A proof of the independence of the surface integral with respect to parameterization

The surface integral should be independent of the way in which we parameterize the surface: suppose we have a parameterization of a surface Σ by $\mathbf{r} = \mathbf{r}(u, v)$ where (u, v) lie in a region U. Suppose that we have another parameterization given by $u = u(w, z)$ and $v = v(w, z)$ where (w, z) lie in a region W.
Theorem.

$$\begin{aligned}\int\int_\Sigma f(\mathbf{r})\,dS &= \int\int_U f(\mathbf{r}(u,v))\,\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\|\,du\,dv \\ &= \int\int_W f(\mathbf{r}(u(w,z),\, v(w,z))\,\|\frac{\partial \mathbf{r}}{\partial w}\times\frac{\partial \mathbf{r}}{\partial z}\|\,dw\,dz.\end{aligned}$$

Proof. We need to compute $\partial\mathbf{r}/\partial w \times \partial\mathbf{r}/\partial z$ in terms of $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$:

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial w} &= \frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial v}{\partial w}, \\ \frac{\partial \mathbf{r}}{\partial z} &= \frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial v}{\partial z}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial w}\times\frac{\partial \mathbf{r}}{\partial z} &= \left(\frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial v}{\partial w}\right)\times\left(\frac{\partial \mathbf{r}}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \mathbf{r}}{\partial v}\frac{\partial v}{\partial z}\right) \\ &= \left(\frac{\partial u}{\partial w}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial w}\right)\left(\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right) \\ &= \frac{\partial(u,v)}{\partial(w,z)}\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\end{aligned}$$

so

$$\begin{aligned}&\int\int_W f(\mathbf{r}(u(w,z),\, v(w,z))\,\|\frac{\partial \mathbf{r}}{\partial w}\times\frac{\partial \mathbf{r}}{\partial z}\|\,dw\,dz \\ &\quad= \int\int_W f(\mathbf{r}(u(x,z),\, v(w,z))\,\left|\frac{\partial(u,v)}{\partial(w,z)}\right|\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|\,dw\,dz \\ &\quad= \int\int_U f(\mathbf{r}(u,v))\,\left\|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right\|\,du\,dv\end{aligned}$$

as we wanted.

A proof of Stoke's theorem

Theorem. *Suppose that Σ is an oriented surface which is parameterized by $\mathbf{r} = \mathbf{r}(u,v)$ for (u,v) in a region U in the plane. Suppose that the normal vector $\mathbf{n}$ for Σ at $\mathbf{r}(u,v)$ is in the direction of $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v$. Then*

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int\int_\Sigma curl\, \mathbf{F}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r})\, dS$$

where C is the boundary of Σ (oriented consistently with Σ).

Proof. Suppose that the boundary C of Σ is the image of the boundary C' of U in the plane. Then

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_{C'} \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv\right) \\
&= \int\int_U \left(\frac{\partial}{\partial u}\left(\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\partial \mathbf{r}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{F}(\mathbf{r}(u,v)) \cdot \frac{\partial \mathbf{r}}{\partial u}\right)\right) du\, dv \\
&= \int\int_U \left(\frac{\partial}{\partial u}\left(\mathbf{F}(\mathbf{r}(u,v))\right) \cdot \frac{\partial \mathbf{r}}{\partial v} - \frac{\partial}{\partial v}\left(\mathbf{F}(\mathbf{r}(u,v))\right) \cdot \frac{\partial \mathbf{r}}{\partial u}\right) du\, dv
\end{aligned}$$

Note that

$$\frac{\partial}{\partial u}\left(\mathbf{F}(\mathbf{r}(u,v))\right) = \frac{\partial \mathbf{F}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \mathbf{F}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \mathbf{F}}{\partial z}\frac{\partial z}{\partial u}$$

Then

$$\int\int_U \left(\frac{\partial}{\partial u}\left(\mathbf{F}(\mathbf{r}(u,v))\right) \cdot \frac{\partial \mathbf{r}}{\partial v} - \frac{\partial}{\partial v}\left(\mathbf{F}(\mathbf{r}(u,v))\right) \cdot \frac{\partial \mathbf{r}}{\partial u}\right) du\, dv =$$

$$\int\int_U \left(\frac{\partial \mathbf{F}}{\partial x} \cdot \left(\frac{\partial x}{\partial u}\frac{\partial \mathbf{r}}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial \mathbf{r}}{\partial u}\right) + \cdots + \frac{\partial \mathbf{F}}{\partial z} \cdot \left(\frac{\partial z}{\partial u}\frac{\partial \mathbf{r}}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial \mathbf{r}}{\partial u}\right)\right) du\, dv$$

Using $\mathbf{F}(\mathbf{r}) = M(\mathbf{r})\mathbf{i} + N(\mathbf{r})\mathbf{j} + P(\mathbf{r})\mathbf{k}$, the first term of the above integrand is

$$\begin{aligned}
\frac{\partial M}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}\right) + \frac{\partial N}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) &+ \frac{\partial P}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right) \\
&= \mathbf{k}\frac{\partial N}{\partial x} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) - \mathbf{j}\frac{\partial P}{\partial x} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \\
&= \left(\mathbf{k}\frac{\partial N}{\partial x} - \mathbf{j}\frac{\partial P}{\partial x}\right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)
\end{aligned}$$

Note that $\mathbf{k}(\partial N/\partial x) - \mathbf{j}(\partial P/\partial x)$ is the part of curl $\mathbf{F}$ that involves derivatives with respect to x. Adding the the other terms in the integrand, we get

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int\int_U \text{curl } \mathbf{F}(\mathbf{r}(u,v)) \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du\, dv \\ &= \int\int_\Sigma \text{curl}\, \mathbf{F}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r})\, dS \end{aligned}$$