The b-functions of quiver semi-invariants

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Outline

1. b-functions and prehomogeneous spaces
2. Semi-invariants of quivers
3. b-functions via reflection functors
4. Rational singularities
We work over $\mathbb{C}$.

Let $V$ be an $n$-dimensional vector space.

Let $D$ the algebra of differential operators on $V$, i.e. the Weyl algebra

$$D = \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle.$$

$D[s] := D \otimes_{\mathbb{C}} \mathbb{C}[s]$. 

Theorem-Definition (J. Bernstein)

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-zero polynomial. Then there is a differential operator $P(s) \in D[s]$ and non-zero polynomial $b(s) \in \mathbb{C}[s]$ such that

$$P(s) \cdot f^{s+1}(x) = b(s) \cdot f^s(x)$$
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\[
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\]

- The functions \( b(s) \) satisfying such a relation form an ideal of \( \mathbb{C}[s] \), whose monic generator we denote by \( b_f(s) \). We call \( b_f(s) \) the \( b \)-function (or Bernstein-Sato polynomial) of \( f \).
Example

\[ f = x, \text{ then } b_f = (s + 1) \text{ by} \]

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Example

\( f = x^2 + y^3 \), then

\[
P(s) = \frac{1}{12} y \partial_x^2 \partial_y + \frac{1}{27} \partial_y^3 + s \frac{1}{4} \partial_x + \frac{3}{8} \partial_x^2
\]

\[
b_f(s) = (s + 1)(s + \frac{5}{6})(s + \frac{7}{6})
\]
Theorem (M. Kashiwara)

All roots of $b_f(s)$ are negative rational numbers.
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Note that $-1$ is always a root. One of the various applications of $b$-functions:

Theorem (M. Saito)

Assume $f$ is reduced. Then $Z(f) := f^{-1}(0)$ has rational singularities iff $-1$ is the largest root of $b_f(s)$ and has multiplicity 1.

We note that there is a more general result for reduced complete intersections.
Example

Take $X = (x_{ij})$ an $n \times n$ generic matrix of variables, and $\partial X$ is the matrix formed by the partial derivatives $\frac{\partial}{\partial x_{ij}}$.

Take $f = \det X$, then $b_f(s) = (s + 1) \cdots (s + n)$ by Cayley’s formula

$$\det \partial X \cdot (\det X)^{s+1} = (s + 1) \cdots (s + n) \cdot (\det X)^s$$
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Reason: $f \in \mathbb{C}[V]$ is a semi-invariant for a prehomogeneous vector space, i.e. there is an action of a reductive group $G$ on $V$ such that there is a dense orbit, and there is a character $\sigma : G \rightarrow \mathbb{C}^*$ s.t.

$$g \cdot f = \sigma(g) \cdot f$$
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Let $f^* \in \mathbb{C}[V^*]$ be the dual semi-invariant of weight $\sigma^{-1}$ (which we view as a differential operator).
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Let $f^* \in \mathbb{C}[V^*]$ be the dual semi-invariant of weight $\sigma^{-1}$ (which we view as a differential operator).

Then the following equation comes for free:

$$f^* \cdot f^{s+1} = b(s) \cdot f^s.$$ 

One can prove $b(s)$ is a polynomial with $\deg b(s) = \deg f$, and $b(s)$ is indeed the $b$-function of $f$ (i.e. its minimal).
Let $Q$ be a quiver without oriented cycles, $\beta$ a dimension vector, and consider the group

$$GL(\beta) := \prod_{x \in Q_0} GL(\beta_x)$$

acting on the representation space

$$Rep(Q, \beta) := \bigoplus_{a \in Q_1} Hom(\mathbb{C}^{\beta ta}, \mathbb{C}^{\beta ha}).$$

For any two representations $V$ and $W$, we have Ringel’s exact sequence:

$$0 \to \text{Hom}_Q(V, W) \xrightarrow{i} \bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) \to$$

$$\bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha)) \xrightarrow{p} \text{Ext}_Q(V, W) \to 0$$
Take dimension vectors $\alpha, \beta$, such that $\langle \alpha, \beta \rangle = 0$. For a representation $V$ with $\dim V = \alpha$, we define the semi-invariant $c^V \in \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle}$ by

$$c^V(W) := \det d^V_W$$

**Theorem (H. Derksen - J. Weyman, A. Schofield - M. Van den Bergh)**

The ring of semi-invariants $\text{SI}(Q, \beta)$ is spanned by the semi-invariants $c^V$, with $\langle \dim(V), \beta \rangle = 0$. 
• An orbit $O_W$ is dense iff $\text{Ext}_Q(W, W) = 0$. Then call $\beta = \dim W$ a prehomogeneous dimension vector, and $W$ a generic representation (or partial tilting module).
• The left perpendicular category $\perp W$ of a generic rep. is equivalent to the category of representations of a quiver without oriented cycles.

**Theorem (A. Schofield)**

Let $\beta$ be prehomogeneous and $W$ the generic representation, and take $V_1, \ldots, V_k$ the simple objects of the category $\perp W$. Then

$$\text{SI}(Q, \beta) = \mathbb{C}[c^{V_1}, \ldots, c^{V_k}].$$
Definition (Reflection Functors)

For \( x \in Q_0 \) sink (or source), we form a new quiver \( c_x Q \) by reversing all arrows ending in \( x \). Also define the map

\[
c_x : \mathbb{Z}^n \rightarrow \mathbb{Z}^n
\]

\[
c_x(\beta)_y = \begin{cases} 
\beta_y & \text{if } x \neq y, \\
-\beta_x + \sum_{\text{edges } x \rightarrow z} \beta_z & \text{if } x = y.
\end{cases}
\]

For an admissible ordering \( i_1, \ldots, i_n \) of sinks, let
\[
c = c_{i_n} \cdots c_{i_1} = -E^{-1}E^t
\]
be the Coxeter transformation, where \( E \) is the Euler matrix of \( Q \).
Theorem (V. Kac)

We have the isomorphisms of rings of semi-invariants:

\[
\begin{align*}
\text{SI}(Q, \beta) &\cong \text{SI}(c x Q, c x(\beta)), \quad \text{when } c x(\beta)_x > 0, \\
\text{SI}(Q, \beta) &\cong \text{SI}(c x Q, c x(\beta)) \otimes \mathbb{C}[\det \beta_x], \quad \text{when } c x(\beta)_x = 0, \\
\text{SI}(Q, \beta) &\cong \text{SI}(Q, \beta - \beta_x \epsilon_x), \quad \text{when } c x(\beta)_x < 0.
\end{align*}
\]

These isomorphisms respect weight spaces:

\[
\text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle} \cong \text{SI}(c x Q, c x(\beta))_{\langle c x(\alpha), \cdot \rangle}
\]
Theorem

Let $\beta$ be a prehomogeneous dimension vector, and $f \in \text{SI}(Q, \beta)_{\langle \alpha, \cdot \rangle}$ a semi-invariant. Then the $b$-function satisfies the formula

$$b_f(s) = b_{c(f)}(s) \prod_{x \in Q_0} \frac{[s]^{c(\alpha)_x}}{[s]^{c(\beta)_x}}$$

where $c$ is the Coxeter transformation. Here we use the notation

$$[s]^d_a := \prod_{i=1}^a \prod_{j=0}^{d-1} (ds + i + j).$$
Let $\beta$ be a prehomogeneous dimension vector, and $f \in SI(Q, \beta)_{\langle \alpha, \cdot \rangle}$ a semi-invariant. Then the b-function satisfies the formula

$$b_f(s) = b_{c(f)}(s) \prod_{x \in Q_0} \frac{[s]^{c(\alpha)_x}_{\beta_x}}{[s]^{c(\alpha)_x}_{c(\beta)_x}}$$

where $c$ is the Coxeter transformation. Here we use the notation

$$[s]_d^a := \prod_{i=1}^{a} \prod_{j=0}^{d-1} (ds + i + j).$$

Further, for $a \leq b$ we introduce the notation

$$[s]_{d,a,b}^d := \prod_{i=b-a+1}^{b} \prod_{j=0}^{d-1} (ds + i + j).$$
Example

\[ D_4 : \]
\[ 2 \]
\[ \downarrow \]
\[ 1 \rightarrow 4 \leftarrow 3 \]

Take \( c^V \in \text{SI}(Q, \beta) \), where \( V \) be the indecomposable of dimension \( \alpha = (1, 1, 1, 1) \). Then \( \langle \alpha, \beta \rangle = 0 \) gives \( \beta = (\beta_1, \beta_2, \beta_3, \beta_4) \) with \( \beta_1 + \beta_2 + \beta_3 = 2\beta_4 \).

\[ c^V = \det \begin{pmatrix} X & 0 & 0 & I_{\beta_4} \\ 0 & Y & 0 & I_{\beta_4} \\ 0 & 0 & Z & I_{\beta_4} \end{pmatrix} \]

\[ \downarrow \]
\[ 1 \rightarrow \beta_1 \leftarrow \beta_3 \rightarrow 1 \]
\[ \beta_2 \]

\[ 1 \rightarrow \beta_4 \leftarrow 1 \]
First, \( c_V \neq 0 \) iff \( \beta_i \leq \beta_4 \), for \( i = 1, 2, 3 \).

Using the inequalities, we can write \( b(s) \) as a polynomial

\[
b(s) = \left[ s^{\beta_4} \right] \cdot \left[ s^{\beta_4 - \beta_1} \right] \cdot \left[ s^{\beta_4 - \beta_2} \right] \cdot \left[ s^{\beta_4 - \beta_3} \right]
\]
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Using the inequalities, we can write $b(s)$ as a polynomial

$$b(s) = [s]_{\beta_4} \cdot [s]_{\beta_4 - \beta_1, \beta_2} \cdot [s]_{\beta_4 - \beta_2, \beta_3} \cdot [s]_{\beta_4 - \beta_3, \beta_1}$$
The \( b \)-function is

\[
b(s) = \begin{bmatrix} s \\ \beta_1, \beta_1 \end{bmatrix} \beta_3, \beta_6 - \beta_5 \begin{bmatrix} s \\ 2 \beta_6 - \beta_3, \beta_1 \end{bmatrix} \beta_4, \begin{bmatrix} s \\ 2 \beta_6 - \beta_1, \beta_3 \end{bmatrix} \beta_5.
\]

with \( \beta_1 + \beta_2 + \beta_3 + \beta_4 + 2\beta_5 = 3\beta_6 \) and inequalities

\[
\begin{align*}
\beta_6 & \leq \beta_4 + \beta_5, \\
\beta_2 + \beta_5, \\
\beta_2 + \beta_4, \\
\beta_1 + \beta_3 + \beta_5
\end{align*}
\]

\[
\beta_6 \geq \beta_1 + \beta_5, \beta_3 + \beta_5.
\]
E₆:

\[
\begin{array}{c}
\beta_1 \\ \downarrow \\
\beta_2 \\ \downarrow \\
\beta_6 \\ \leftarrow \\
\beta_4 \\ \leftarrow \\
\beta_3
\end{array}
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\beta_6 \geq \beta_1 + \beta_5, \beta_3 + \beta_5.
\]

The \( b \)-function is

\[
b(s) = [s]_{\beta_1 + \beta_4 + \beta_5 - \beta_6} [s]_{\beta_3 + \beta_3 + \beta_5 - \beta_6} [s]_{\beta_6 - \beta_5 - \beta_3} [s]_{\beta_3} [s]_{\beta_6 - \beta_5 - \beta_1} [s]_{\beta_4} \cdot
\]

\[
\cdot [s]_{\beta_1 + \beta_3 + \beta_5 - \beta_6} [s]_{\beta_6 - \beta_5 - \beta_3} [s]_{\beta_1 + \beta_3 + \beta_5 - \beta_6} [s]_{\beta_3} [s]_{\beta_6 - \beta_5} \cdot
\]

\[
\cdot [s]_{\beta_6 - \beta_3 - \beta_1 + \beta_4 + \beta_5 - \beta_6} [s]_{\beta_5}
\]
**Theorem**

Let $Q$ be a Dynkin or Euclidean quiver, and $\beta$ a prehomogeneous dimension vector. Then all semi-invariants in $SI(Q, \beta)$ are reducible by reflections to constant functions (hence we can compute their $b$-functions).
Theorem

Let $Q$ be a Dynkin or Euclidean quiver, and $\beta$ a prehomogeneous dimension vector. Then all semi-invariants in $\text{SI}(Q, \beta)$ are reducible by reflections to constant functions (hence we can compute their $b$-functions).

Lemma

Let $Q$ be a Dynkin quiver, $V$ a sincere generic representation. Then any simple object $S$ in the perpendicular category $\perp V$ (or $V \perp$) satisfies

$$\dim S \leq \dim V$$
Corollary

Let $Q$ be a Dynkin quiver, then any codimension 1 orbit closure in $\text{Rep}(Q, \beta)$ (i.e. zero set of an irreducible semi-invariant) has rational singularities (in particular, is normal). This is true for Euclidean quivers if the prehomogeneous $\beta$ is large enough.
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Let $Q$ be a Dynkin quiver, then any codimension 1 orbit closure in $\text{Rep}(Q, \beta)$ (i.e. zero set of an irreducible semi-invariant) has rational singularities (in particular, is normal). This is true for Euclidean quivers if the prehomogeneous $\beta$ is large enough.

Example

Let $Q$ be the Kronecker quiver $1 \rightarrow 2$

$\beta = (k \cdot n, k \cdot (n + 1)), \alpha = (n + 1, n + 2)$

where $k, n$ are arbitrary positive integers. The $b$-function is

$$b(s) = \prod_{i=1}^{n} [s]_{2k,(i+1)k}^i$$

The zero-set has rational singularities iff $k > 1$. 
Some things to do:

- Extend arguments of rational singularities to zero-sets of more semi-invariants, in particular the null-cone (partially done)
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- Investigate the combinatorics behind $b(s)$, in particular the reduction to a polynomial of $b(s)$
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- Extend arguments of rational singularities to zerosets of more semi-invariants, in particular the null-cone (partially done)
- Investigate the combinatorics behind $b(s)$, in particular the reduction to a polynomial of $b(s)$
- Find other techniques that work in wild cases, and other reductive groups (the are already quite a few)
Thank you!