An Isometry Theorem for Incidence Algebras

David Meyer

University of Missouri
Joint work with Killian Meehan

Conference on Geometric Methods in Representation Theory 5

November 19, 2017

Generalized Persistence Modules

Informally, a **generalized persistence module** is a representation of a partially ordered set P with values in a category $\mathcal{D}$.

More precisely, if P is a poset and $\mathcal{D}$ is a category, a generalized persistence module M with values in $\mathcal{D}$ assigns

- an object $M(x)$ of $\mathcal{D}$ for each $x \in P$,
- a morphism $M(x \leq y)$ in $Mor_{\mathcal{D}}(M(x), M(y))$ for each $x \leq y$ in P
- satisfying $M(x \leq z) = M(y \leq z) \circ M(x \leq y)$ for all $x, y, z \in P$ with $x \leq y \leq z$.

Persistent homology uses generalized persistence modules to attempt to discern the genuine topological properties of a finite data set.

When P is a finite poset and $\mathcal{D}$ is K-mod, generalized persistence modules for P are modules for the incidence algebra $I(P)$ of P

Persistent Homology

Quick summary of Persistent Homology:

- Suppose D is a finite data set contained in a metric space with undetermined topological features.
- The data set D is associated to the Vietoris-Rips complex $(C_\epsilon)_{\epsilon \in \mathbb{R}}$
- For each $\epsilon > 0$, let C_ϵ be the abstract simplicial complex whose k-simplices are determined by data points $x_1, x_2, ...x_{k+1} \in D$ where $d(x_i, x_j) \leq \epsilon$ for all $1 \leq i, j \leq k+1$.
- Clearly $\delta < \epsilon$, $C_\delta \hookrightarrow C_\epsilon$, thus $\epsilon \to C_\epsilon$ is a generalized persistence module.
- We apply the functor $H(-, K)$ to $\epsilon \to C_\epsilon$ to obtain $f(\epsilon) = H(C_\epsilon, K)$.

Persistent Homology

As ϵ increases generators for H are born and die, as cycles appear and become boundaries.

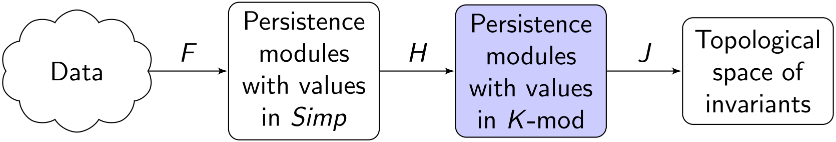
In persistent homology, one takes the viewpoint that true topological features of the data set can be distinguished from noise by looking for generators of homology which "persist" for a long period of time.

Informally, one "keeps" an indecomposable summand of f when it corresponds to a wide interval. Conversely, cycles which disappear quickly after their appearance (narrow ones) are interpreted as noise and disregarded.

By passing to the jump discontinuities of the Vietoris-Rips complex, one obtains a representation of equioriented $\mathbb{A}_n$.

Workflow/Stability

Here is the workflow diagram for Persistent Homology



Typically, one endows the collection of generalized persistence modules with values in K-mod with a metric structure.

Soft stability theorems concerns the continuity of $H \circ F$.

Hard stability concerns the continuity of J.

Algebraic Stability

In an algebraic stability theorem, one endows the collection of generalized persistence modules $\mathcal{D}^P$ with two (potentially) different metrics, and the identity is shown to be a contraction or an isometry.

A **bottleneck metric** is a way of defining a metric on the collection of finite multisubsets of a fixed set Σ. A bottleneck metric comes from a metric d on Σ, and function $W : \Sigma \to (0, \infty)$, with $|W(\sigma) - W(\tau)| \leq d(\sigma, \tau)$ for all $\sigma, \tau \in \Sigma$.

A matching between two multisubsets S, T of Σ is a bijection $f : S' \to T'$ between multisubsets $S' \subseteq S$ and $T' \subseteq T$. For $\epsilon \in (0, \infty)$, a matching f is an ϵ-matching if

- $\text{for all } s \in S, W(s) > \epsilon \implies s \in S'$
- $\text{for all } t \in T, W(t) > \epsilon \implies t \in T'$, and
- $d(s, f(s)) \leq \epsilon$, for all $s \in S$.

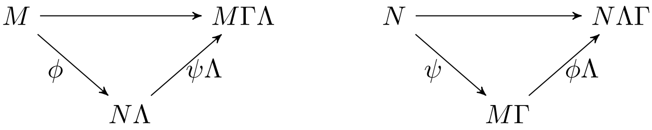
Then, $D_B(S, T) := \inf\{\epsilon : \text{there exists an } \epsilon\text{-matching between } S, T\}$.

Algebraic Stability

The other metric is an **interleaving metric**. An interleaving metric comes from,

- a monoid $\mathcal{T}(P)$ that acts on the category $\mathcal{D}^P$. That is, if M is in $\mathcal{D}^P$ and $\Lambda \in \mathcal{T}(P)$, then $M\Lambda \in \mathcal{D}^P$, and if $g \in \mathrm{Hom}(M, N)$, then $g\Lambda \in \mathrm{Hom}(M\Lambda, N\Lambda)$, and
- a metric d' on P.

Then, if $M, N \in \mathcal{D}^P$, $\Lambda, \Gamma \in \mathcal{T}(P)$ a (Λ, Γ)-interleaving between M, N are two morphisms $\phi \in \mathrm{Hom}(M, N\Lambda), \psi \in \mathrm{Hom}(N, M\Gamma)$ such that



Algebraic Stability

The metric d' on P gives rise to a notion of height on the monoid $\mathcal{T}(P)$.

Thus we may define the interleaving distance between $M, N \in \mathcal{D}^P$, as $D(M, N) = \inf\{\epsilon : \exists(\Lambda, \Gamma)\text{-interleaving between } M, N, h(\Lambda), h(\Gamma) \leq \epsilon\}$.

Theorem (Bauer and Lesnick)

Let $P = (0, \infty)$, $\mathcal{D} = K$-mod, $\mathcal{T}(P) = ([0, \infty), +)$. Then the interleaving metric D equals the bottleneck metric D_B.

This suggests the following representation-theoretic analogue of the isometry theorem of Bauer and Lesnick.

Let P be a finite poset and let K be a field. Choose a full subcategory $\mathcal{C} \subseteq I(P)$-mod, and let

- D be the interleaving distance restricted to $\mathcal{C}$, and
- D_B be a bottleneck metric on $\mathcal{C}$ which incorporates some algebraic information.

Prove that $Id : (\mathcal{C}, D) \to (\mathcal{C}, D_B)$ is an isometry.

Isometry Theorem 1

Theorem (Meehan, M.)

Let P be an n-Vee and let $\mathcal{C}$ be the full subcategory of $I(P)$-modules consisting of direct sums of convex modules. Let $(a, b) \in \mathbb{N} \times \mathbb{N}$ be a weight and let D denote interleaving distance (corresponding to the weight (a, b)) restricted to $\mathcal{C}$.

Set $W(M) = \min\{\epsilon : \mathrm{Hom}(M, M\Gamma\Lambda) = 0, \Gamma, \Lambda \in \mathcal{T}(\mathcal{P}), h(\Gamma), h(\Lambda) \leq \epsilon\}$, and let D_B be the bottleneck distance on $\mathcal{C}$ corresponding to the interleaving distance and W. Then, the identity is an isometry from $(\mathcal{C}, D) \xrightarrow{Id} (\mathcal{C}, D_B)$.

Isometry Theorem 2

Theorem (Meehan, M.)

Let X denote $\mathbb{A}_n$ with arbitrary choice of weights, let $Sh(X)$ be its shift refinement ($Sh(X) = \mathbb{A}_{n+k}$). Let $\mathcal{C}$ be the full subcategory of $I(Sh(X))$-mod given by the image of $I(X)$-mod $\xrightarrow{j} I(Sh(X))$-mod, and let $D^{Sh(X)}$, $D_B^{Sh(X)}$ be the interleaving metric and bottleneck metric respectively restricted to $\mathcal{C}$. Then, j is a contraction and $Id : \mathcal{C} \to \mathcal{C}$ is an isometry.

Theorem (Meehan, M.)

Let I, M be persistence modules (for $\mathbb{R}$). Then,

$$\lim_{X \in \Delta(D)} \left(D^X(\delta^X I, \delta^X M) \right) = D(I, M).$$

Current/Future Related Research

An algebraic stability theorem for $\mathbb{A}_n$ with arbitrary orientation. The metrics are an interleaving metric and a bottleneck metric using a weighted graph metric on the Auslander-Reiten Quiver of $\mathbb{A}_n$.

Analysis of the variety of interleavings between two generalized persistence modules.

Thank You

THANK YOU!