Finite Subgroups of $Gl_2(\mathbb{C})$ and Universal Deformation Rings

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Goal

Goal : Find connections between **fusion** and **universal deformation rings.**

Two elements of a subgroup N of a finite group Γ are said to be **fused** if they are conjugate in Γ, but not in N.

The study of **fusion** arises in trying to relate the local structure of Γ to its global structure. Fusion is also important to understanding the representation theory of Γ.

Universal deformation rings of irreducible mod p representations of Γ can be viewed as providing a universal generalization of Brauer character theory of Γ.

My aim is to connect fusion to this universal generalization.

Universal Deformation Rings

- Let Γ be a finite group
- Let V be an absolutely irreducible $\mathbb{F}_p\Gamma$-module.

By Mazur, V has a so-called universal deformation ring $R(\Gamma, V)$.

> The ring $R(\Gamma, V)$ is characterized by the property that the isomorphism class of every lift of V over a complete local commutative Noetherian ring R with residue field $\mathbb{F}_p$ arises from a unique local ring homomorphism $\alpha : R(\Gamma, V) \to R$.
>
> (A lift of V to R is a pair (M, ϕ) where M is a finitely generated $R\Gamma$-module that is free over R, and $\phi : \mathbb{F}_p \otimes_R M \to V$ is an isomorphism of $\mathbb{F}_p\Gamma$-modules)

Setup

Let G be a finite group which admits a faithful two-dimensional irreducible complex representation. We associate to G an odd prime p, such that

- $\mathbb{F}_p G$ is semisimple
- $\mathbb{F}_p$ is a sufficiently large field for G

Consider a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \arrow[r, "\iota"] & \Gamma \arrow[r, "\pi"] & G \arrow[r] \arrow[ll, bend left, "\phi"] & 1 \end{tikzcd}$$

where

- The action of G on $N \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ corresponds to an irreducible representation ϕ

Question

We call the **fusion** of N in Γ the collection of tuples $(n_1, n_2) \in N \times N$, where n_1 and n_2 are fused in Γ.
We try to answer the following question:

Question

Let Σ be some subset of isoclasses of two-dimensional, absolutely irreducible $\mathbb{F}_p\Gamma$-modules. Consider the function

$$\Sigma \to \{\text{local rings}\}, \text{ which sends } V \to R(\Gamma_\phi, V).$$

Can the graph of this function be used to detect the fusion of N in Γ?

Answer

The function $V \to R(\Gamma_\phi, V)$ is nonconstant in this context exactly when the representation ϕ is trivial on the center of G.

When the function $V \to R(\Gamma_\phi, V)$ is not trivial, knowledge of its graph can be used to determine the fusion of N in Γ.

Specifically, we obtain the correspondence

$$\text{Fusion of } \phi \leftrightsquigarrow \{ker(\rho) : \rho \text{ abs. irr. and } R(\Gamma, V_\rho) \ncong \mathbb{Z}_p\}.$$

Answer

Theorem (M.)

Let G be a finite irreducible subgroup of $Gl_2(\mathbb{C})$. Let p be an odd prime such that $\mathbb{F}_pG$ is semisimple, and $\mathbb{F}_p$ is a sufficiently large field for G. Let ϕ be an irreducible action of G on $N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let $\Gamma = \Gamma_\phi$ be the corresponding semidirect product. Then, the following two statements are equivalent,

i. *ϕ is trivial on the center of G*
ii. *there exists a V with $R(\Gamma, V) \ncong \mathbb{Z}_p$.*

Theorem (M.)

Let G be a finite irreducible subgroup of $Gl_2(\mathbb{C})$. Let p be an odd prime such that $\mathbb{F}_pG$ is semisimple, and $\mathbb{F}_p$ is a sufficiently large field for G. Let ϕ be an irreducible action of G on $N = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and let $\Gamma = \Gamma_\phi$ be the corresponding semidirect product. Suppose that ϕ is trivial on the center of G. Then one can determine the fusion of N in Γ from the set $\{ker(\rho) : R(\Gamma, V_\rho) \ncong \mathbb{Z}_p\}$.

Sketch

Make use of the following results:

Proposition (M.)

Let ϕ be the action of G on N, $\tilde{\phi}$ denote the contragredient representation of ϕ. Let V be an absolutely irreducible $\mathbb{F}_p\Gamma$-module. Then,

$$\mathrm{H}^2(\Gamma, \mathrm{Hom}_{\mathbb{F}_p}(V, V)) \cong [(W_{\tilde{\phi}} \otimes V^* \otimes V) \oplus (W_{\tilde{\phi}\wedge\tilde{\phi}} \otimes V^* \otimes V)]^G.$$

(For any representation θ, W_θ denotes the $\mathbb{F}_p\Gamma$-module associated to θ)

Theorem (Dickson)

If $G \subseteq GL_2(\mathbb{F}_p)$ is a semisimple subgroup, then its image in $PGL_2(\mathbb{F}_p)$ is either cyclic, dihedral, or isomorphic to A_4, A_5, or S_4.

Sketch

So we have the following;

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathbb{Z}/m\mathbb{Z} \arrow[d, "\iota"] & \\
0 \arrow[r] & \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \arrow[r, "\iota"] & \Gamma \arrow[r, "\pi"] & G \arrow[r] \arrow[d, "\pi"] \arrow[ll, bend left, "\phi"] & 1 \\
 & & & H \arrow[d] & \\
 & & & 0 &
\end{tikzcd}
$$

Sketch

- Reduce to the case where H is dihedral and use the faithful irreducible complex representation to construct a presentation of G
- When ϕ is trivial on $Z(G)$, ϕ corresponds to a two-dimensional representation of a dihedral group G
- Explicitly construct a representation with universal deformation ring different from $\mathbb{Z}_p$
- Show that the representations with universal deformation ring different from $\mathbb{Z}_p$ are a full orbit of the character group of G
- Associate to the kernels of each of these representations a linear diophantine equation with coefficients in a cyclic group, and use the character group of G to make a combinatorial argument

Thank You

THANK YOU!