Versal Deformation Rings of Modules over Brauer Tree Algebras

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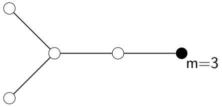
Brauer trees

Let k be an algebraically closed field of arbitrary characteristic.

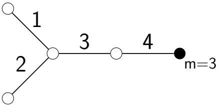
Definition

A Brauer tree T is a finite, connected, undirected, acyclic graph together with a counterclockwise ordering of the edges emanating from each vertex, along with a single exceptional vertex assigned with a positive integer value, called the multiplicity.



Brauer tree algebras

A finite dimensional k-algebra Λ is called a Brauer tree algebra if there is a related Brauer tree $T(\Lambda)$ which encodes all projective indecomposable Λ-modules.



For this Brauer tree, the related Brauer tree algebra Λ has the following projective indecomposable modules:

$$P_1\colon \begin{array}{c}1\\2\\3\\1\end{array} \qquad P_2\colon \begin{array}{c}2\\3\\1\\2\end{array} \qquad P_3\colon \begin{array}{ccc} & 3 & \\ 1 & & 4\\ 2 & & \\ & 3 & \end{array} \qquad P_4\colon \begin{array}{ccc} & 4 & \\ 3 & & 4\\ & & 4 \\ & 4 & \end{array}$$

Some properties of Brauer tree algebras

Brauer tree algebras are:

- special biserial
- symmetric, thus self injective
- finite representation type

For any finite group G with $char(k)|\#G$, if the group ring kG is of finite representation type, then kG is a direct sum of Brauer tree algebras and matrix algebras over k.

The versal deformation ring of a module

Let $\hat{\mathcal{C}}$ be the category of complete local Noetherian commutative k-algebras, and $R \in Ob(\hat{\mathcal{C}})$. Let V be a finitely generated Λ-module.

Definition

A lift of V over R is an $R \otimes_k \Lambda$-module M which is free as an R-module together with a Λ-module isomorphism $\phi : k \otimes_R M \to V$.

We say V has a versal deformation ring $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ if every isomorphism class of lifts of V over every $R \in Ob(\hat{\mathcal{C}})$ arises from a (not necessarily unique) k-algebra homomorphism from $R(\Lambda, V)$ to R. In addition, when $R = k[\epsilon]/(\epsilon^2)$, the k-algebra homomorphism is unique.

Properties

For a Brauer tree algebra Λ, we have that every indecomposable Λ-module V has a versal deformation ring.

The versal deformation ring for V is of the form $k[[t_1, \ldots, t_n]]/J$ for some ideal J, where the following properties hold:

- the dimension of $\mathrm{Ext}^1_\Lambda(V, V)$ is the minimal number of necessary variables t_i
- the dimension of $\mathrm{Ext}^2_\Lambda(V, V)$ is an upper bound on the minimal number of generators for the ideal J

Properties of $R(\Lambda, V)$ for Brauer tree algebras

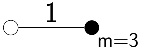
For a Brauer tree algebra Λ with e edges and exceptional vertex with multiplicity m, there are exactly me^2 non-projective indecomposable Λ-modules.

The stable Auslander-Reiten (AR) quiver is an e-tube, and because Λ is symmetric, the syzygy functor Ω induces an automorphism of the AR quiver, and for any non-projective indecomposable Λ-module V, $\Omega^2(V)$ is the AR-translate of V.

The versal deformation ring of V is uniquely determined by the length of the shortest path from V to the closest boundary on the stable Auslander-Reiten quiver of Λ.

Example

Consider the Brauer tree T with one edge and multiplicity 3:



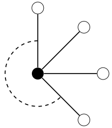
The related Brauer tree algebra Λ is $k[\alpha]/(\alpha^4)$, and the only indecomposable Λ-modules are uniserial of length $1 \leq i \leq 4$.

Consider the module $V = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}$, for which $\dim_k \mathrm{Ext}^1_\Lambda(V, V) = 2$. The versal deformation ring $R(\Lambda, V)$ for this module is $k[t_1, t_2]/(t_1^2 t_2 + t_2^2, t_1^3 + 2t_1 t_2)$.

A particular lift over $R(\Lambda, V)$ has α acting as the matrix $\left(\begin{smallmatrix} 0 & t_2 \\ 1 & t_1 \end{smallmatrix}\right)$, and the entries of α^4 determine the ideal for $R(\Lambda, V)$.

Results

Let Λ be the Brauer tree algebra related to the star Brauer tree with e edges and multiplicity m, and let U_i be the uniserial module with top 1 of length $1 \leq i \leq e$.



$$U_i: \begin{matrix} 1 \\ 2 \\ \vdots \\ i \end{matrix}$$

Define $V_0 = \begin{matrix} U_e \\ U_e \end{matrix}$, and $V_j = \begin{matrix} U_e \\ U_e \\ U_j \end{matrix}, 1 \leq j < e$.

These modules (given sufficiently large m) and their Ω-translates are all the Λ-modules with $\dim_k \mathrm{Ext}^1_\Lambda(V, V) = 2$.

Define $g_n = \sum_{i=1}^{\lceil \frac{n}{2} \rceil} \binom{n-i}{i-1} t_1^{n+1-2i} t_2^{i-1}$, $f_n = t_2 g_{n-1}$ and $J_n =< f_n, g_n >$.

Theorem

For $e = 1, m \geq 3$, we have:

$$R(\Lambda, V_0) \cong k[t_1, t_2]/J_{m+1}$$

For $e > 1, m = 4$, we have:

$$R(\Lambda, V_0) \cong k[t_1, t_2]/J_4$$

For $e > 1, m \geq 5$, we have:

$$R(\Lambda, V_j) \cong \begin{cases} k[t_1, t_2]/J_m & : j = 0, 1 \\ k[t_1, t_2]/J_{m-1} & : 2 \leq j < e \end{cases}$$

Thank you!