GENERATING INVARIANTS IN POSITIVE CHARACTERISTIC

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ABSTRACT. The ring of invariants for a rational representation of a reductive group is finitely generated by the results of Hilbert, Nagata and Haboush. However, the proof is not constructive in nature. While finding a minimal set of generators remains a difficult question, one could ask instead for an upper bound on the degrees of generators. The best known bounds in characteristic zero are due to Derksen, and it remains an open question in positive characteristic.

We outline a strategy to compute bounds in positive characteristic when the coordinate ring has a good filtration. Using this strategy, we are able to obtain strong bounds for invariant rings associated to quivers for arbitrary characteristic.

This is joint work with Harm Derksen.

1. MATRIX INVARIANTS

We fix an algebraically closed field K. We look at the action of $G = \mathrm{GL}_n$ on $V = \mathrm{Mat}_{n,n}^m$ by simultaneous conjugation. More precisely,

$$g \cdot (X_1, X_2, \dots, X_m) = (gX_1g^{-1}, gX_2g^{-1}, \dots, gX_mg^{-1}).$$

The ring of polynomial invariants $K[V]^G$ is called the ring of matrix invariants. The following results are known about the ring of matrix invariants.

Theorem 1.1 (Procesi, 1976)**.** *Assume* $\mathrm{char}(K) = 0$. *The invariants of the form* $Tr(X_{i_1}X_{i_2}\dots X_{i_r})$ *generate the invariant ring* $K[V]^G$

This result was extended by Donkin to all characteristics, if one takes the coefficients of the characteristic polynomial rather than the traces. Let σ_j denote the coefficient of t^j in the characteristic polynomial.

Theorem 1.2 (Donkin, 1992)**.** *The invariants of the form* $\sigma_j(X_{i_1}X_{i_2}\dots X_{i_r})$ *generate the invariant ring* $K[V]^G$ *in all characteristics.*

Observe that the generating sets given above are infinite. Finding a minimal set of generators is perhaps hopeless, one could ask instead for a bound on the degree of generators. We recall some definitions that were also given in Harm Derksen's talks.

Definition 1.3. $\beta_G(V) = \min\{d \mid K[V]_{\leq d} \text{ generates } K[V]\}$.

For the case of matrix invariants, there is a good bound on $\beta_G(V)$ due to Razmyslov.

Theorem 1.4 (Razmyslov, 1974)**.** *Assume* $\mathrm{char}(K) = 0$. *Then we have* $\beta_G(V) \leq n^2$.

A lower bound of $n(n+1)/2$ has been shown by Kuzmin, and hence the above bound is quite a good one in characteristic 0. Razmslov's approach can be seen as showing that the traces of monomials of degree $\geq n^2$ are decomposable, i.e., can be rewritten from traces of smaller degrees. This approach however struggles to give a good bound in positive characteristic.

Theorem 1.5 (Domokos, 2002). *We have* $\beta_G(V) = O(n^7 m^n)$

As is evident, the bound in positive characteristic is much worse than in characteristic 0. A natural question to ask is if can improve the bound, in particular, could we get a bound that is polynomial in n and m.

2. Degree bounds in characteristic 0

In characteristic 0, degree bounds for generators of invariant rings can be given by the methods of Popov and Derksen. We would like to adapt their method to other characteristic. Hence I would like to outline the strategy and identify the key ingredients required.

The starting point of the method is the celebrated Hochster-Roberts theorem. For the rest of this section, let G be a linearly reductive group in characteristic 0, and let V be a finite dimensional rational representation of G

Theorem 2.1 (Hochster-Roberts). *The invariant ring* $K[V]^G$ *is Cohen-Macaulay.*

Definition 2.2. A collection of homogeneous algebraically independent invariants $\{f_1, f_2, \ldots, f_r\}$ is called a homogeneous system of parameters (hsop) if $K[V]^G$ is a finite module over $K[f_1, f_2, \ldots, f_r]$.

The Cohen-Macaulay condition tells us that for any hsop $\{f_1, f_2, \ldots, f_r\}$, $K[V]^G$ is a free module over $K[f_1, f_2, \ldots, f_r]$. Let $g_1, g_2, \ldots, g_s$ be the free module generators. In particular, notice that the collection $\{f_i, g_j\}$ gives us a generating set, even though it may not be a minimal generating set.

The next major ingredient is the following result of Kempf.

Theorem 2.3 (Kempf). *We have* $\deg(g_j) \leq \sum_{i=1}^{r} \deg(f_i)$.

The above result of Kempf is sometimes stated as "The Hilbert series of the invariant ring is a proper rational function". It is easy to see that the above formulation is an equivalent one.

With this result, one realizes that to get a bound on the degree of generating invariants, it suffices to produce a hsop. This was first done by Popov. Derksen improved Popov's results, and showed that in fact it suffices to find invariants defining the null cone. We will not recall this as we do not have an explicit need for this.

Recall from Derksen's talk that the null cone is the zero set in V of all homogeneous invariants of positive degree.

Hence, the three main ingredients required are:

- $K[V]^G$ is Cohen-Macaulay.
- Kempf's result on the Hilbert series.
- A hsop, or atleast a set of invariants defining the null cone.

3. Good Filtrations

In positive characteristic, the only linearly reductive groups are finite groups, tori, and extensions of tori by finite groups. Classical groups such as GL_n and SL_n are not linearly reductive, and hence the Hochster-Roberts theorem is unavailable in the cases of interest to us. However, the theory of good filtrations gives us an alternative.

Definition 3.1. Let G be a reductive group. A G-module V is called a good G-module if there exists a filtration $\cdots \subset V_i \subset V_{i+1} \subset \ldots$ such that the successive quotients V_{i+1}/V_i are dual Weyl modules.

For the case of GL_n, dual Weyl modules are just Schur modules. It turns out that a result similar to Hochster-Roberts theorem exists if the coordinate ring of a representation has a good filtration.

Theorem 3.2 (Hashimoto 2001)**.** *Let G be a reductive group and V a finite dimensional representation. If $K[V]$ is a good G-module, then $k[V]^G$ is strongly F-regular, and hence Cohen-Macaulay.*

$K[\mathrm{Mat}_{n,n}^m]$ has a good filtration as a GL_n-module. Hence the ring of matrix invariants is Cohen-Macaulay in all characteristics. Further, it follows from the theory of good filtrations that the Hilbert series is independent of the characteristic, and hence Kempf's result will still be true! However, a hsop, or a set of invariants defining the null cone is absent. For this reason, we turn to the situation of matrix semi-invariants.

4. Matrix semi-invariants

Consider the left-right action of $G = SL_n \times SL_n$ on $V = \mathrm{Mat}_{n,n}^m$ as follows:

$$(A, B) \cdot (X_1, \ldots, X_m) = (AX_1B^{-1}, \ldots, AX_mB^{-1}).$$

We recall the main results from Derksen's talks.

Theorem 4.1 (Derksen,M.)**.** *There exist a hsop $f_1, f_2, \ldots, f_r$ with $\deg(f_i) = n(n-1)$.*

Further, observe that $r = \dim(K[V]^G) \leq mn^2$. Using Kempf's result, we get a bound of mn^4 on the degree of generating invariants.

$K[\mathrm{Mat}_{n,n}^m]$ has a good filtration as an $SL_n \times SL_n$-module as well. Hence Hashimoto's result is valid, as well as Kempf's result. The above theorem on the hsop is true in all characteristics, and hence the bound of mn^4 works in all characteristics.

Theorem 4.2 (Derksen,M.)**.** *We have $\beta_G(V) \leq mn^4$.*

5. Matrix invariants revisited

We wish to use the bound for matrix semi-invariants to get a bound for matrix invariants. To do so, we use a trick that was first used by Domokos.

We consider the map $\mathrm{Mat}_{n,n}^m \hookrightarrow \mathrm{Mat}_{n,n}^{m+1}$ given by $(X_1, \ldots, X_m) \mapsto (X_1, \ldots, X_m, I)$. It is easy to check that a surjection of invariant rings $K[\mathrm{Mat}_{n,n}^{m+1}]^{\mathrm{SL}_n \times \mathrm{SL}_n} \twoheadrightarrow K[\mathrm{Mat}_{n,n}^m]^{GL_n}$ follows. This map is degree non-increasing, so a bound on the degree of generators for matrix invariants follows.

Theorem 5.1 (Derksen,M.)**.** *We have $\beta_{GL_n}(\mathrm{Mat}_{n,n}^m) \leq (m+1)n^4$.*

Thus, we have successfully given a polynomial bound for matrix invariants!