Variety of semi-conformal vectors in a vertex operator algebra

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1. Vertex algebras and conformal structure

A vertex operator algebra is a vertex algebra with a conformal structure.

A vertex operator is an operator valued function on the Riemann sphere.

1.1. Fields Given a vector space V (over $\mathbb{C}$), set

$$V[[z^{\pm 1}]] = \{f(z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \mid v_n \in V\}$$

$$V((z)) = \{\sum_{n \in \mathbb{Z}} v_n z^{-n-1} \mid v_n \in V, v_n = 0 \text{ for } n >> 0\}$$

Given two vector spaces V and W, denote

$$\begin{aligned}&\mathrm{Hom}(V, W(z)) \\ &= \{\phi(z) \in \mathrm{Hom}(V, W)[[z^{\pm 1}]] \mid \phi(z)(v) \in W((z)), \forall v \in V\}\end{aligned}$$

Note that V is finite dimensional if and only if

$$\mathrm{Hom}(V, W((z))) = \mathrm{Hom}(V, W)((z)).$$

A field on V is an element in Hom$(V, V((z)))$.

Remark 1. Given two fields $\phi(z), \psi(z) \in$ Hom$(V, V((z))$ the composition $\phi(z) \circ \psi(z)$ does not make any sense. This raises the equation of operator product expansion (OPE) problem in conformal field theory (we will not discuss the locality property)

1.2. Vertex algebras

Definition 1. A vertex algebra is a vector space V together with a map

(1) (state-field correspondence)

$$Y(\cdot, z) : V \to \text{Hom}(V, V((z))).$$

(2) (vacuum) $\mathbf{1} \in V$ satisfying the following:

(a) (Commutativity): For any $v, u \in\in V$, there is an $N(u, v) > 0$ such that

$$(z_1 - z_2)^{N(u,v)}[Y(u, z_1), Y(v, z_2)] = 0$$

(b) (Associativity) For any $v, w \in V$, there is $l(u, w) > 0$ such that

$$(z_1 + z_2)^{l(u,w)} Y(Y(u, z_1)v, z_2)w$$
$$= (z_1 + z_2)^{l(u,w)} Y(u, z_1 + z_2) Y(v, z_2) w$$

(c) $Y(\mathbf{1}, z) = Id$, $Y(v, z)\mathbf{1} = v + D(v)z + \cdots$.

with $D \in \mathrm{End}(V)$ and

$$[D, Y(v,z)] = Y(D(v), z) = \frac{d}{dz} Y(v,z)$$

Example **1.** If A is a commutative associative algebra with identity $\mathbf{1}$, then A is a vertex algebra with

$$Y(a,z) = l_a$$

with l_a being the left multiplication on A by $a \in A$ and $D = 0$.

A vertex algebra is denoted by $(V, Y, \mathbf{1})$.
For each $v \in V$, we denote

$$Y(v,z) = \sum_n v_n z^{-n-1}, \quad v_n \in \mathrm{End}(V)$$

1.3. Conformal structures

Definition **2.** A conformal structure on a vertex algebra $(V, Y, \mathbf{1})$ is an element $\omega \in V$ such that

$$Y(\omega, z) = \sum_n \omega_n z^{-n-1} = \sum_n L(n) z^{-n-2}$$

with ($\omega_{n+1} = L(n)$)

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3 - m}{12} \delta_{m+n,0} c \,\mathrm{Id}\,.$$

This means that the operators $\{L(n) \mid n \in \mathbb{Z}\}$ defines a module structure on V for the Virasoro Lie algebra $\mathcal{V}ir$.

The vector ω is called a Virasoro vector or a conformal vector.

Remark 2. On a vertex algebra $(V, Y, \mathbf{1})$, there can be many different conformal structures. The moduli space of conformal structures on a vertex algebra in general has not been well studied yet.

1.4. Vertex operator algebras

Definition 3. A vertex operator algebra is a vertex algebra $(V, Y, \mathbf{1})$ with a conformal structure $\omega \in V$ such that

(i) The operator $L(0) : V \to V$ is semi-simple with integer eigenvalues and finite dimensional eigenspaces $V_n = \ker(L(0) - n)$, i.e.,

$$V = \oplus_{n \in \mathbb{Z}} V_n$$

(ii) $V_n = 0$ if $n << 0$.

(iii) $L(-1) = D$.

Remark 3. Let A be a commutative algebra over $\mathbb{C}$. Then A is a vertex operator algebra if and only if A is finite dimensional. In this case, $A = A_0$.

Example 2. (Heisenberg vertex algebra) $\widehat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}]$ is a commutative associative algebra. Any $f(z) = \sum_n a_n t^n \in \mathbb{C}[[t, t^{-1}]]$ defines an element

$$\phi_f(z) = \sum_{n \in \mathbb{Z}} (a_n t^n) z^{-n-1} \in \mathrm{End}(\widehat{\mathfrak{h}})[[z, z^{-1}]]$$

with $a_n t^n : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}$ by multiplication. Then $\phi_f(z)$ is in $\mathrm{Hom}(\widehat{\mathfrak{h}}, \widehat{\mathfrak{h}}((z)))$ if and only if $a_n = 0$ for $n >> 0$, i. e., $f \in \mathbb{C}((t^{-1}))$.

On $\widehat{\mathfrak{h}}$, one defines a skew symmetric bilinear form

$$(f, g) = \mathrm{res}_t(f' g).$$

Then V becomes a $\mathbb{Z}$-graded Lie algebra (Heisenberg Lie algebra) with commutator

$$[f, g] = (f, g)1 \in \widehat{\mathfrak{h}}$$

$$[t^n, t^m] = n\delta_{n+m,0}1$$

Define:

$$V = V_{\widehat{\mathfrak{h}}}(l, 0) = U(\widehat{\mathfrak{h}}) \otimes_{U(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}_l.$$

This is an induced module for the Heisenberg Lie algebra $\widehat{\mathfrak{h}}$. It has a unique vertex algebra structure extending

$$Y(a, z) = \sum_n (a t^n) z^{-n-1}$$

with the Lie algebra element at^n acting on the module V.

Example 3. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. For example any finite dimensional reductive Lie algebra $\mathfrak{g}$ has this property. In particular the abelian Lie algebra $\mathfrak{h} = \mathbb{C}^{\oplus d}$ with the standard symmetric bilinear form. Then

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm}] + \mathbb{C}\mathbf{c}$$

is a $\mathbb{Z}$-graded Lie algebra with

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m+n,0}\langle x, y \rangle \mathbf{c}.$$

For any $l \in \mathbb{C}$, $\mathbb{C}_l$ is a module for the Lie algebra $\hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}\mathbf{c}$ with $\mathbf{c}$ acting by l and $\mathfrak{g}$ acts trivially.

Then induced $\hat{\mathfrak{g}}$-module

$$V_{\hat{\mathfrak{g}}}(l, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} \mathbb{C}_l$$

has a unique vertex algebra structure extending

$$Y(x, z) = \sum_n (x \otimes t^n) z^{-n-1}$$

with $x \otimes t^n$ in $\hat{\mathfrak{g}}$ acting on the module $V_{\hat{\mathfrak{g}}}(l, 0)$.

Remark 4. The category of modules for the vertex algebra $V_{\hat{\mathfrak{g}}}(l, 0)$ corresponds to the category of modules considered in Kazhdan-Lusztig in their construction of the tensor product (which is different from usual tensor of representations of Lie algebras). This tensor product reflects the fusion properties of vertex algebras.

1.5. Constructing conformal structures, Casimir Elements

In the above setting, take an orthonormal basis v^i in $\mathfrak{g}$ with respect to the symmetric form $\langle \cdot, \cdot \rangle$ and define the Casimir element

$$\Omega = \sum_i v^i v^i \in U(\mathfrak{g})$$

which is always in the center of $U(\mathfrak{q})$. Uk ander the adjoint $\mathfrak{g}$-module structure, $\mathfrak{g}$ is a $U(\mathfrak{g})$-module and assume there there is an $h \in \mathbb{C}$ such that

$$\Omega(x) = 2hx \quad \forall x \in \mathfrak{g}$$

h is called dual Coxeter number of a simple Lie algebra. If $l \in \mathbb{C}$ such that $l + h \neq 0$, then

$$\omega = \frac{1}{2(l+h)} \sum_i (v^i t^{-1})^2 \mathbf{1} \in V_{\hat{\mathfrak{g}}}(l, 0)$$

is a conformal structure and $L(0)$ action on $V_{\hat{\mathfrak{g}}}(l, 0)$ is the standard degree operator.

Remark **5.** When $\mathfrak{g}$ is the Lie algebra of diagonal $n \times n$ matrices. Then $h = 0$. For $l \neq 0$, the vertex operator algebra $V_{\widehat{\mathfrak{g}}}(l, 0)$ is called the Heisenberg vertex operator algebra.

When $\mathfrak{g}$ is a finite dimensional simple Lie algebra, $l \neq -h$, the vertex operator algebra $V_{\widehat{\mathfrak{g}}}(l, 0)$ is called the universal affine vertex operator algebra.

Remark 6. In general $V_{\widehat{\mathfrak{g}}}(l, 0)$ is a highest weight module for the affine Lie algebra $\widehat{\mathfrak{g}}$ which has a unique simple quotient module

$$L_{\widehat{\mathfrak{g}}}(l, 0) = V_{\widehat{\mathfrak{g}}}(l, 0) / \text{ unique max submodule}$$

which is also a vertex operator algebra. This is the case when l is a positive integer. In this case, the category of $L_{\widehat{\mathfrak{g}}}(l, 0)$-modules is semisimple with finitely irreducibles. **Such VOA is called rational**.

It is expected that for any vertex operator algebra $(V, Y, \mathbf{1})$, the category of representations is a tensor category. When V is rational, the representation category is modular tensor category.

1.6. Homomorphisms of vertex operator algebras

We will denote a vertex operator algebra (VOA) by $(V, Y, \omega, \mathbf{1}$ When $Y, \omega, \mathbf{1}$ are understood, one will only use V to denote a vertex operator algebra (or a vertex algebra).

Definition 4. A vertex algebra homomorphism

$$f : (V, Y^V, \mathbf{1}^V) \to (W, Y^W, \mathbf{1}^W)$$

is a linear map

$$f : V \to W$$
$$f(Y^V(v,z)u) = Y^W(f(v),z)f(u), \quad \forall u, v \in V$$
$$f(\mathbf{1}^V) = \mathbf{1}^W.$$

Note that automatically $f \circ D^V = D^W \circ f$

Definition 5. A vertex operator algebra homomorphism $f : (V, Y^V, \omega^V, \mathbf{1}^V) \to (W, Y^W, \omega^W, \mathbf{1}^W)$ is a vertex algebra homomorphism and additionally $f(\omega^V) = \omega^W$

Remark 7. If $f : (V, Y^V, \omega^V, \mathbf{1}^V) \to (W, Y^W, \omega^W, \mathbf{1}^W)$ is only a homomorphism of vertex algebra, then we always have $f \circ L^V(-1) = L^W(-1) \circ f$.

f is a vertex operator algebra homomorphism if and only if

$$f \circ L^V(n) = L^W(n) \circ \quad \forall n \in \mathbb{Z}$$

2. Semi-conformal vectors and semi-conformal subalgebras of a vertex operator algebra

2.1.Semi-conformal homomorphisms

Definition 6. Let $(V, Y^V, \omega^V, \mathbf{1}^V)$ and $(W, Y^W, \omega^W, \mathbf{1}^W)$ be two vertex operator algebras. A vertex algebra morphism $f : V \to W$ is said to be semi-conformal if

$$f \circ \omega_n^V = \omega_n^W \circ f \quad \forall n \geq 0.$$

Note that f is conformal if and only if

$$f \circ \omega_n^V = \omega_n^W \circ f \quad \forall n$$

if and only if

$$f \circ \omega_{-1}^V = \omega_{-1}^W \circ f$$

.

Remark 8. Noting that for any vertex algebra homomorphism f, we always have $f \circ L^V(-1) = L^W(-1)$. Thus f is *semi-conformal* if and only if $f \circ L^V(n) = L^W(n) \circ f$, for all $n \geq 0$

Thus there are two categories of vertex operator algebras using conformal morphisms and semi-conformal morphisms respectively. One is a subcategory (not full) of the other.

Theorem 1 (Jiang-L)**.** *Any surjective semi-conformal homomorphism between two vertex operator algebras is conformal.*

Corollary 1. *The automorphisms and isomorphisms in these two categories are the same. Thus the problem of classifications of vertex operator algebras in these two categories are the same.*

2.2. Vertex operator subalgebras

Given a vertex algebra $(W, Y, \mathbf{1})$. The vertex subalgebra is a subspace $U \subseteq W$ such that $Y(u, z)U \subseteq U((z))$ for all $u \in U$ and $\mathbf{1} \in U$.

But when we talk about vertex operator subalgebra U, it is vertex subalgebra with a conformal structure ω^U.

Classically, one would require that $\omega^U = \omega^W$. But most of the constructions will involve vertex operator subalgebras that does not preserve this property.

Definition 7. A vertex subalgebra U of $(W, Y, \omega, \mathbf{1})$ with conformal structure ω^U is said to be semi-conformal if the inclusion map is semi-conformal.

Theorem 2 (Jiang-L). *On a vertex subalgebra U of a vertex operator algebra $(W, Y, \omega, \mathbf{1})$, the conformal structure ω^U making U semi-conformal is unique.*

Thus we can talk about semi-conformal vertex subalgebra without mentioning what the conformal structure is!

2.3. Semi-conformal vectors

Definition 8. An element ω' in W is called a semi-conformal vector if there is a vertex subalgebra U such that $\omega' \in U$ defines a conformal structure on U making (U, ω') a semi-conformal subalgebra.

For a vertex operator algebra (W, ω), we define

$$\text{ScAlg}(W, \omega) = \{(U, \omega') \mid (U, \omega') \text{ a semi-conf. subalg.}\};$$
$$\text{Sc}(W, \omega) = \{\omega' \in W \mid \omega' \text{ a,semi-conf. vector}\};$$

Theorem 3 (Chu-L). *For any vertex operator algebra* $(W, Y, \omega, \mathbf{1})$, Sc$(W, \omega)$ *is a Zariski closed subset of* W_2, *thus an algebraic variety.*

3. Coset constructions in conformal field theory

Given any vertex algebra $(W, Y, \mathbf{1})$, any subset $S \subseteq W$ define the centralizer

$$C_W(S) = \{w \in W \mid [Y(w, z_1), Y(u, z_2)] = 0, \forall u \in S\}$$

Note that: $[Y(w, z_1), Y(u, z_2)] = 0$ if and only if $w_n u_m = u_m w_n$ for all $m, n \in Z$.

The following standard facts are obvious:

- $C_W(S)$ is always a vertex subalgebra;
- $C_W(S) = C_W(< S >)$, where $< S >$ is the vertex subalgebra generated by S.

Not obvious but is true:

$$C_W(S) = \{w \in W \mid w_n(u) = 0, \forall n \geq 0, u \in S\}$$
$$= \{w \in W \mid u_n(w) = 0, \forall n \geq 0, u \in S\}$$

Theorem 4 (Chu-Lin). *If (U, ω') is semi-conformal vertex subalgebra of (W, ω), then $C_W(U)$ is also a semi-conformal vertex subalgebra with conformal structure $\omega - \omega'$.*

Theorem 5 (Chu-Lin). *For any semi-conformal vertex subalgebra (U, ω'), the contralizer $C_W(U)$ does not depend on U, but on the conformal element ω' only. i.e., for any two semi-conformal vertex subalgebras (U, ω') and (U', ω') with the same ω', then $C_W(U) = C_W(U')$.*

Corollary 2. Sc(W, ω) *has a poset structure and an order reversing involution $\omega \mapsto \bar{\omega}$.*

The projection map ScAlg$(W, \omega) \to$ Sc(W, ω) $\quad (U, \omega') \mapsto \omega'$ has two sections

$$\omega' \mapsto < \omega' > \quad \text{and} \quad \omega' \mapsto U(\omega') = C_W(C_W(\omega').$$

$< \omega' >$ is a minimal model (Virasoro vertex operator algebra) which does not reflect the properties of W as much as $U(\omega)$ does.

U defines a cosheaf of vertex operator algebras on Sc(W, ω).

The automorphism group $G = \mathrm{Aut}(W, \omega)$ acts on both $\mathrm{ScAlg}(W, \omega)$ and $\mathrm{Sc}(W, \omega)$. We will also be interested in determining G-orbit structures.

4. Affine Constructions

Recall that for a finite dimensional Lie algebra $\mathfrak{g}$ with non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ such that the Casimir element Ω acting on $\mathfrak{g}$ by constant $2h$.

For any subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\langle \cdot, \cdot \rangle_{\mathfrak{a}}$ is non-degenerate, then $\mathfrak{a}(-1) = (\mathfrak{a} \otimes t^{-1})\mathbf{1}$ generates a vertex subalgebra in $V_{\widehat{\mathfrak{g}}}(l, 0)$ (or in $L_{\widehat{\mathfrak{g}}}(l, 0)$) and it is semi-conformal. Its centralizer is also semi-conformal.

Theorem 6 (Chu-L). *Let $\mathfrak{h} = \mathbb{C}^d$ be the abelian Lie algebra with standard symmetric non-degenerate bilinear form. Then $\widehat{\mathfrak{h}}$ is the (affine) Heisenberg Lie algebra of rank d. For any level $l \neq 0$.*

$$\mathrm{Sc}(V_{\widehat{\mathfrak{h}}}(l, 0) = \{\mathfrak{h}' \leq \mathfrak{h} \mid \mathfrak{h}' \text{ is a nondegenerate subspace}\}$$

Theorem 7 (Chu-L). *Let (W, ω) be a simple vertex operator algebra of CFT type ($W = \sum_{n=0}^{\infty} W_n$ with $W_0 = \mathbb{C}\mathbf{1}$) and generated by W_1. If for any $\omega' \in \mathrm{Sc}(W, \omega)$ one have*

$$W = U(\omega') \otimes C_W(U(\omega'))$$

and the maximal chain length in $\mathrm{Sc}(W,\omega)$ *is* $\dim W_1$*, then* $W \cong V_{\widehat{\mathfrak{h}}}(l,0)$ *with* $\dim \mathfrak{h} = \dim W_1$ *with standard* ω*.*

Remark 9. On $V_{\widehat{\mathfrak{h}}}(l,0)$ there are other conformal structures with the same $L(0)$ which can also be characterized in such way.

Here $\mathfrak{a}$ can be the Cartan subalgebra $\mathfrak{h}$. $\mathfrak{a}$ can also be any Levi-subalgebra in case that $\mathfrak{g}$ is a simple Lie algebra or any $\mathfrak{a} = \mathfrak{g}^\sigma$ for an involution $\sigma \in \mathrm{Aut}(\mathfrak{g})$.

Conjecture 1 (Dong)**.** *If* (W,ω) *rational and* U *is a rational semi-conformal vertex subalgebra, then* $C_W(U)$ *is also a rational.*

Example 4. If $\mathfrak{g}$ if finite dimensional simple Lie algebra and $\mathfrak{h}$ is the Cartan subalgebra, then $C_{L_{\widehat{\mathfrak{g}}}}(\mathfrak{h}t^{-1}\mathbf{1})$ is called the parafermion vertex operator algebra. it was conjectured that it rational. Jing-Lin proved the special case. It was recently proved by Dong.

Example 5. An even lattice L is a free abelian group of finite rank with positive definite symmetric bilinear $\mathbb{Z}$-value form such that all vectors even length. There is a way to construct vertex operator algebra V_L. This vertex operator algebra is always rational.

If L is root lattice of an semisimple Lie algebra $\mathfrak{g}$, then

$V_L = V_{\hat{\mathfrak{g}}}(1,0)$ (level one). If L' is positive definite sublattice such that $(L')^{\perp}$ is also positive definite, then $V_{L'}$ both $V_{(L')^{\perp}}$ are semi-conformal subvertex algebras and $V_{L'} \otimes V_{(L')^{\perp}}$ are conformal subalgebras of V_L.

Example 6. L and L' are two even lattices then $L \otimes L'$ is also even lattices. There are many ways of embedding L and L' into $L \otimes L'$. Their centralizers of $V_{L'}$ in $V_{L \otimes L'}$ are all semi-conformal subalgebras. This arguments was also used in the Schur-Weyl duality and level-rank duality. More generally on representations of symmetric pairs, (Mirror dual?)

THANK YOU!