Prime Spectra of 2-Categories

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Overview

2-Categories

Definition

A **2-category** *is a category enriched over the category of small categories.*

So a 2-category $\mathcal{T}$ has:

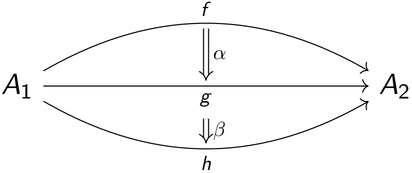
- Objects, denoted by A_1, A_2 etc;
- 1-morphisms between objects, denoted f, g, h, etc; set of 1-morphisms from A_1 to A_2 denoted $\mathcal{T}(A_1, A_2)$;
- 2-morphisms between 1-morphisms, denoted α, β, γ, etc; set of 2-morphisms from f to g denoted $\mathcal{T}(f, g)$.

2-Categories

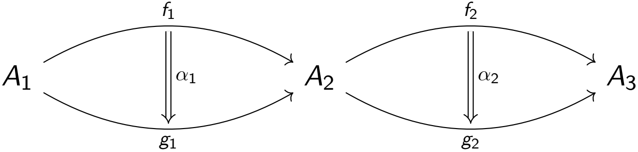
Composition of 1-morphisms:

$$A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3.$$

Vertical composition of 2-morphisms $\alpha \circ \beta$:

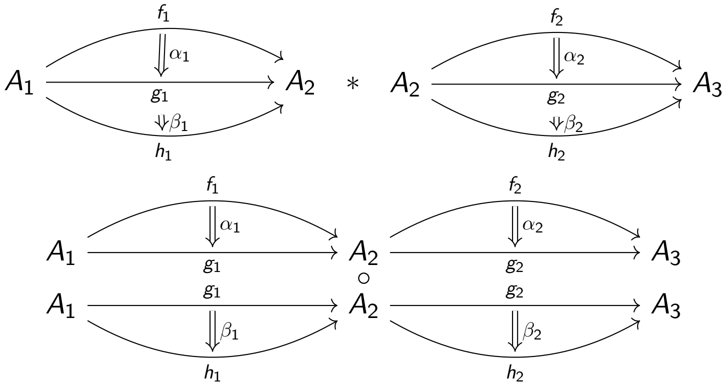


Horizontal composition of 2-morphisms $\alpha_2 * \alpha_1$:



2-Categories

$$(\alpha_1 \circ \beta_1) * (\alpha_2 \circ \beta_2) = (\alpha_1 * \alpha_2) \circ (\beta_1 * \beta_2):$$



Exact categories

Definition

A 1-category is called **exact** *if:*

- *It is additive;*
- *It has a set of distinguished short exact sequences*

$$A_1 \to A_2 \to A_3$$

  that obey some axioms.

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Exact categories

Some exact 1-categories:

- An additive category with short exact sequences defined by

$$A_1 \to A_1 \oplus A_3 \to A_3;$$

- Abelian categories with traditional short exact sequences ($\ker g \cong \operatorname{im} f$);
- Full subcategories of abelian categories closed under extension.

Definition

A 2-category $\mathcal{T}$ is **exact** *if each set $\mathcal{T}(A, B)$ is itself an exact 1-category.*

Grothendieck group

Definition

Suppose $\mathcal{C}$ is an exact 1-category. Then the **Grothendieck group** *of $\mathcal{C}$, denoted $K_0(\mathcal{C})$, is defined by:*

- *Take the free abelian group on objects of $\mathcal{C}$;*
- *For every exact sequence*

$$0 \to A_1 \to A_2 \to A_3 \to 0,$$

quotient by the relation $[A_1] + [A_3] = [A_2]$.

Grothendieck group

Definition

Suppose $\mathcal{T}$ is an exact 2-category. Then the **Grothendieck group** *of $\mathcal{T}$, denoted $K_0(\mathcal{T})$ is defined as the 1-category with:*

- *Objects the same as $\mathcal{T}$;*
- *Set of morphisms from X to Y given by $K_0(\mathcal{T}(X, Y))$, the Grothendieck group of the 1-category $\mathcal{T}(X, Y)$.*
- *Composition of morphisms induced from composition of morphisms in $\mathcal{T}$.*

Positive part of the Grothendieck group

Definition

The **positive part of the Grothendieck group** *of an exact 1-category $\mathcal{C}$, denoted $K_0(\mathcal{C})_+$, is defined as the subset of $K_0(\mathcal{C})$ forming a monoid under addition generated by the indecomposable objects.*

In other words, while the Grothendieck group has all elements of the form

$$\sum_i \lambda_i [b_i], \lambda_i \in \mathbb{Z},$$

the positive part of the Grothendieck group has elements of the form

$$\sum_i \lambda_i [b_i], \lambda_i \in \mathbb{N}.$$

Positive part of the Grothendieck group

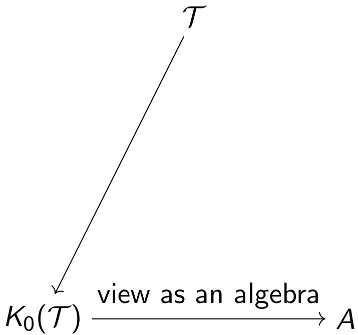
Definition

The **positive part of the Grothendieck group** *of an exact 2-category* $\mathcal{T}$*, denoted* $K_0(\mathcal{T})_+$*, has the same objects as* $\mathcal{T}$*, with hom spaces* $K_0(\mathcal{T})_+(X, Y)$ *defined by* $K_0(\mathcal{T}(X, Y))_+$*.*

Strong categorification

- Let A an algebra with orthogonal idempotents e_i with $1 = e_1 + e_2 + \ldots + e_n$.
- $A = \bigoplus e_i A e_j$.
- Consider A as a category: an object for each e_i, set of morphisms from i to j given by $e_i A e_j$.
- Composition of morphisms given by multiplication.

Strong categorification



Strong categorification

Definition

*We call B_+ a $\mathbb{Z}_+$-**ring** if B_+ has a basis (as a monoid) $\{b_i\}$ with relations $b_i b_j = \sum m_{i,j}^k b_k$ where all coefficients are positive. Elements are all positive linear combinations of basis elements, multiplication is extended from basis elements.*

So we can view Grothendieck groups of 2-categories as $\mathbb{Z}$-algebras, and positive Grothendieck groups as $\mathbb{Z}_+$-rings.

Ideals

Definition

Let $\mathcal{T}$ be an exact 2-category where composition of 1-morphisms is an exact bifunctor. We call $\mathcal{I}$ a **thick ideal** *of $\mathcal{T}$ if:*

- *$\mathcal{I}$ is a full subcategory of $\mathcal{T}$ such that if in $\mathcal{T}(X, Y)$ we have an exact sequence of 1-morphisms*

$$0 \to f_1 \to f_2 \to f_3 \to 0,$$

then f_2 is in $\mathcal{I}$ if and only if f_1 and f_2 are in $\mathcal{I}$;

- *$\mathcal{I}$ is an ideal: if $f \in (X, Y)$ is $\in \mathcal{I}$ and $g \in \mathcal{T}(Y, Z)$ then $g \circ f \in \mathcal{I}$; and if $h \in \mathcal{T}(W, X)$ then $f \circ h \in \mathcal{I}$.*

Ideals

Definition

Suppose $\mathcal{M}$ is any subset of 1-morphisms and 2-morphisms of a 2-category $\mathcal{T}$. Then we define the **thick ideal generated by** *$\mathcal{M}$, denoted $\langle \mathcal{M} \rangle$, to be the smallest thick ideal that contains $\mathcal{M}$, which is the intersection of all thick ideals containing $\mathcal{M}$.*

Definition

Suppose B_+ is a $\mathbb{Z}_+$-ring. Then $I \subset B_+$ is a **thick ideal** *if $a + b$ is in I if and only if a and b are in I, and we also have that if i is in I, then ai and ia are in I for every $a \in B_+$.*

Prime and completely prime ideals

Definition

We call $\mathcal{P}$ a **prime** *of $\mathcal{T}$ if $\mathcal{P}$ is a thick ideal of $\mathcal{T}$ such that if $\mathcal{I}$ and $\mathcal{J}$ are thick ideals in $\mathcal{T}$, then if $\mathcal{I} \circ \mathcal{J} \subset \mathcal{P}$, then either $\mathcal{I} \subset \mathcal{P}$ or $\mathcal{J} \subset \mathcal{P}$. We call $\mathcal{I}$* **completely prime** *if it is a thick ideal such that $f \circ g \in \mathcal{I}$ implies either $f \in \mathcal{I}$ or $g \in \mathcal{I}$.*

Definition

The set of all primes $\mathcal{P}$ of a 2-category $\mathcal{T}$ is called the **spectrum of** $\mathcal{T}$ *and is denoted* $\mathrm{Spec}(\mathcal{T})$.

Prime and completely prime ideals

Definition

Suppose B_+ is a $\mathbb{Z}_+$-ring. Then we call P a **prime** *if P is a thick ideal, and $IJ \subset P$ implies I or J is in P for all thick ideals I and J.*

General results

We obtain many results with respect to $\text{Spec}(\mathcal{T})$ that correspond to the prime spectra of noncommutative rings.

Theorem

A thick ideal $\mathcal{P}$ is prime if and only if: for all 1-morphisms m, n of $\mathcal{T}$ with $m \circ \mathcal{T} \circ n \in \mathcal{P}$, either $m \in \mathcal{P}$ or $n \in \mathcal{P}$.

This corresponds to the result in the classical theory:

Theorem

An ideal P of a ring R is prime if and only if: for all $x, y \in R$, if $xRy \subset P$ then x or y is in P.

General results

Theorem

A thick ideal $\mathcal{P}$ is prime if and only if: for all thick ideals $\mathcal{I}, \mathcal{J}$ properly containing $\mathcal{P}$, we have that $\mathcal{I} \circ \mathcal{J} \not\subset \mathcal{P}$.

Theorem

Every maximal thick ideal is prime.

Theorem

The spectrum of an exact 2-category $\mathcal{T}$ is nonempty.

Relationship between the spectra

Lemma

There is a bijection between $\mathrm{Spec}(\mathcal{T})$ *and* $\mathrm{Spec}(K_0(\mathcal{T})_+)$.

Let $\mathcal{T}$ be a categorification of A. Consider the map $\phi : \mathrm{Spec}(K_0(\mathcal{T})_+) \to \mathrm{Ideals}(K_0(\mathcal{T})) = A$ defined by $\phi(P) = \{x - y : x, y \in P\}$.

Lemma

In general, ϕ *is not a map* $\mathrm{Spec}(K_0(\mathcal{T})_+) \to \mathrm{Spec}(K_0(\mathcal{T}))$.

Example: let H be a Hopf algebra, $\mathcal{T}$ be the category of finitely generated H-modules. Then $\{0\}$ is completely prime in $K_0(\mathcal{T})_+$ but not in $K_0(\mathcal{T})$.

Relationship between the spectra

$$\mathrm{Spec}(\mathcal{T})$$

$$\updownarrow$$

$$\mathrm{Spec}(K_0(\mathcal{T})_+) \xrightarrow{\phi} \mathrm{Ideals}(K_0(\mathcal{T}))$$

Lemma

Let $\mathcal{T}$ be a categorification of A. If $\phi(P)$ is a prime in $K_0(\mathcal{T})$, and $\mathcal{P}$ is the prime in $\mathcal{T}$ corresponding to P, then $A/\phi(P)$ is categorified by the Serre quotient $\mathcal{T}/\mathcal{P}$.

Coordinate rings of Richardson varieties

Definition

Suppose G is a connected simple Lie group, $B_\pm$ opposite Borel subgroups, and W the Weyl group. Then the **Richardson variety** *of u and $w \in W$ is*

$$R_{u,w} = B_- \cdot uB_+ \cap B_+ \cdot wB_+ \subset G/B_+.$$

Individually, $B_- \cdot uB_+$ and $B_+ \cdot wB_+$ are called **Schubert cells**.

Coordinate rings of Richardson varieties

Theorem (Yakimov)

$$G/B_+ = \bigsqcup_{\substack{u \leq w \\ u,w \in W}} R_{u,w}.$$

Applications of Richardson varieties:

- Representation theory (Richardson, Kazhdan, Lusztig, Postnikov);
- Total positivity (Lusztig);
- Poisson geometry (Brown, Goodearl, and Yakimov);
- Algebraic geometry (Knutson, Lam, Speyer);
- Cluster algebras (Leclerc).

Coordinate rings of Richardson varieties

We restrict to $u = 1$ for simplicity.
Let $U_q(\mathfrak{n}_+)$ denote the subset of $U_q(\mathfrak{g})$ generated by the E_i Chevalley generators.

Theorem (Yakimov)

If T is a maximal torus of G, then T acts on $U_q(\mathfrak{n}_+)$ via an algebra automorphism. The T-invariant prime ideals are parametrized by elements of W.

Theorem (Yakimov)

$U_q(\mathfrak{n}_+)/I_w$ is a quantization of the coordinate ring $\mathbb{C}[R_{1,w}]$.

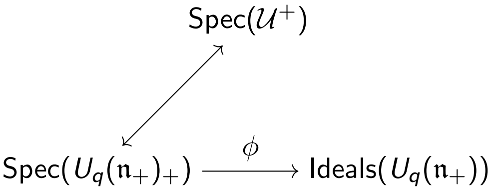
Coordinate rings of Richardson varieties

We want to produce a categorification of $U_q(\mathfrak{n}_+)/I_w$.

Theorem (Khovanov and Lauda)

There exists a categorification $\mathcal{U}^+$ of $U_q(\mathfrak{n}_+)$ that is a tensor category of modules of KLR-algebras.

Current work



We are currently working on showing that I_w is a prime in $\mathrm{Spec}(U_q(\mathfrak{n}_+))$ corresponding to a prime in $\mathrm{Spec}(U_q(\mathfrak{n}_+)_+)$. Then if $\mathcal{I}_w$ is the prime in $\mathrm{Spec}(\mathcal{U}^+)$ corresponding to I_w, then

$$\mathcal{U}^+/\mathcal{I}_w$$

will categorify quantization of the coordinate ring of the Richardson variety.

Conclusion

Thanks for listening!