On the Structure of Generalized Symmetric Spaces of $\mathrm{SL}_n(\mathbb{F}_q)$

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Motivation

Real symmetric spaces were introduced by É. Cartan as a special class of homogeneous Riemannian manifolds.

Later generalized by M. Berger who gave classifications of the irreducible semisimple symmetric spaces.

The goal of this talk is to explore the structure of generalized symmetric spaces for $G = \mathrm{SL}_n(k)$ where $k = \mathbb{F}_q$.

Previous Work

Classified the involutions of $\mathrm{SL}_2(k)$ where $\mathrm{char}(k) \neq 2$.

Described the extended symmetric space, R, and the generalized symmetric space, Q, related to $\mathrm{SL}_2(k)$.

Proved the following:

Theorem

Let k be a finite field of odd characteristic. Then $R = Q$ for any involution of the group $\mathrm{SL}_2(k)$.

Does this result extend to $\mathrm{SL}_n(k)$?

Outline

Classify the involutions of $\mathrm{SL}_n(k)$ where $\mathrm{char}(k) \neq 2$.

Determine the relationship between R and Q for two conjugacy classes of involutions of $\mathrm{SL}_n(k)$.

Provide the relationship between R and Q for the remaining conjugacy classes of involutions of $\mathrm{SL}_n(k)$.

Discuss the corresponding results when $\mathrm{char}(k) = 2$.

What are generalized symmetric spaces?

Definition

Let G be a group and $\theta \in Aut(G)$. Then θ is an **involution** if θ has order 2.

Let G be a group and θ be an involution of G.

Definition

The $\mathbf{fixed - point\ group}$ is the set of elements given by $H = \{g \in G | \theta(g) = g\}$.

Definition

The **generalized symmetric space** is the set G/H.

What are generalized symmetric spaces?

$\tau : G \to G$ given by $\tau(g) = g\theta(g)^{-1}$

τ induces an isomorphism of the coset space G/H onto $\tau(G)$

Generalized symmetric space $G/H \cong \{g\theta(g)^{-1} | g \in G\} = Q$

What are extended symmetric spaces?

Let G be a group and θ be an involution of G.

Definition

The **extended symmetric space** is the set of elements given by $R = \{g \in G | \theta(g) = g^{-1}\}$.

Relationship between R and Q

In general, $Q \subseteq R$.

$$\theta\left(g\theta(g)^{-1}\right) = \theta(g)g^{-1} = \left(g\theta(g)^{-1}\right)^{-1}.$$

However, typically $Q \neq R$.

For example, consider the involution $\theta : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ defined by $\theta(A) = (A^T)^{-1}$. Then $Q = \{AA^T \mid A \in \mathrm{SL}_2(\mathbb{R})\}$ and $R = \{A \in \mathrm{SL}_2(\mathbb{R}) \mid A = A^T\}$. Clearly, $Q \subset R$ but $Q \neq R$.

Main Tool: Twisted Conjugation

$SL_n(k)$ acts on R by twisted conjugation: $SL_n(k) \times R \to R$

$$g.r = gr\theta(g)^{-1}$$

Equivalence Relation on R:

$$r_1 \sim r_2 \text{ if and only if } g.r_1 = r_2 \text{ for some } g \in SL_n(k)$$

Orbit of $r \in R$:

$$[r] = SL_n(k).r = \{gr\theta(g)^{-1} | g \in SL_n(k)\}$$

$$[I_n] = SL_n(k).I_n = \{g\theta(g)^{-1} | g \in SL_n(k)\} = Q$$

Main Tool: Twisted Conjugation

The twisted conjugacy classes partition R

$$R - Q = \{r \in R | gr\theta(g)^{-1} \neq I_n \text{ for all } g \in \mathrm{SL}_n(k)\}$$

The involutions of $\mathrm{SL}_n(\mathbb{F}_q)$

Three kinds of involutions:

- *Inner Involutions* $\mathrm{Inn}_x(g) = xgx^{-1}$

$$\begin{aligned} R - Q &= \{r \in R \,|\, g(rx)g^{-1} \neq x \text{ for all } g \in \mathrm{SL}_n(k)\} \\ &= \{r \in R \,|\, rx \text{ is } \textbf{not similar to } x \text{ under } \mathrm{SL}_n(k)\} \end{aligned}$$

- *Outer Involution* $\theta(g) = g^{-T}$

$$\begin{aligned} R - Q &= \{r \in R \,|\, g(r)g^{T} \neq I_n \text{ for all } g \in \mathrm{SL}_n(k)\} \\ &= \{r \in R \,|\, r \text{ is } \textbf{not congruent to } I_n \text{ under } \mathrm{SL}_n(k)\} \end{aligned}$$

- *A composition of the two*

Inner involutions of $\mathrm{SL}_n(\mathbb{F}_q)$

Theorem (Helminck, Wu, Dometrius)

Let k be a finite field of odd characteristic and $n > 2$. The isomorphism classes for involutions of $\mathrm{SL}_n(k)$ include Inn_{Y_i} where Y_i is the block-diagonal matrix $\mathrm{diag}(I_{n-i}, -I_i)$ for $i \in \left\{1, 2, \cdots, \lceil \frac{n-1}{2} \rceil\right\}$. If n is odd, these are the only isomorphism classes of involutions. If n is even, there is one additional isomorphism class, namely, Inn_L where

$$L = \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ s_p & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ s_p & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ s_p & 0 \end{pmatrix}\right)$$

and s_p is any non-square in the field.

Inner involutions of $SL_n(\mathbb{F}_q)$

Recall for $\text{Inn}_x(g) = xgx^{-1}$

$$\begin{aligned} R - Q &= \{r \in R \mid g(rx)g^{-1} \neq x \text{ for all } g \in SL_n(k)\} \\ &= \{r \in R \mid rx \text{ is } \textbf{not similar to } x \text{ under } SL_n(k)\} \end{aligned}$$

Tools:

- Rational canonical form
- (Waterhouse, 1984) If k is a finite field, then every matrix similar to a matrix A over k is actually similar to A over $SL_n(k)$, except for those cases in which there is an integer m such that (i) $k^* \neq (k^*)^m$ and (ii) all the invariant factors of A are m^{th} powers.

Inner involutions of $\mathrm{SL}_n(\mathbb{F}_q)$

Consider $\mathrm{Inn}_{Y_i}(g) = Y_i g Y_i^{-1}$ where $Y_i = \mathrm{diag}(I_{n-i}, -I_i)$ for $i \in \{1, 2, \ldots, \lceil \frac{n-1}{2} \rceil\}$.

$$\begin{aligned} R &= \left\{X \in \mathrm{SL}_n(k) | Y_i X Y_i^{-1} = X^{-1}\right\} \\ &= \left\{X \in \mathrm{SL}_n(k) | (XY_i)^2 = I_n\right\} \\ &= \{X \in \mathrm{SL}_n(k) | \text{ minimal poly of } XY_i \text{ divides } (\lambda+1)(\lambda-1)\} \\ &= \left\{ X \in \mathrm{SL}_n(k) \;\middle|\; \begin{array}{l} XY_i \text{ is similar to } Y_j = \mathrm{diag}(I_{n-j}, -I_j) \text{ over } k \\ \text{for some } j \text{ in } \{0, 1, 2, \ldots, n\} \end{array} \right\} \end{aligned}$$

$$\begin{aligned} Q &= \left\{X \in \mathrm{SL}_n(k) | X = PY_iP^{-1}Y_i^{-1} \text{ for some } P \in \mathrm{SL}_n(k)\right\} \\ &= \{X \in \mathrm{SL}_n(k) | XY_i \text{ is similar to } Y_i \text{ over } \mathrm{SL}_n(k)\} \\ &= \{X \in \mathrm{SL}_n(k) | XY_i \text{ is similar to } Y_i \text{ over } k\} \end{aligned}$$

Inner involutions of $\mathrm{SL}_n(\mathbb{F}_q)$

$$
\begin{aligned}
R - Q &= \left\{ X \in \mathrm{SL}_n(k) \,\middle|\, \begin{array}{l} XY_i \text{ is similar to } Y_j \text{ over } k \text{ for some } j \\ \text{in } \{0, 1, 2, \ldots, n\} \text{ with } \ j \neq i \end{array} \right\} \\
&= \bigcup_{\substack{j \in \{0,1,\ldots,n\} - \{i\} \\ j \equiv i \bmod 2}} \left\{ AY_i^{-1} | A \in \mathrm{cl}(Y_j) \right\}.
\end{aligned}
$$

Example

$\mathrm{SL}_3(k)$ has only one inner involution, Inn_{Y_1}.

$$
\begin{aligned}
R &= \{X \in \mathrm{SL}_3(k) | XY_1 \text{ is similar to } Y_1 \text{ or } Y_3 \text{ over } k\} \text{ and} \\
Q &= \{X \in \mathrm{SL}_3(k) | XY_1 \text{ is similar to } Y_1 \text{ over } k\}.
\end{aligned}
$$

$R - Q = \{AY_1^{-1} | A \in \mathrm{cl}(Y_3)\} = \left\{-I_3 Y_1^{-1}\right\} = \{\mathrm{diag}(-1, -1, 1)\}$ as $\mathrm{cl}(Y_3) = \mathrm{cl}(-I_3) = \{-I_3\}$.

Outer involutions of $SL_n(\mathbb{F}_q)$

Theorem (Helminck, Wu, Dometrius)

Let k be a finite field of odd characteristic, s_p be a representative of the non-square class of $k^*/(k^*)^2$, I_j be the $j \times j$ identity matrix, Inn_G represent conjugation by a matrix G, and M be the matrix $\begin{pmatrix} I_{n-1} & 0 \\ 0 & s_p \end{pmatrix}$ for an integer $n > 2$. Furthermore, for even n define the matrix J as $\begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$. If n is odd, then there are two isomorphism classes of outer involutions for $SL_n(k)$; representatives are θ_1 given by $\theta_1(X) = X^{-T}$ and $\theta_1 \circ \mathrm{Inn}_M$. If n is even, then there are three isomorphism classes of outer involutions for $SL_n(k)$; representatives are given by θ_1, $\theta_1 \circ \mathrm{Inn}_M$, and $\theta_1 \circ \mathrm{Inn}_J$.

Outer Involutions of $\mathrm{SL}_n(k)$

Consider the outer involution θ_1 with $\theta_1(g) = g^{-T}$.

$$\begin{aligned} R - Q &= \{r \in R \,|\, g(r)g^T \neq I_n \text{ for all } g \in \mathrm{SL}_n(k)\} \\ &= \{r \in R \,|\, r \text{ is } \textbf{not congruent to } I_n \text{ over } \mathrm{SL}_n(k)\} \end{aligned}$$

Tools:

- (Albert, 1938) For k be a finite field of odd characteristic and $A \in \mathrm{GL}_n(k)$.
  - (1) If A is symmetric, then A is congruent over k to a diagonal matrix.
  - (2) If $n = 2\ell$ and A is skew-symmetric, then A is congruent over k to the block matrix $J = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$.
- (BHKSWZ) Every diagonal matrix in $\mathrm{SL}_n(k)$ is congruent to I_n over $\mathrm{SL}_n(k)$.

Outer Involutions of $\mathrm{SL}_n(k)$

θ_1 with $\theta_1(g) = g^{-T}$

$$\begin{aligned} R - Q &= \{r \in R \mid g(r)g^T \neq I_n \text{ for all } g \in \mathrm{SL}_n(k)\} \\ &= \{r \in R \mid r \text{ is } \textbf{not congruent to } I_n \text{ over } \mathrm{SL}_n(k)\} \end{aligned}$$

What do matrices in R look like?

$$R = \{r \in \mathrm{SL}_n(k) \mid r^{-T} = r^{-1}\}$$
$$= \{r \in \mathrm{SL}_n(k) \mid r^T = r\}$$

- By Albert r is congruent to a diagonal matrix d
- By BHKSWZ d is congruent to I_n

The symmetric spaces of $SL_n(\mathbb{F}_q)$

Involution	Result
$\theta_1(X) = X^{-T}$	$R = Q$
$\theta_1 \circ \mathrm{Inn}_M$	$R = Q$
$\theta_1 \circ \mathrm{Inn}_J$	$R \neq Q, \ \lvert R\rvert = 2\lvert Q\rvert$
Inn_L	$R = Q$
Inn_{Y_i}	$R \neq Q$

Inner involutions of $\mathrm{SL}_n(\mathbb{F}_q)$, $\mathrm{char}(\mathbb{F}_q) = 2$

Theorem (Swartz):

For k a finite field of characteristic two and $n > 2$, there are $\lfloor \frac{n}{2} \rfloor$ isomorphism classes of inner involutions of $\mathrm{SL}_n(k)$ with representatives Inn_{L_i} for $i \in \left\{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\right\}$ where

$$L_i = \mathrm{diag}\left(\underbrace{\begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ c^2 & 0 \end{pmatrix}}_{i \text{ copies}}, cI_{n-2i} \right) \tag{1}$$

with c any element in k^*.

Outer involutions of $\mathrm{SL}_n(\mathbb{F}_q)$, $\mathrm{char}(\mathbb{F}_q) = 2$

Theorem (Swartz):

Suppose k is a finite field of field of characteristic two, I_j be the $j \times j$ identity matrix, and Inn_G represent conjugation by a matrix G. Furthermore, for even n define the matrix J as $\begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}$.

If n is odd, then there is one isomorphism class of outer involutions for $\mathrm{SL}_n(k)$ with representative θ_1 given by $\theta_1(X) = X^{-T}$. If n is even, then there are two isomorphism classes of outer involutions for $\mathrm{SL}_n(k)$; representatives are given by θ_1 and $\theta_3 = \theta_1 \circ \mathrm{Inn}_J$.

A similar result does not hold when $\mathrm{char}(k) = 2$.

The only involution of $\mathrm{SL}_2(\mathbb{Z}_2)$ is $\mathrm{Inn}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ [Schwartz], which gives rise to

$$Q(\mathrm{SL}_2(\mathbb{Z}_2)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\} \quad \text{and}$$

$$R(\mathrm{SL}_2(Z_2)) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

The symmetric spaces of $SL_n(\mathbb{F}_q)$, $char(\mathbb{F}_q) = 2$

Involution	Result
θ_1	n odd, $R = Q$; n even, $R \neq Q$
$\theta_1 \circ Inn_M$	$R \neq Q$
$\theta_1 \circ Inn_J$	$R \neq Q$
Inn_{L_i}	$R \neq Q$

Thank you!