Donaldson–Thomas invariants for A-type square product quivers

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Quantum dilogarithm series and pentagon identity

Definition 1

For a variable z, the **quantum dilogarithm series** in $\mathbb{Q}(q^{1/2})[[z]]$ is

$$\mathbb{E}(z) = 1 + \sum_{n=1}^{\infty} \frac{(-z)^n q^{n^2/2}}{\prod_{i=1}^{n}(1-q^i)}.$$

Theorem (Pentagon identity)

In the algebra $\mathbb{Q}(q^{1/2})[[y_1, y_2]]/(y_2 y_1 - q y_1 y_2)$ *we have*

$$\mathbb{E}(y_1)\,\mathbb{E}(y_2) = \mathbb{E}(y_2)\,\mathbb{E}(-q^{-1/2} y_2 y_1)\,\mathbb{E}(y_1).$$

This identity is often credited to Schützenberger (1953) but appeared more or less in the form above in the work of Faddeev–Kashaev (1994) as a quantum mechanical generalization of a dilogarithm function defined first by Euler, and then refined by Rogers (1907).

We seek generalizations of this identity.

Quivers

- Let $Q = (Q_0, Q_1)$ be a quiver with vertex set Q_0 and arrow set Q_1.
- For $a \in Q_1$ let $ta, ha \in Q_0$ respectively denote its head and tail (target and source) vertex.
- For any dimension vector γ we have the representation space

$$\mathbf{M}_\gamma = \bigoplus_{a \in Q_1} \mathrm{Hom}(\mathbb{C}^{\gamma(ta)}, \mathbb{C}^{\gamma(ha)})$$

  with action of the algebraic group $\mathbf{G}_\gamma = \prod_{i \in Q_0} \mathrm{GL}(\mathbb{C}^{\gamma(i)})$ by base-change at each vertex.
- For dimension vectors $\gamma_1, \gamma_2 \in \mathbb{N}^{Q_0}$ let χ denote the **Euler form**:

$$\chi(\gamma_1, \gamma_2) = \sum_{i \in Q_0} \gamma_1(i)\gamma_2(i) - \sum_{a \in Q_1} \gamma_1(ta)\gamma_2(ha).$$

- Let λ denote its opposite anti-symmetrization

$$\lambda(\gamma_1, \gamma_2) = \chi(\gamma_2, \gamma_1) - \chi(\gamma_1, \gamma_2).$$

Quantum algebra of Q

Let $q^{1/2}$ be an indeterminate and q denote its square. The **quantum algebra** $\mathbb{A}_Q$ of the quiver is the $\mathbb{Q}(q^{1/2})$-algebra

- generated by the symbols y_γ, one for each dimension vector γ;
- subject to the relation

$$y_{\gamma_1+\gamma_2} = -q^{-\frac{1}{2}\lambda(\gamma_1,\gamma_2)} y_{\gamma_1} y_{\gamma_2}.$$

Remark

The elements y_γ form a $\mathbb{Q}(q^{1/2})$-vector space basis.
The elements $y_{\mathbf{e}_i}$ form a set of algebraic generators.
(Where $\mathbf{e}_i$ is the dimension vector with 1 at the i-th vertex and zeroes elsewhere)

Observe we that the relation above also implies that

$$y_{\gamma_1} y_{\gamma_2} = q^{\lambda(\gamma_1,\gamma_2)} y_{\gamma_2} y_{\gamma_1}.$$

Example: A_2

Remark

Notice that

$$\lambda(\mathbf{e}_i, \mathbf{e}_j) = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

Consider the quiver $1 \longleftarrow 2$ and let $y_{\mathbf{e}_i} = y_i$. Then

$$y_2\, y_1 = q\, y_1\, y_2 \qquad\qquad y_{\mathbf{e}_1+\mathbf{e}_2} = -q^{-1/2} y_2\, y_1$$

Thus the pentagon identity says that

$$\mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(y_{\mathbf{e}_1+\mathbf{e}_2})\mathbb{E}(y_1).$$

The left-hand side gives an ordering of the **simple** roots of A_2;
the right-hand side gives an ordering for the **positive** roots of A_2.

Generalizing the pentagon identity

Definition 2

A **Dynkin quiver** is an orientation of a type A, D, or E Dynkin diagram. By Gabriel's Theorem, these are exactly the **representation finite** quivers, i.e. for which there are only finitely many $\mathbf{G}_\gamma$-orbits in $\mathbf{M}_\gamma$.

- For each $i \in Q_0$, there is a simple root α_i, which is identified with the dimension vector $\mathbf{e}_i$.
- Since each positive root $\beta = \sum_i d_i^\beta \alpha_i$ for some positive integers d_i^β, these are also identified with dimension vectors.

Theorem (Reineke (2010), Rimányi (2013))

For Dynkin quivers Q there exist orderings on the simple and positive roots such that

$$\overset{\curvearrowright}{\prod_{\alpha \text{ simple}}} \mathbb{E}(y_\alpha) = \overset{\curvearrowright}{\prod_{\beta \text{ positive}}} \mathbb{E}(y_\beta).$$

where "$\curvearrowright$" indicates the products are taken in the specified orders.

Donaldson–Thomas invariant

Theorem (Reineke (2010), Rimányi (2013))

For Dynkin quivers Q there exist orderings on the simple and positive roots such that

$$\prod_{\alpha \text{ simple}}^{\rightarrow} \mathbb{E}(y_\alpha) = \prod_{\beta \text{ positive}}^{\rightarrow} \mathbb{E}(y_\beta).$$

where the arrows indicate the products are taken in the specified orders.

The common value of both sides above is the **Donaldson–Thomas invariant** $\mathbb{E}_Q$ of the quiver Q.
It is known that the identity above is a consequence of the Pentagon Identity.

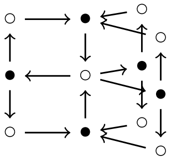
Square products

The **square product** of two Dynkin quivers is formed by the process below: (Here we do the example $A_3 \,\square\, D_4$)

- Assign **alternating** orientations to A_3 and D_4, e.g.

$$A_3\colon\ 1 \longleftarrow 2 \longrightarrow 3 \quad \text{and} \quad D_4\colon\ 1 \longleftarrow 2 \begin{array}{l} \nearrow 3 \\ \searrow 4 \end{array}$$

- make a grid of vertices $A_3 \times D_4$ (use matrix notation to name locations)
- reverse the arrows in the full sub-quivers $\{i\} \times D_4$ and $A_3 \times \{j\}$ whenever i is a sink in A_3 and j is a source in D_4.
- The result is the diagram of oriented squares:



- The "$\circ$" nodes are called **odd**, the "$\bullet$" nodes are called **even**.

Example: $A_2 \square A_2$

- Begin with $(1 \leftarrow 2) \times (1 \rightarrow 2)$.
- For $u, v \in Q_0$, let $y_{\mathbf{e}_u} = y_u$;
- let $y_{\mathbf{e}_u+\mathbf{e}_v} = y_{u+v}$.

$$\begin{array}{ccc} (11) & \longleftarrow & (12) \\ \downarrow & & \uparrow \\ (21) & \longrightarrow & (22) \end{array}$$

Theorem (Keller (2011,2013), A.–Rimányi (2016))

We have the following identity of quantum dilogarithm series

$$\begin{aligned} &\mathbb{E}(y_{(12)})\mathbb{E}(y_{(21)})\mathbb{E}(y_{(11)+(12)})\mathbb{E}(y_{(21)+(22)})\mathbb{E}(y_{(11)})\mathbb{E}(y_{(22)}) \\ &\qquad = \mathbb{E}(y_{(11)})\mathbb{E}(y_{(22)})\mathbb{E}(y_{(11)+(21)})\mathbb{E}(y_{(12)+(22)})\mathbb{E}(y_{(12)})\mathbb{E}(y_{(21)}). \end{aligned}$$

The common value of both sides is the Donaldson–Thomas invariant $\mathbb{E}_{Q,W}$ where W is the superpotential determined by traversing the oriented cycle once.

The left-hand side comes from an ordering on **horizontal positive roots**; the right-hand side comes from an ordering on **vertical positive roots**.

The general statement

Let $\Phi(A_N)$ denote the set of positive roots of type A_N;
let $\Delta(A_N)$ denote the set of simple roots (this is identified with $(A_N)_0$).

Theorem (A.–Rimányi (2016))

For the square product $A_n \,\square\, A_m$ we have the identity

$$\overrightarrow{\prod_{(i,\phi)\in\Delta(A_n)\times\Phi(A_m)}} \mathbb{E}(y_{(i,\phi)}) = \overrightarrow{\prod_{(\psi,j)\in\Phi(A_n)\times\Delta(A_m)}} \mathbb{E}(y_{(\psi,j)})$$

How to prove?

Theorem (A.–Rimányi (2016))

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- Method 1. Cluster theory and combinatorics
  - Find a **maximal green sequence** of quiver mutations
  - Keller (2011, 2013) describes how, from this, one can algorithmically write down the factors on each side
  - The result must be the DT-invariant $\mathbb{E}_{Q,W}$
- Method 2. Topology and geometry (our method)
  - For each γ, stratify $\mathbf{M}_\gamma$.
  - Use spectral sequence for stratification to relate Poincaré series for cohomology of each strata.

Stratify the representation space

- Recall that by Gabriel's theorem, a Dynkin quiver with dimension vector $\mathbf{d}$ has finitely many $\mathbf{G_d}$ orbits in $\mathbf{M_d}$.
- In fact, each orbit corresponds to a vector $(m_\beta)_{\beta \in \Phi}$ such that

$$\mathbf{d} = \textstyle\sum_\beta m_\beta \beta.$$

- Fix a dimension vector γ for $A_n \square A_m$ and form strata in $\mathbf{M}_\gamma$ as follows.
  - For each $i \in \Delta(A_n)$, fix a *Dynkin quiver orbit* along the corresponding row.
  - Allow complete freedom in the maps along vertical arrows of the quiver.
  - Call this a **horizontal stratum**.
  - There are finitely many of these.
  - Similarly define **vertical strata** by fixing orbits along columns corresponding to $j \in \Delta(A_m)$.

Example: $A_2 \square A_2$

Fix the dimension vector $\gamma = \left(\begin{smallmatrix}2 & 2\\ 1 & 1\end{smallmatrix}\right)$

η	2 0 0 0 0 1	1 1 1 0 0 1	2 0 0 1 1 0	1 1 1 1 1 0	0 2 2 0 0 1	0 2 2 1 1 0
codim$(\eta; \mathbf{M}_\gamma)$	0	1	1	2	4	5

Table: The six horizontal strata.

θ	1 1 1 1 0 0	1 2 1 0 0 1	2 1 0 1 1 0	2 2 0 0 1 1
codim$(\theta; \mathbf{M}_\gamma)$	0	2	2	4

Table: The four vertical strata.

Equivariant cohomology spectral sequence

Let $G \circlearrowright X$ and let $X = \bigcup_j \eta_j$ be a stratification by G-invariant subvarieties. Form

$$F_i = \bigcup_{\mathrm{codim}_{\mathbb{R}}(\eta_j) \leqslant i} \eta_j$$

and obtain a topological filtration

$$F_0 \subset F_1 \subset \cdots \subset F_{\dim_{\mathbb{R}}(X)} = X.$$

Apply the Borel construction for equivariant cohomology to obtain

$$B_G F_0 \subset B_G F_1 \subset \cdots \subset B_G X.$$

There is an associated spectral sequence in cohomology $E_\bullet^{p,q}$.

Remark

The application of this spectral sequence goes at least back to Atiyah & Bott (1983), to study Yang–Mills equations.

Rapid-decay cohomology from superpotential

Let X be a complex manifold/variety and $f : X \to \mathbb{C}$ a regular function. For $t \in \mathbb{R}$, set $S_t = \{z \in \mathbb{C} : \Re[z] < t\}$.

Definition 3

The **rapid-decay cohomology** $\mathrm{H}^*(X; f)$ is the limit as $t \to -\infty$ of the cohomology of the pair $\mathrm{H}^*(X, f^{-1}(S_t))$.
Fortunately, this stabilizes at some finite $t_0 \ll 0$. And...if X has a G-action, an equivariant version can be defined.

On $\mathbf{M}_\gamma$ we have a natural choice of regular function as follows.

- Assign the sum over oriented square paths p, $W = -\sum_p p$ as a superpotential on Q. ($W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$)
- Define a regular function $W_\gamma : \mathbf{M}_\gamma \to \mathbb{C}$ by

$$(f_a)_{a \in Q_1} \in \mathbf{M}_\gamma \longmapsto -\sum_p \mathrm{Tr}(f_p)$$

where f_p means the *composition* around the oriented square p.

The big idea

Theorem (A.–Rimányi)

The spectral sequence $E_\bullet^{ij}$ *(in rapid decay cohomology) converges to* $\mathrm{H}^*_{\mathbf{G}_\gamma}(\mathbf{M}_\gamma; W_\gamma)$ *and*

- *the spectral sequence degenerates at the* E_1 *page;*
- *taking the direct sum over all horizontal strata* η

$$E_1^{ij} = \bigoplus_{\operatorname{codim}_{\mathbb{R}}(\eta;\mathbf{M}_\gamma)=i} \mathrm{H}^j_{\mathbf{G}_\eta}(\eta; W_\gamma) = \bigoplus_{\operatorname{codim}_{\mathbb{R}}(\eta;\mathbf{M}_\gamma)=i} \mathrm{H}^{j-\mathbf{w}(\eta)}(B\mathbf{G}_\eta);$$

- *taking the direct sum over all vertical strata* θ

$$E_1^{ij} = \bigoplus_{\operatorname{codim}_{\mathbb{R}}(\theta;\mathbf{M}_\gamma)=i} \mathrm{H}^j_{\mathbf{G}_\theta}(\theta; W_\gamma) = \bigoplus_{\operatorname{codim}_{\mathbb{R}}(\theta;\mathbf{M}_\gamma)=i} \mathrm{H}^{j-\mathbf{w}(\theta)}(B\mathbf{G}_\theta).$$

$\mathbf{G}_\eta$ (resp. $\mathbf{G}_\theta$) is an "isotropy subgroup" for η (resp. θ).
Picture please...

Convergence of $E_\bullet^{p,q}$ for $A_2 \square A_2$ with $\gamma = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

horizontal strata								vertical strata					
0 / 2 2 / 1 1 / 0	0 / 2 2 / 0 0 / 1	—	1 / 1 1 / 1 1 / 0	2 / 0 0 / 1 1 / 0	1 / 1 1 / 0 0 / 1	2 / 0 0 / 0 0 / 1		1 1 / 1 1 / 0 0	—	1 2 / 1 0 / 0 1	2 1 / 0 1 / 1 0	—	2 2 / 0 0 / 1 1
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
188	80	0	210	50	56	9	160	56	0	130	130	0	188
108	50	0	126	34	35	6	87	35	0	80	80	0	108
58	30	0	70	22	20	4	43	20	0	46	46	0	58
28	16	0	35	13	10	2	18	10	0	24	24	0	28
12	8	0	15	7	4	1	6	4	0	11	11	0	12
4	3	0	5	3	1	↑	1	1	0	4	4	0	4
1	1	0	1	1	↑ w	w	0	↑ w	0	1	1	0	1
5	4	3	2	1	1	0		0	1	2	2	3	4

codimensions

Wrap-up

Recall for $A_2 \square A_2$ the identity

$$\mathbb{E}(y_{(12)})\mathbb{E}(y_{(21)})\mathbb{E}(y_{(11)+(12)})\mathbb{E}(y_{(21)+(22)})\mathbb{E}(y_{(11)})\mathbb{E}(y_{(22)}) \\ = \mathbb{E}(y_{(11)})\mathbb{E}(y_{(22)})\mathbb{E}(y_{(11)+(21)})\mathbb{E}(y_{(12)+(22)})\mathbb{E}(y_{(12)})\mathbb{E}(y_{(21)}).$$

Our theorem is that the identity above encodes the picture on the previous page *simultaneously* for *all* dimension vectors.

Other questions/projects:

- Find a combinatorial Rosetta stone between stratifications and maximal green sequences.
- Play the game above with different stratifications
- Complete the picture above for $A_n \square D_m$, $A_n \square E_m$, $D_n \square E_m$, etc.