A Conjecture of Victor Kac

Conference on Geometric Methods in Invariant Theory

Daniel Kline (joint work with Calin Chindris)

November 28, 2016

University of Missouri - Columbia

Kac's Conjecture

Action of $\mathrm{GL}(\beta)$ on $\mathrm{rep}(Q, \beta)$

Definition

1. $\mathrm{rep}(Q, \beta) = \prod_{a \text{ arrow of } Q} \mathrm{Mat}_{\beta(ha) \times \beta(ta)}(K)$

2. $\mathrm{GL}(\beta) = \prod_{i \text{ vertex of } Q} \mathrm{GL}(\beta(i))$

There is a natural action of $\mathrm{GL}(\beta)$ on $\mathrm{rep}(Q, \beta)$ by simultaneous conjugation:

$$(g \cdot V)(a) = g(ha) \cdot V(a) \cdot g(ta)^{-1}$$

Algebra of Semi-invariants

$$\mathrm{SL}(\beta) := \prod_{i \text{ vertex of } Q} SL(\beta(i)).$$

The *algebra of semi-invariants* is:

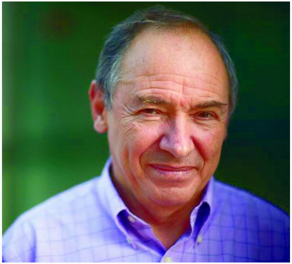
$$\mathrm{SI}(Q,\beta) = K[\mathrm{rep}(Q,\beta)]^{\mathrm{SL}(\beta)}$$

Locally Semi-Simple Representations

- Hilbert's 14th problem $\implies$ $\mathsf{SI}(Q,\beta)$ is finitely generated.
- $\mathsf{SI}(Q,\beta)$ defines the affine quotient variety $\mathrm{rep}(Q,\beta)//\mathrm{SL}(\beta)$.
- $V \in \mathrm{rep}(Q,\beta)$ is called *locally semi-simple* if:

$$\mathrm{SL}(\beta)V = \overline{\mathrm{SL}(\beta)V}.$$

Kac's Conjecture



Victor Kac

"It seems that in the case of finite and tame oriented graphs...a representation is [locally] semisimple if and only if its endomorphism ring is semisimple." (page 161, *Infinite Root Systems, Representations of Graphs and Invariant Theory II*, Journal of Algebra, 78, 1982)

Stability

Fact

There is an epimorphism of abelian groups: $(\mathbb{Z}^{Q_0}, +) \twoheadrightarrow X^*(GL(\beta))$, *where* $\theta \mapsto \chi_\theta$, *defined by:*

$$\chi_\theta\left((g(i))_{i\in Q_0}\right) := \prod_{i\in Q_0} \det(g(i))^{\theta(i)}$$

Fact

$\mathrm{SI}(Q,\beta) \cong \bigoplus_{\theta\in\mathbb{Z}^{Q_0}} \mathrm{SI}(Q,\beta)_\theta$ *where*

$$\mathrm{SI}(Q,\beta)_\theta = \{f \in K[\mathrm{rep}(Q,\beta)] | g\cdot f = \theta(g)f, \forall g \in GL(\beta)\}.$$

Stability

Definition

Let $V \in rep(Q, \beta)$, $\theta \in \mathbb{Z}^{Q_0}$, *and* $GL(\beta)_\theta \colon = \ker(\chi_\theta)$.

a) *We say that* V *is* θ*-semi-stable if there exist* $n \in \mathbb{Z}_{\geq 1}$ *and* $f \in \mathrm{SI}(Q, \beta)_{n\theta}$ *such that* $f(V) \neq 0$.

b) *We say that* V *is* θ*-stable if* V *is* θ*-semi-stable, and* $\mathrm{GL}(\beta)_\theta \cdot V$ *is a closed orbit of dimension* $\dim GL(\beta) - 2$.

Theorem (King, 1993)

Let $V \in \mathrm{rep}(Q, \beta)$ *and* $\theta \in \mathbb{Z}^{Q_0}$.

1. V *is* θ*-semi-stable if* $\theta(\underline{\dim}\, V) = 0$ *and* $\theta(\underline{\dim}\, V') \leq 0$ *for all* $V' \leq V$.
2. V *is* θ*-stable if* $\theta(\underline{\dim}\, V) = 0$ *and* $\theta(\underline{\dim}\, V') < 0$ *for all proper* $V' \leq V$

Locally Semi-Simple Representations and Stability

Theorem

Let $V \in \mathrm{rep}(Q, \beta)$ *with*

$$V \simeq \bigoplus_{i=1}^{r} V_i^{m_i}$$

a decomposition of V *into pairwise non-isomorphic indecomposable representations* $V_1, \ldots, V_r$*, with multiplicities* $m_1, \ldots, m_r \geq 1$*. Then the following are equivalent:*

a) V *is locally semi-simple;*

b) *there exists a common weight* θ *of* Q *such that each* V_i *is* θ*-stable.*

Semi-Simple Endomorphism Rings

Definition

A sequence of representations $V_1, \dots, V_r$ *is called an orthogonal Schur sequence if all the representations* V_i *are Schur and* $\operatorname{Hom}(V_i, V_j) = 0$ *for* $i \neq j$.

Theorem

Let A *be a* K*-algebra and* V *an* A*-module. Let*

$$V \cong \bigoplus_{i=1}^{r} V_i^{m_i}$$

be a decomposition of V *into pairwise non-isomorphic indecomposable* A*-modules* $V_1, \dots, V_r$ *with multiplicities* $m_1, \dots, m_r \geq 1$. *Then* $End_A(V)$ *is a semi-simple* K*-algebra if and only if* $V_1, \dots, V_r$ *form an orthogonal Schur sequence.*

One Direction of Kac's Conjecture

Corollary

Let Q be any acyclic quiver and $V \in \mathrm{rep}(Q, \beta)$. If V is locally semi-simple, then $\mathrm{End}_Q(V)$ is semi-simple.

Key question: *Given an orthogonal Schur sequence, does there exists a common weight θ such that each representation is θ-stable?*

Orthogonal Schur Sequences and Stability Weights

Non-regular Case

Definition

A sequence $V_1, \ldots, V_r$ *is called an exeptional sequence if each* V_i *is exceptional and* $\mathrm{Hom}_Q(V_i, V_j) = \mathrm{Ext}^1_Q(V_i, V_j) = 0$ *for* $i < j$.

Proposition (Derksen-Weymen)

Let Q *be a quiver and* $\mathcal{L} = (V_1, \ldots, V_r)$ *an orthogonal exceptional sequence of representations of* Q. *Then there exists a weight* θ *such that* V_i *is* θ*-stable for all* $1 \leq i \leq r$.

Proposition

a) *When* Q *is Dynkin, any orthogonal Schur sequence has a common stability weight.*

b) *When* Q *is Euclidean, any orthogonal Schur sequence containing at least one non-regular representation has a common stability weight.*

$$\mathcal{R}(Q) = \operatorname{rep}(Q)^{ss}_{\langle\delta,\cdot\rangle}$$

Lemma

Let X be a regular simple representation. Then:

i) *X is Schur;*
ii) *$\tau^i(X)$ is regular simple for all i;*
iii) *X is τ-periodic;*
iv) *$\tau(X) \cong X$ if and only if $\underline{\dim}\, X = r\delta$, for some $r \in \mathbb{Z}_{\geq 0}$;*
v) *if X has period p, then $\underline{\dim}\, X + \underline{\dim}\, \tau(X) + \ldots + \underline{\dim}\, \tau^{p-1}(X) = \delta$.*

Indecomposable Regular Representations

Definition

A regular representation X is called *regular uniserial* if all of the regular subrepresentations of X lie in a chain:

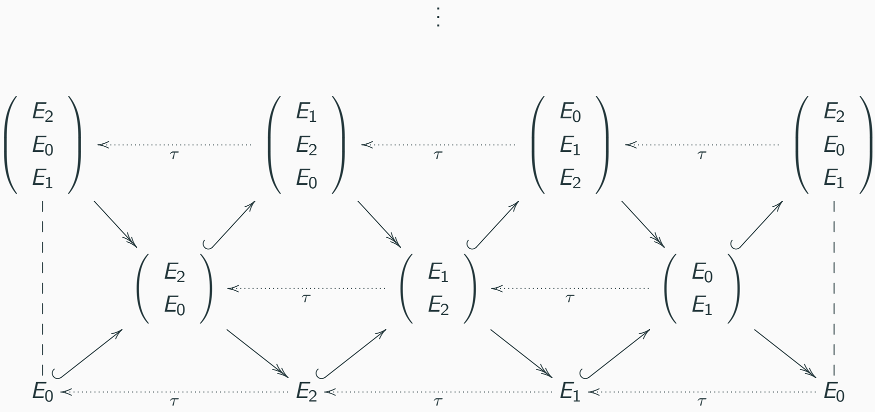
$$0 = X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_{r-1} \subsetneq X_r = X$$

In this case, X has regular simple composition factors $X_1, X_2/X_1, \ldots, X_r/X_{r-1}$, *regular length* $r\ell(X) := r$, *regular socle* $\mathrm{rSoc}(X) := X_1$ and *regular top* $\mathrm{rTop}(X) := X/X_{r-1}$.

Theorem

Every indecomposable regular representation X is regular uniserial. Moreover, if E is the regular top of X, then the compositions factors of X are precisely $E, \tau(E), \ldots, \tau^{\ell}(E)$ where $\ell + 1 = r\ell(X)$.

Tube of period 3



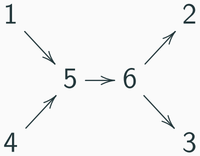
Regular Case

Proposition

Let Q be a Euclidean quiver. Then given any orthogonal Schur sequence of regular representations $V_1, \ldots, V_r$ there exists a weight θ such that each V_i is θ-stable.

Example

Let Q be the $\widetilde{\mathbb{D}}_5$ quiver:



The three non-homogeneous regular tubes of Q are generated by the following regular simples:

$$\mathcal{T}_1 = \left\langle E_1 = \begin{array}{ccccccc} K & & & & & & K \\ & \searrow^{id} & & & & \nearrow^{id} & \\ & & K & \xrightarrow[id]{} & K & & \\ & \nearrow_{id} & & & & \searrow_{id} & \\ K & & & & & & K \end{array}, E_2 = \begin{array}{ccccccc} 0 & & & & & & 0 \\ & \searrow & & & & \nearrow & \\ & & K & \rightarrow & 0 & & \\ & \nearrow & & & & \searrow & \\ 0 & & & & & & 0 \end{array}, E_3 = \begin{array}{ccccccc} 0 & & & & & & 0 \\ & \searrow & & & & \nearrow & \\ & & 0 & \rightarrow & K & & \\ & \nearrow & & & & \searrow & \\ 0 & & & & & & 0 \end{array} \right\rangle,$$

$$\mathcal{T}_2 = \left\langle L_1 = \begin{array}{ccccccc} K & & & & & & K \\ & \searrow^{id} & & & & \nearrow^{id} & \\ & & K & \xrightarrow[id]{} & K & & \\ & \nearrow & & & & \searrow & \\ 0 & & & & & & 0 \end{array}, L_2 = \begin{array}{ccccccc} 0 & & & & & & 0 \\ & \searrow & & & & \nearrow & \\ & & K & \xrightarrow[id]{} & K & & \\ & \nearrow_{id} & & & & \searrow_{id} & \\ K & & & & & & K \end{array} \right\rangle,$$

$$\mathcal{T}_3 = \left\langle Y_1 = \begin{array}{ccccccc} K & & & & & & 0 \\ & \searrow^{id} & & & & \nearrow & \\ & & K & \xrightarrow[id]{} & K & & \\ & \nearrow & & & & \searrow_{id} & \\ 0 & & & & & & K \end{array}, Y_2 = \begin{array}{ccccccc} 0 & & & & & & K \\ & \searrow & & & & \nearrow^{id} & \\ & & K & \xrightarrow[id]{} & K & & \\ & \nearrow_{id} & & & & \searrow & \\ K & & & & & & 0 \end{array} \right\rangle.$$

Consider the orthogonal Schur sequence

$$\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3,$$

where:

$$\mathcal{L}_0 = \left\{ V_1 = \begin{array}{ccccccc} K & & & & & & K \\ & \searrow^{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & & & \nearrow^{[1\ 1]} & \\ & & K^2 & \xrightarrow[id]{} & K^2 & & \\ & \nearrow_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & & & \searrow_{[1\ 2]} & \\ K & & & & & & K \end{array} \right\},\ \mathcal{L}_1 = \left\{ V_2 = \begin{array}{ccccccc} K & & & & & & K \\ & \searrow^{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & & & \nearrow^{[1\ 0]} & \\ & & K^2 & \xrightarrow[id]{} & K^2 & & \\ & \nearrow_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & & & \searrow_{[1\ 1]} & \\ K & & & & & & K \end{array} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \right\},$$

$$\mathcal{L}_2 = \left\{ V_3 = \begin{array}{ccccccc} K & & & & & & K \\ & \searrow^{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & & & \nearrow^{[1\ 1]} & \\ & & K^2 & \xrightarrow[id]{} & K^2 & & \\ & \nearrow_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & & & & \searrow_{[1\ 1]} & \\ K & & & & & & K \end{array} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, V_4 = E_2 \right\},$$

and $\mathcal{L}_3 = \{V_5 = Y_1, V_6 = Y_2\}$.

$$
\begin{bmatrix} \underline{\dim} E_1 \\ \underline{\dim} E_2 \\ \underline{\dim} E_3 \\ \underline{\dim} L_1 \\ \underline{\dim} Y_1 \end{bmatrix} \cdot \theta = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.
$$

The general solution of this system is $(t, 2-t, 1-t, t-1, 0, -1)$ for $t \in \mathbb{R}$. When $t = 1$, we get $\theta = (1, 1, 0, 0, 0, -1)$

Now set:

$$\sigma = \theta + 2\langle \delta, \cdot \rangle = (3, -1, -2, 2, 0, -1).$$

Then each V_i is σ-stable and $V = \bigoplus_{i=1}^{6} V_i$ is locally semi-simple.

Example

$K_3 := \bullet_1 \Rrightarrow \bullet_2$. Let $V = \mathbb{C}^2 \underset{c}{\overset{a}{\underset{b}{\Rrightarrow}}} \mathbb{C}^2$, with maps

$V(a) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ V(b) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, V(c) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$

Theorem (Main Result)

Let Q be an acyclic quiver. Then the following statements are equivalent:

(i) *Q is tame;*

(ii) *a Q-representation V is locally semi-simple if and only if $\mathrm{End}_Q(V)$ is semi-simple.*

Thank you!