A Conjecture of Victor Kac

Conference on Geometric Methods in Invariant Theory

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November 28, 2016

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Kac’s Conjecture
Action of $GL(\beta)$ on $\text{rep}(Q, \beta)$

Definition

1. $\text{rep}(Q, \beta) = \prod_{a \text{ arrow of } Q} \text{Mat}_{\beta(ha) \times \beta(ta)}(K)$

2. $GL(\beta) = \prod_{i \text{ vertex of } Q} GL(\beta(i))$

There is a natural action of $GL(\beta)$ on $\text{rep}(Q, \beta)$ by simultaneous conjugation:

$$(g \cdot V)(a) = g(ha) \cdot V(a) \cdot g(ta)^{-1}$$
SL(\(\beta\)) := \prod_{i \text{ vertex of } Q} SL(\beta(i)).

The algebra of semi-invariants is:

\[ SI(Q, \beta) = K[\text{rep}(Q, \beta)]^{SL(\beta)} \]
• Hilbert’s 14th problem $\implies$ $\text{SI}(Q, \beta)$ is finitely generated.

• $\text{SI}(Q, \beta)$ defines the affine quotient variety $\text{rep}(Q, \beta)//\text{SL}(\beta)$.

• $V \in \text{rep}(Q, \beta)$ is called \textit{locally semi-simple} if:

$$\text{SL}(\beta)V = \overline{\text{SL}(\beta)V}.$$
Kac’s Conjecture

“It seems that in the case of finite and tame oriented graphs...a representation is [locally] semisimple if and only if its endomorphism ring is semisimple.” (page 161, Infinite Root Systems, Representations of Graphs and Invariant Theory II, Journal of Algebra, 78, 1982)

Victor Kac
Stability

**Fact**

There is an epimorphism of abelian groups: \((\mathbb{Z}^{Q_0}, +) \to X^{*}(GL(\beta))\), where \(\theta \mapsto \chi_{\theta}\), defined by:

\[
\chi_{\theta} ((g(i))_{i \in Q_0}) := \prod_{i \in Q_0} \det(g(i))^{\theta(i)}
\]

**Fact**

\(\text{SI}(Q, \beta) \cong \bigoplus_{\theta \in \mathbb{Z}^{Q_0}} \text{SI}(Q, \beta)_{\theta}\) where

\[
\text{SI}(Q, \beta)_{\theta} = \{ f \in K[\text{rep}(Q, \beta)] | g \cdot f = \theta(g)f, \forall g \in GL(\beta) \}.
\]
Stability

Definition

Let $V \in \text{rep}(Q, \beta)$, $\theta \in \mathbb{Z}^{Q_0}$, and $GL(\beta)_{\theta} : = \ker(\chi_{\theta})$.

a) We say that $V$ is $\theta$-semi-stable if there exist $n \in \mathbb{Z}_{\geq 1}$ and $f \in \text{SI}(Q, \beta)_{n\theta}$ such that $f(V) \neq 0$.

b) We say that $V$ is $\theta$-stable if $V$ is $\theta$-semi-stable, and $GL(\beta)_{\theta} \cdot V$ is a closed orbit of dimension $\dim GL(\beta) - 2$.

Theorem (King, 1993)

Let $V \in \text{rep}(Q, \beta)$ and $\theta \in \mathbb{Z}^{Q_0}$.

1. $V$ is $\theta$-semi-stable if $\theta(\dim V) = 0$ and $\theta(\dim V') \leq 0$ for all $V' \leq V$.

2. $V$ is $\theta$-stable if $\theta(\dim V) = 0$ and $\theta(\dim V') < 0$ for all proper $V' \leq V$. 
Theorem

Let $V \in \text{rep}(Q, \beta)$ with

$$V \simeq \bigoplus_{i=1}^{r} V_{i}^{m_{i}}$$

a decomposition of $V$ into pairwise non-isomorphic indecomposable representations $V_{1}, \ldots, V_{r}$, with multiplicities $m_{1}, \ldots, m_{r} \geq 1$. Then the following are equivalent:

a) $V$ is locally semi-simple;

b) there exists a common weight $\theta$ of $Q$ such that each $V_{i}$ is $\theta$-stable.
Semi-Simple Endomorphism Rings

Definition

A sequence of representations $V_1, \ldots, V_r$ is called an orthogonal Schur sequence if all the representations $V_i$ are Schur and $\text{Hom}(V_i, V_j) = 0$ for $i \neq j$.

Theorem

Let $A$ be a $K$-algebra and $V$ an $A$-module. Let

$$V \cong \bigoplus_{i=1}^{r} V_i^{m_i}$$

be a decomposition of $V$ into pairwise non-isomorphic indecomposable $A$-modules $V_1, \ldots, V_r$ with multiplicities $m_1, \ldots, m_r \geq 1$. Then $\text{End}_A(V)$ is a semi-simple $K$-algebra if and only if $V_1, \ldots, V_r$ form an orthogonal Schur sequence.
Corollary
Let $Q$ be any acyclic quiver and $V \in \text{rep}(Q, \beta)$. If $V$ is locally semi-simple, then $\text{End}_Q(V)$ is semi-simple.

Key question: Given an orthogonal Schur sequence, does there exists a common weight $\theta$ such that each representation is $\theta$-stable?
Orthogonal Schur Sequences and Stability Weights
Non-regular Case

Definition

A sequence $V_1, \ldots, V_r$ is called an exceptional sequence if each $V_i$ is exceptional and $\text{Hom}_Q(V_i, V_j) = \text{Ext}_Q^1(V_i, V_j) = 0$ for $i < j$.

Proposition (Derksen-Weymen)

Let $Q$ be a quiver and $\mathcal{L} = (V_1, \ldots, V_r)$ an orthogonal exceptional sequence of representations of $Q$. Then there exists a weight $\theta$ such that $V_i$ is $\theta$-stable for all $1 \leq i \leq r$.

Proposition

a) When $Q$ is Dynkin, any orthogonal Schur sequence has a common stability weight.

b) When $Q$ is Euclidean, any orthogonal Schur sequence containing at least one non-regular representation has a common stability weight.
\[ \mathcal{R}(Q) = \text{rep}(Q)^{ss}_{\langle \delta, \cdot \rangle} \]

**Lemma**

Let \( X \) be a regular simple representation. Then:

i) \( X \) is Schur;

ii) \( \tau^i(X) \) is regular simple for all \( i \);

iii) \( X \) is \( \tau \)-periodic;

iv) \( \tau(X) \cong X \) if and only if \( \dim X = r\delta \), for some \( r \in \mathbb{Z}_{\geq 0} \);

v) if \( X \) has period \( p \), then \( \dim X + \dim \tau(X) + \ldots + \dim \tau^{p-1}(X) = \delta \).
Definition

A regular representation $X$ is called regular uniserial if all of the regular subrepresentations of $X$ lie in a chain:

$$0 = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_{r-1} \subsetneq X_r = X$$

In this case, $X$ has regular simple composition factors $X_1, X_2/X_1, \ldots, X_r/X_{r-1}$, regular length $r\ell(X) := r$, regular socle $r\text{Soc}(X) := X_1$ and regular top $r\text{Top}(X) := X/X_{r-1}$.

Theorem

Every indecomposable regular representation $X$ is regular uniserial. Moreover, if $E$ is the regular top of $X$, then the compositions factors of $X$ are precisely $E, \tau(E), \ldots, \tau^\ell(E)$ where $\ell + 1 = r\ell(X)$. 
Tube of period 3

\[
\begin{pmatrix}
E_2 \\
E_0 \\
E_1
\end{pmatrix}
\overset{\tau}{\rightarrow}
\begin{pmatrix}
E_1 \\
E_2 \\
E_0
\end{pmatrix}
\overset{\tau}{→}
\begin{pmatrix}
E_0 \\
E_1 \\
E_2
\end{pmatrix}
\overset{\tau}{→}
\begin{pmatrix}
E_2 \\
E_0 \\
E_1
\end{pmatrix}
\]

E_0 \overset{\tau}{→} E_2 \overset{\tau}{→} E_1 \overset{\tau}{→} E_0
Proposition

Let $Q$ be a Euclidean quiver. Then given any orthogonal Schur sequence of regular representations $V_1, \ldots, V_r$ there exists a weight $\theta$ such that each $V_i$ is $\theta$-stable.
Example

Let $Q$ be the $\tilde{D}_5$ quiver:
The three non-homogeneous regular tubes of $Q$ are generated by the following regular simples:

\[ \mathcal{T}_1 = \left\langle E_1 = \begin{array}{ccc} K & \xrightarrow{id} & K \\ \downarrow & \downarrow & \downarrow \\ K & \xrightarrow{id} & K \end{array} \right\rangle, \quad E_2 = \begin{array}{ccc} K & \rightarrow & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & \rightarrow & 0 \end{array} \right\rangle, \quad \mathcal{T}_3 = \left\langle E_3 = \begin{array}{ccc} 0 & \rightarrow & 0 \\ \downarrow & \downarrow & \downarrow \\ 0 & \rightarrow & K \end{array} \right\rangle, \]

\[ \mathcal{T}_2 = \left\langle L_1 = \begin{array}{ccc} K & \xrightarrow{id} & K \\ \downarrow & \downarrow & \downarrow \\ 0 & \xrightarrow{id} & 0 \end{array} \right\rangle, \quad L_2 = \begin{array}{ccc} K & \xrightarrow{id} & 0 \\ \downarrow & \downarrow & \downarrow \\ K & \xrightarrow{id} & K \end{array} \right\rangle, \]

\[ \mathcal{T}_3 = \left\langle Y_1 = \begin{array}{ccc} K & \xrightarrow{id} & K \\ \downarrow & \downarrow & \downarrow \\ 0 & \xrightarrow{id} & K \end{array} \right\rangle, \quad Y_2 = \begin{array}{ccc} K & \xrightarrow{id} & 0 \\ \downarrow & \downarrow & \downarrow \\ K & \xrightarrow{id} & 0 \end{array} \right\rangle. \]
Consider the orthogonal Schur sequence
\[ \mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3, \]
where:
\[
\mathcal{L}_0 = \begin{cases}
V_1 = \begin{cases}
K \xrightarrow{[1 \ 0]} K^2 \xrightarrow{id} K^2 \xrightarrow{[1 \ 2]} K \\
K \xrightarrow{[0 \ 1]} \end{cases} & \\
K \xrightarrow{[0 \ 1]} K \xrightarrow{[1 \ 2]} K \xrightarrow{[1 \ 1]} K
\end{cases},
\mathcal{L}_1 = \begin{cases}
V_2 = \begin{cases}
K \xrightarrow{[1 \ 0]} K^2 \xrightarrow{id} K^2 \xrightarrow{[1 \ 1]} K \\
K \xrightarrow{[0 \ 1]} \end{cases} & \\
K \xrightarrow{[0 \ 1]} K \xrightarrow{[1 \ 1]} K \xrightarrow{[1 \ 0]} K
\end{cases} = \left( \begin{array}{c}
L_1 \\
L_2
\end{array} \right),
\mathcal{L}_2 = \begin{cases}
V_3 = \begin{cases}
K \xrightarrow{[1 \ 0]} K^2 \xrightarrow{id} K^2 \xrightarrow{[1 \ 1]} K \\
K \xrightarrow{[0 \ 1]} \end{cases} & \\
K \xrightarrow{[0 \ 1]} K \xrightarrow{[1 \ 1]} K \xrightarrow{[1 \ 1]} K
\end{cases} = \left( \begin{array}{c}
E_1 \\
E_2 \\
E_3
\end{array} \right), V_4 = E_2
\]
and \( \mathcal{L}_3 = \{ V_5 = Y_1, V_6 = Y_2 \} \).
\[
\begin{bmatrix}
\text{dim } E_1 \\
\text{dim } E_2 \\
\text{dim } E_3 \\
\text{dim } L_1 \\
\text{dim } Y_1
\end{bmatrix}
\cdot \theta =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
-1 \\
1 \\
0
\end{bmatrix}.
\]
The general solution of this system is \((t, 2-t, 1-t, t-1, 0, -1)\) for \(t \in \mathbb{R}\). When \(t = 1\), we get \(\theta = (1, 1, 0, 0, 0, -1)\)

Now set:

\[
\sigma = \theta + 2\langle \delta, \cdot \rangle = (3, -1, -2, 2, 0, -1).
\]

Then each \(V_i\) is \(\sigma\)-stable and \(V = \bigoplus_{i=1}^{6} V_i\) is locally semi-simple.
Example

$K_3 := \bullet_1 \rightarrow \bullet_2$. Let $V = \mathbb{C}^2$, with maps $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$V(a) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $V(b) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $V(c) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Theorem (Main Result)

Let $Q$ be an acyclic quiver. Then the following statements are equivalent:

(i) $Q$ is tame;

(ii) a $Q$-representation $V$ is locally semi-simple if and only if $\text{End}_Q(V)$ is semi-simple.
Thank you!