Singularities of Dual Varieties Associated to Exterior Representations

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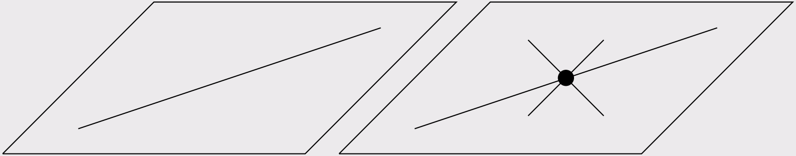
Outline

Projectivization

Let V be a vector space over $\mathbb{C}$, V^* be its dual. The set of one dimensional subspaces of V is called *projectivization of* V and denoted by $\mathbb{P}(V)$. For each point in $\mathbb{P}(V)$ we can associate a hyperplane. After regarding those hyperplanes as points, *dual projective space* $\mathbb{P}(V)^* \cong \mathbb{P}(V^*)$ is obtained.

Picture of Duality



Point in $(\mathbb{P}^2)^*$ Line in $(\mathbb{P}^2)^*$

Dual Variety

Let $X \subset \mathbb{P}^N$ be a projective variety. *Dual variety* $X^\vee \subset \left(\mathbb{P}^N\right)^*$ is defined as the closure of the set of all tangent hyperplanes to X.

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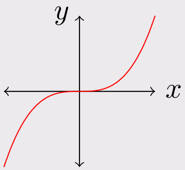
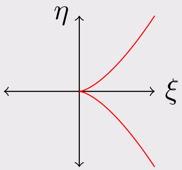
(2)



$y = x^3$



$4\xi^3 = 27\eta^2$

Determinant

Consider Segre embedding:

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$[x_0 : x_1] \times [y_0 : y_1] \mapsto [x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1]$$

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Multilinear form $f = ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1$

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$\frac{\partial f}{\partial x_0} = ay_0 + by_1 = 0,\qquad \frac{\partial f}{\partial x_1} = cy_0 + dy_1 = 0$

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$$\frac{\partial f}{\partial y_0} = ax_0 + cx_1 = 0, \qquad \frac{\partial f}{\partial y_1} = bx_0 + dx_1 = 0$$

System of equations have a nontrivial solution if and only if

$$ad - bc = 0$$

Hyperdeterminant

Segre Embedding

Consider the Segre embedding:
$X = \mathbb{P}^{k_1} \times \ldots \times \mathbb{P}^{k_r} \hookrightarrow \mathbb{P}^{(k_1+1)\ldots(k_r+1)-1}$
where each $\mathbb{P}^{k_j}$ is projectivization of $V_j^* = \mathbb{C}^{k_j+1}$. If $X^\vee$ is a hypersurface then its defining equation is called *hyperdeterminant* which is a homogeneous polynomial function on $V_1 \otimes \ldots \otimes V_r$.

Examples

If $r = 2$, $k_1 = k_2$ then hyperdeterminant is classical determinant.
The first nontrivial was case founded by Cayley, when $r = 3$, $k_i = 1$: $\Delta\left(\det|Ax + By|\right)$ where A, B are 2×2 matrices, x, y are variables to take discriminant.

Coordinate System

Choose a coordinate system $x^j = \left(x_0^j, \ldots, x_{k_j}^j\right)$ on each V_j^*, then $F \in V_1 \otimes \ldots \otimes V_r$ is represented after restriction on X by a multilinear form:
$F\left(x^1, \ldots, x^r\right) = \sum_{i_1,\ldots,i_r} a_{i_1,\ldots,i_r} x_{i_1}^1 \cdots x_{i_r}^r$

$F \in X^\vee \Leftrightarrow$ system of equations $\quad F\left(x\right) = \frac{\partial F(x)}{\partial x_i^j} = 0$
(for all i,j) has a nontrivial solution for some $x = \left(x^1, \ldots, x^r\right)$.

Remark

Hyperdeterminant of format $(k_1, \ldots, k_r)$ exists iff $k_j \leq \sum_{i \neq j} k_i$.

Dual Grassmannian

Let X be Grassmanian of k dimensional subspaces of n dimensional vector space V. Consider the Plücker embedding: $G(k,V) \hookrightarrow \mathbb{P}\left(\bigwedge^k V\right)$. After choosing coordinate matrix:

$$K = \begin{bmatrix} 1 & 0 & \cdots & 0 & x^1_{k+1} & x^1_{k+2} & \cdots & x^1_n \\ 0 & 1 & \cdots & 0 & x^2_{k+1} & x^2_{k+2} & \cdots & x^2_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & x^k_{k+1} & x^k_{k+2} & \cdots & x^k_n \end{bmatrix}.$$

$$F(A,K) = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} a_{i_1 \ldots i_k} \eta_{i_1 \ldots i_k}$$

where $\eta_{i_1 \ldots i_k}$ is the minor of K indexed by $(i_1, \ldots, i_k)$.

$F \in G(k,n)^\vee \Leftrightarrow$ system of equations $\quad F(x) = \frac{\partial F(x)}{\partial x^j_i} = 0$ (for all i,j) has a nontrivial solution for some x.

Segre-Plücker Embedding

$$X = \mathbb{P}\left(\bigwedge^{k_1}\mathbb{C}^{N_1}\right)\times\ldots\times\mathbb{P}\left(\bigwedge^{k_r}\mathbb{C}^{N_r}\right)\mapsto\mathbb{P}\left(\bigwedge^{k_1}\mathbb{C}^{N_1}\otimes\ldots\otimes\bigwedge^{k_r}\mathbb{C}^{N_r}\right)$$

$N_i \geq 2k_i$. For each component we have Plücker embedding like above. Then take the Segre embedding. Generic form becomes:

$$F = \sum a_{I_1;\ldots;I_r}\eta^1_{I_1}\cdots\eta^r_{I_r}$$

where I_j is the index set of $\bigwedge^{k_j}\mathbb{C}^{N_j}$ of size k_j. Again

$$F \in X^\vee \Leftrightarrow \text{system of equations} \quad F(x) = \frac{\partial F(x)}{\partial x^j_i} = 0$$

(for all i,j) has a nontrivial solution for some x .

For the analysis of singularities the key tool is Hessian matrix.

Definition

Given form F, we define *Hessian* matrix at point $p \in X$ ie. matrix of double partial derivatives

$$H\left(F\right)_p = \| \frac{\partial^2 F}{\partial^{i'}_{j'} \partial^i_j} \|_p$$

evaluated at p for all possible indices i, i', j, j', and $\partial^i_j = \partial x^i_j$.

Definition

The *cusp* component is the subvariety of $X^\vee$ such that determinant of Hessian matrix vanishes. Formally:

$$X_{cusp} := \{F \mid \exists p \in X \text{ s.t } \mathbb{P}T_pX \subset F \text{ and } \det H(F)|_p = 0\}$$

Definition

The *node* component is the subvariety of $X^\vee$ which is the set of forms such that $F(p) = F(q) = 0$ for two distinct points $p, q \in X$. Formally:

$$X_{node} := \overline{\{F \mid \exists p, q \in X \quad \text{such that} \quad \mathbb{P}T_pX, \mathbb{P}T_qX \subset F\}}$$

Summary of Results

Representation	Cusp	Node	Jth Node
Hyperdeterminant $\mathbb{C}^{k_1} \otimes \ldots \otimes \mathbb{C}^{k_r}$	WZ	WZ	WZ
Dual Grassmannian $\bigwedge^k \mathbb{C}^N$	M,S	H,M,S	S
$\bigwedge^k \mathbb{C}^N \otimes \mathbb{C}^M$	S	S	S
$\bigwedge^{k_1} \mathbb{C}^{N_1} \otimes \ldots \otimes \bigwedge^{k_r} \mathbb{C}^{N_r}$	partial	S	partial

WZ: Weyman, Zelevinsky 1996

H: Holweck 2011, M: Maeda 2001

S: Sen

Problem-Cusp Type

Let's recall definition of cusp variety: points of dual variety such that determinant of Hessian vanishes. Now problem reduces to the following linear algebra problem:
What are the homogeneous polynomial factors of determinant of Hessian matrix?

Theorem

Assume that the determinant of the Hessian associated to form F, $F \in X^\vee$ is irreducible and $X^\vee$ does not have finitely many orbits. Then X_{cusp} is irreducible hypersurface in $X^\vee$.

There is a natural action of the group $G = SL\left(\mathbb{C}^{N_1}\right) \times \ldots \times SL\left(\mathbb{C}^{N_r}\right)$ on the form.

Problem-Node Type

Analysis of the node component reduces to the following linear algebra problem:
When does there exist two invertible Hessian matrices satisfying certain conditions?

Theorem

Generic node component for $\bigwedge^{k_1}\mathbb{C}^{N_1}\otimes\ldots\otimes\bigwedge^{k_r}\mathbb{C}^{N_r}$ *is always codimension one except the following* 10 *cases:*

$\bigwedge^3\mathbb{C}^6$, $\bigwedge^3\mathbb{C}^7$,$\bigwedge^3\mathbb{C}^8$
$\bigwedge^2\mathbb{C}^4\otimes\mathbb{C}^2$, $\bigwedge^2\mathbb{C}^4\otimes\mathbb{C}^3$,$\bigwedge^2\mathbb{C}^4\otimes\mathbb{C}^4$, $\bigwedge^2\mathbb{C}^4\otimes\mathbb{C}^5$
$\bigwedge^2\mathbb{C}^5\otimes\mathbb{C}^3$,$\bigwedge^2\mathbb{C}^5\otimes\mathbb{C}^4$, $\bigwedge^2\mathbb{C}^6\otimes\mathbb{C}^2$

Hessian of $G(3,6)$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & -a_1 & 0 & a_3 & -b_1 & 0 & b_3 \\ 0 & 0 & 0 & -a_2 & -a_3 & 0 & -b_2 & -b_3 & 0 \\ & & & & & & & & \\ & & & & & & & & \end{bmatrix}$$

Hessian of $G(3,6)$

$$\left[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & a_1 & a_2 & 0 & b_1 & b_2 \\
0 & 0 & 0 & -a_1 & 0 & a_3 & -b_1 & 0 & b_3 \\
0 & 0 & 0 & -a_2 & -a_3 & 0 & -b_2 & -b_3 & 0 \\
0 & -a_1 & -a_2 & 0 & 0 & 0 & 0 & c_1 & c_2 \\
a_1 & 0 & -a_3 & 0 & 0 & 0 & -c_1 & 0 & c_3 \\
a_2 & a_3 & 0 & 0 & 0 & 0 & -c_2 & -c_3 & 0 \\
 & & & & & & & & \\
 & & & & & & & &
\end{array}\right.$$

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$$\begin{bmatrix} 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & -a_1 & 0 & a_3 & -b_1 & 0 & b_3 \\ 0 & 0 & 0 & -a_2 & -a_3 & 0 & -b_2 & -b_3 & 0 \\ 0 & -a_1 & -a_2 & 0 & 0 & 0 & 0 & c_1 & c_2 \\ a_1 & 0 & -a_3 & 0 & 0 & 0 & -c_1 & 0 & c_3 \\ a_2 & a_3 & 0 & 0 & 0 & 0 & -c_2 & -c_3 & 0 \\ 0 & -b_1 & -b_2 & 0 & -c_1 & -c_2 & 0 & 0 & 0 \\ b_1 & 0 & -b_3 & c_1 & 0 & -c_3 & 0 & 0 & 0 \\ b_2 & b_3 & 0 & c_2 & c_3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hessian for Dual Grassmannian

Hessian for $G(k,n)^\vee$

$$\begin{bmatrix} 0 & A_{12} & \cdots & \cdots & A_{1k} \\ -A_{12} & 0 & \cdots & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & 0 & A_{k-1,k} \\ -A_{1k} & -A_{2k} & \cdots & -A_{k-1,k} & 0 \end{bmatrix}$$

A_{ij}'s are skew symmetric blocks of size $(n-k)\times(n-k)$.

Hessian for $\bigwedge^k \mathbb{C}^N \otimes \mathbb{C}^M$

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$$\begin{bmatrix} 0 & A_{12} & \cdots & \cdots & A_{1k} & B_{11} \\ -A_{12} & 0 & \cdots & \cdots & A_{2k} & B_{21} \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & 0 & A_{k-1,k} & B_{k-1,1} \\ -A_{1k} & -A_{2k} & \cdots & -A_{k-1,k} & 0 & B_{k,1} \\ B_{11}^t & B_{21}^t & \cdots & B_{k-1,1}^t & B_{k,1}^t & 0 \end{bmatrix}$$

A_{ij}'s are skew symmetric blocks of size $(N-k)\times(N-k)$.

B_{ij} are generic matrices of size $(N-k)\times(M-1)$.

^{t} denotes transpose.

Hessian in general

Hessian in general $\bigwedge^{k_1} \mathbb{C}^{N_1} \otimes \ldots \otimes \bigwedge^{k_r} \mathbb{C}^{N_r}$

$$\begin{bmatrix} H\left(\bigwedge^{k_1}\mathbb{C}^{N_1}\right) & S_{12} & \cdots & \cdots & S_{1r} \\ S_{12}^t & H\left(\bigwedge^{k_2}\mathbb{C}^{N_2}\right) & \cdots & \cdots & S_{2r} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & H\left(\bigwedge^{k_{r-1}}\mathbb{C}^{N_{r-1}}\right) & S_{r-1,r} \\ S_{1r}^t & S_{2r}^t & \cdots & S_{r-1,r}^t & H\left(\bigwedge^{k_r}\mathbb{C}^{N_r}\right) \end{bmatrix}$$

$H\left(\bigwedge^{k_j} \mathbb{C}^{N_j}\right)$ are Hessian associated to dual Grassmannian of size

$$k_j\left(N_j - k_j\right) \times k_j\left(N_j - k_j\right).$$

S_{ij} are generic matrices of size $k_i\left(N_i - k_i\right) \times k_j\left(N_j - k_j\right)$.

Determinant of Hessian matrix for $\bigwedge^k \mathbb{C}^N \otimes \mathbb{C}^M$ is irreducible, except the cases below:

Representation	N	Factors	Degrees
$\bigwedge^2 \mathbb{C}^N \otimes \mathbb{C}^2$	odd	0	0
	even	$pf^3 \times U$	$3\left(\frac{N}{2}-1\right)+\frac{N}{2}$
$\bigwedge^2 \mathbb{C}^N \otimes \mathbb{C}^3$	odd	U^2	$2(N-1)$
	even	$pf^2 \times U$	$2\left(\frac{N-2}{2}\right)+N$
$\bigwedge^k \mathbb{C}^N \otimes \mathbb{C}^{k(N-k)+1}$	-	U^2	$2k(N-k)$

$\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^4$	$f \times g$	$\deg f = 1, \deg g = 6$
$\bigwedge^2 \mathbb{C}^6 \otimes \mathbb{C}^4$	$f \times g$	$\deg f = 2, \deg g = 9$
$\bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2$	$f^2 \times g$	$\deg f = 3, \deg g = 4$
$\bigwedge^3 \mathbb{C}^7 \otimes \mathbb{C}^2$	$f \times g$	$\deg f = 7, \deg g = 6$
$\bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^3$	$f \times g$	$\deg f = 8, \deg g = 3$

Theorem

Assume that the determinant of the Hessian associated to form F, $F \in X^\vee$ is irreducible and $X^\vee$ does not have finitely many orbits. Then X_{cusp} is irreducible hypersurface in $X^\vee$.

Theorem

X_{cusp} is of codimension 2 for the format $\bigwedge^k \mathbb{C}^N \otimes \mathbb{C}^{k(N-k)+1}$.

Partial Results

Theorem

Assume that the format is not boundary.
The determinant of Hessian matrix is irreducible except the following classes:

$$\bigwedge^2 \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \mathbb{C}^{N_3}$$

$$\bigwedge^3 \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \otimes \mathbb{C}^{N_3}$$

Generic Node Type For $\bigwedge^{k_1} \mathbb{C}^{N_1} \otimes \ldots \otimes \bigwedge^{k_r} \mathbb{C}^{N_r}$

Plainly, we analyze forms which are tangent to the variety at two distinct points. Generic form is:
$F = \sum a_{I_1;\ldots;I_r} \eta^{(1)}_{I_1} \cdots \eta^{(r)}_{I_r}$, where I_j is the index set of $\bigwedge^{k_j} \mathbb{C}^{N_j}$ of size k_j and $I_j \subseteq [1, N_j]$. We define:

$$I_j^{first} = (1, \ldots, k_j)$$

$$I_j^{last} = (N_j - k_j + 1, \ldots, N_j)$$

$$I^{first} = \left(I_1^{first}; \ldots; I_r^{first}\right)$$

$$I^{last} = \left(I_1^{last}; \ldots; I_r^{last}\right)$$

There is a natural action of the group $G = SL\left(\mathbb{C}^{N_1}\right) \times \ldots \times SL\left(\mathbb{C}^{N_r}\right)$ on the form.

$$S := \{F \mid a_{I_1;\ldots;I_r} = 0 \quad \text{whenever} \\ |I^{first} \cap (I_1;\ldots;I_r)| \geq k_1 + \ldots + k_r - 1 \\ \text{or } |I^{last} \cap (I_1;\ldots;I_r)| \geq k_1 + \ldots + k_r - 1 \Big\}$$

$$X_{node} := \overline{G \bullet S}$$

Theorem

Generic node component for $\bigwedge^{k_1}\mathbb{C}^{N_1}\otimes\ldots\otimes\bigwedge^{k_r}\mathbb{C}^{N_r}$ *is always codimension one except:*

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Thank You!

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