa, 2017)

When $\mathcal{A}$ is a dualizing variety and $\mathcal{X} \subseteq \mathcal{A}$ is a functorially finite subcategory, there is a recollement

$$\mathcal{F}_{\mathcal{X}} \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{i}}} \mathcal{A}-\mathrm{mod} \underset{\longleftarrow}{\overset{\longleftarrow}{\xrightarrow{\mathrm{res}_{\mathcal{X}}}}} \mathcal{X}-\mathrm{mod},$$

Theorem

*Let $\mathcal{C}$ be a dualizing **k**-variety. Let $\mathcal{A}$ be a contravariantly finite subcategory of $\mathcal{C}$ and $\mathcal{X} \subseteq \mathcal{A}$ be a functorially finite subcategory of $\mathcal{C}$. Then we have a recollement of abelian categories:*

$$\mathcal{F}_{\mathcal{X}} \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{i} \\ \xleftarrow[coind_{\mathcal{A}}]{} \end{array} \mathcal{A}-\mathrm{mod} \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{\mathrm{res}_{\mathcal{X}}} \\ \xleftarrow[coind_{\mathcal{X}}]{} \end{array} \mathcal{X}-\mathrm{mod}\,.$$

This unifies the previous theorems.

The right adjoint of $\text{res}_{\mathcal{X}}$ is given by the coinduction functor: $coind_{\mathcal{X}} F := \text{Hom}(\text{Hom}_{\mathcal{A}}(\mathcal{X}, -), F)$.
Since, suppose T is the right adjoint of $\text{res}_{\mathcal{X}}$, then

$$\begin{aligned} coind_{\mathcal{X}} F &= \text{Hom}((\mathcal{X}, -), F) = \text{Hom}(\text{res}_{\mathcal{X}}(\mathcal{A}, -), F) \\ &= \text{Hom}((\mathcal{A}, -), TF) = TF. \end{aligned}$$

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