THE NUMERICAL SOLUTION OF LAPLACE'S EQUATION IN THREE DIMENSIONS

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Abstract. We consider the Dirichlet problem for Laplace's equation, on a simply-connected three-dimensional region with a smooth boundary. This problem is easily converted to the solution of a Fredholm integral equation of the second kind, based on representing the harmonic solution as a double layer potential function. We solve this integral equation formulation by using Galerkin's method, with spherical harmonics as the basis functions. This approach leads to small linear systems, and once the Galerkin coefficients for the region have been calculated, the computation time is small for any particular boundary function. The major disadvantage of the method is the calculation of the Galerkin coefficients, each of which is a four-fold integral with a singular integrand. Theoretical and computational details of the method are presented.

1. Introduction. A numerical method will be described and analyzed for the solution of the interior Dirichlet problem for Laplace's equation in three dimensions. The problem is first reformulated using a standard integral equation, and this is then solved using Galerkin's method.

Let $D$ be an open bounded simply-connected region in $\mathbb{R}^3$, and assume its boundary $S$ is sufficiently smooth. The degree of smoothness will be discussed in §2, but it must be a Lyapunov surface [9, p. 1]. The interior Dirichlet problem is

$$ \Delta u(A) = 0, \quad A = (x, y, z) \in D, $$

$$ u(P) = f(P), \quad P \in S, $$

with $f$ a given function.

The solution $u$ can be represented as a double layer potential,

$$ u(A) = \int_S \rho(Q) \frac{\partial}{\partial \nu_Q} \left( \frac{1}{r_{QA}} \right) d\sigma(A), \quad A \in D, $$

in which $\rho(Q)$ is called a double layer density function. In the integral $r_{QA} = |r_{QA}|$, and $r_{QA}$ is the vector from $Q$ to $A$. The symbol $\partial / \partial \nu_Q$ denotes the normal derivative at the point $Q$, in the direction of $\nu_Q$, the inner normal to $S$ at $Q$.

By letting $A$ tend to a point $P \in S$, we obtain an integral equation for the unknown function $\rho(Q)$:

$$ 2\pi \rho(P) + \int_S \rho(Q) \frac{\partial}{\partial \nu_Q} \left( \frac{1}{r_{QP}} \right) d\sigma(Q) = f(P), \quad P \in S. $$

The use of (1.2) and (1.3) is a well-known approach to the existence theory for (1.1); for example, see [7, p. 334], [9], [10], [13, Chap. 12]. The properties of the solution $\rho$ and of the integral operator in (1.3) will be reviewed in §2. But we note that the kernel of the integral operator has a singularity of order $1/r_{OP}$.

The equations (1.2) and (1.3) have been used as a basis for the numerical solution of (1.1); for example, see [10], [19] and the references contained therein. But most of this work has been of the finite element type. The surface $S$ is approximated by triangulation, and the approximate surface is usually piecewise linear. The solution $\rho$ is

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approximated by a constant or linear function on each triangular element of the surface. This numerical procedure is very general and quite flexible, but it has the disadvantage of leading to large linear systems of equations. It is also slowly convergent in most situations.

We will solve (1.3) using Galerkin’s method, and the basis functions for the method will be the spherical harmonics on the unit sphere. When both the surface \( S \) and the boundary function \( f \) are sufficiently smooth, this numerical method will lead to quite small linear systems as well as a rapidly convergent sequence of approximants to \( \rho \) and \( u \). The main disadvantage of the method is in the calculation of the Galerkin coefficients; this is discussed in § 5.

The paper begins with a discussion of (1.3) and the harmonic function \( u \) of (1.2). This will include results on the solvability of (1.3) and on the smoothness of the function \( \rho \). The spherical harmonics are defined in § 3, and convergence results on their use in the approximation of functions are given. The Galerkin method is defined in § 4, and convergence rates are derived. The practical implementation of the numerical method is covered in § 5; numerical examples are given in § 6.

2. The double layer potential. A complete discussion of the solution of potential theory problems on regions with a smooth boundary and using integral equation techniques is given in Günter [9]. We will summarize the results needed here.

The surface \( S \) will be a Lyapunov surface, with Hölder exponent \( \lambda \), \( 0 < \lambda \leq 1 \), for the first derivatives of the local surface representations. This is denoted by \( S \in L_{1,\lambda} \). If the \( k \)th order derivatives of the surface representations are Hölder continuous with exponent \( \lambda \), denote it by \( S \in L_{k,\lambda} \). Finally, assume that the region \( D \) enclosed by \( S \) is simply connected.

The standard Banach spaces for work on \( S \) are \( L^2(S) \) and \( C(S) \), with the usual inner product norm \( \| \cdot \|_2 \) and uniform norm \( \| \cdot \|_\infty \), respectively. If a function \( f \) is \( k \) times continuously differentiable on \( S \) and if the \( k \)th order derivatives are Hölder continuous with exponent \( \lambda \), we say \( f \in H_{k,\lambda}(S) \).

Let the integral operator in (1.1) be denoted by \( \mathcal{K} \). Then (1.3) is written as

\[
(2.1) \quad (2\pi + \mathcal{K})\rho = f.
\]

From [5, p. 359], \( \mathcal{K} \) is a compact operator on \( L^2(S) \) to \( L^2(S) \), and from results in [9, p. 49], it is straightforward to show \( \mathcal{K} \) is compact on \( C(S) \) to \( C(S) \). The Fredholm alternative theorem leads directly to a proof of the unique solvability of (2.1) for every boundary function \( f \), with respect to both \( L^2(S) \) and \( C(S) \). In addition, \( (2\pi + \mathcal{K})^{-1} \) is a bounded operator with respect to both of these spaces.

Smoothness of the double layer. From [9, p. 49],

\[
(2.2) \quad \rho \in C(S), \ S \in L_{1,\lambda} \quad \text{implies} \quad \mathcal{K}\rho \in \begin{cases} H_{0,\lambda}(S), & \lambda < 1, \\ H_{0,\lambda}(S), & \lambda = 1, \end{cases}
\]

with \( 0 < \lambda' < 1 \) arbitrary. Since

\[
\rho = \frac{1}{2\pi} (f - \mathcal{K}\rho), \quad \rho \in S,
\]

we have that

\[
(2.3) \quad f \in H_{0,\lambda}(S), \ S \in L_{1,\lambda} \quad \text{implies} \quad \rho \in H_{0,\lambda}(S),
\]

with \( \lambda' \) chosen as in (2.2).
If \( f \) and \( S \) have greater smoothness, then so does \( \rho \). From [9, p. 312],

\[
\rho \in H_{k,\lambda}(S), S \in L_{k+2,\lambda} \quad \text{implies} \quad \mathcal{K} \rho \in H_{k+1,\lambda}(S),
\]

with \( 0 < \lambda' < \lambda \) arbitrary, \( k \geq 0 \). By simple induction,

\[
f \in H_{k,\lambda}(S), S \in L_{k+1,\lambda} \quad \text{implies} \quad \rho \in H_{k,\lambda}(S).
\]

For results on the differentiability of \( u \) on \( \bar{D} = D \cup S \), see [9, p. 102].

3. Spherical harmonics. Let \( U \) denote the unit sphere in \( \mathbb{R}^3 \),

\[
U = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}.
\]

The spherical harmonics can be defined in a number of ways, and a very extensive discussion of the classical theory is given in MacRobert [12]. If the homogeneous harmonic polynomials of degree \( n \) in \( \mathbb{R}^3 \) are restricted to \( U \), we obtain the spherical harmonics of degree \( n \). If any other polynomial is restricted to \( U \), then the restriction will be called a spherical polynomial. It can be proven that every spherical polynomial of degree \( \leq n \) can be written as a linear combination of spherical harmonics of degree \( \leq n \); see [12, p. 128].

There is a standard basis for the spherical harmonics, and it is orthogonal in \( L^2(U) \). Let \( P_n(u) \) and \( P_n^m(u) \) denote the Legendre polynomials and associated Legendre functions on \([-1, 1] \), \( n \geq 0 \), \( 1 \leq m \leq n \); see [1, Chap. 8] or [12, Chaps. 5, 7]. For \((x, y, z) \in U\), let

\[
(x, y, z) = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)
\]

for \( 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi \). If \( \rho \) is defined on \( U \), then \( \rho(\phi, \theta) \) and \( \rho(x, y, z) \) will be used interchangeably.

If \( \rho \) is a spherical polynomial of degree \( N \), then

\[
\rho(\phi, \theta) = \sum_{n=0}^{N} \left\{ A_n P_n(\cos \theta) + \sum_{m=1}^{n} [A_m^m \cos (m\phi) + B_m^m \sin (m\phi)] P_n^m(\cos \theta) \right\}.
\]

The basis functions

\[
P_n(\cos \theta), \quad P_n^m(\cos \theta) \cos (m\phi), \quad P_n^m(\cos \theta) \sin (m\phi), \quad 1 \leq m \leq n
\]

are spherical harmonics of degree \( n \). For \( 0 \leq n \leq N \), the total number of basis functions is \( d(N) = (N + 1)^2 \).

Using the orthogonality of the spherical harmonics of (3.2), we have

\[
A_n = \frac{2n+1}{4\pi} \int_{0}^{2\pi} \int_{0}^{\pi} \rho(\phi, \theta) P_n(\cos \theta) \sin \theta \, d\theta \, d\phi,
\]

\[
\left[ A_m^m \right] = \frac{2n+1}{2\pi} \frac{(n-m)!}{(n+m)!} \int_{0}^{2\pi} \int_{0}^{\pi} \rho(\phi, \theta) \cos (m\phi) P_n^m(\cos \theta) \sin \theta \, d\theta \, d\phi.
\]

The functions \( P_n^m \) and \( P_n \) can be evaluated very efficiently with recursion relations; see [1, p. 334]. The same is also, of course, true for \( \sin (m\phi) \) and \( \cos (m\phi) \).

Approximation by spherical polynomials. From Gronwall [8] and Ragozin [14], we have the following result for approximation of continuous functions on \( U \). If \( \rho \in H_{k,\lambda}(U) \), then there is a sequence of spherical polynomials \( T_N \) of degree \( \leq N \) for which

\[
\| \rho - T_N \|_{\infty} \leq \frac{K_i}{N^{l+\lambda}}, \quad N \geq 1.
\]
The spherical polynomials are dense in both $C(U)$ and $L^2(U)$. The expansion of $\rho \in L^2(U)$ is called the Laplace series; it is given by (3.1), (3.3) with $N = \infty$. Let $\mathcal{P}_N\rho$ denote the partial sum of the Laplace series of $\rho$, restricted to terms of degree $\leq N$. On $L^2(U)$, $\mathcal{P}_N$ is orthogonal, and thus $\|\mathcal{P}_N\| = 1$. On $C(U)$, it was shown by Gronwall [8, § 2] that

$$\|\mathcal{P}_N\| = \left(\sqrt{\frac{8}{\pi}} + \delta_N\right)\sqrt{N},$$

with $\delta_N \to 0$. Using this, it is easy to show that

$$\rho \in H_{l,\lambda}(U) \quad \text{implies} \quad \|\rho - \mathcal{P}_N\rho\|_\infty \leq \frac{K'_1}{N^{l+\lambda-1/2}}.$$  

The potential problem on the sphere. The spherical harmonics were designed for the analysis of three-dimensional potential theory problems, particularly on the sphere. One of the first interesting results we note is that spherical harmonics furnish the eigenfunctions for the operator $\mathcal{H}$ of (1.3) in the case $S = U$. In particular, we then have

$$\mathcal{H} h = \frac{2\pi}{2n+1} h, \quad n \geq 0,$$

for any spherical harmonic $h$ of degree $n$. Since there are $2n + 1$ linearly independent functions $h$ of degree $n$, of which (3.2) is a basis, we have that $\lambda = 2\pi/(2n+1)$ is an eigenvalue of multiplicity $2n+1$. This furnishes an interesting example of the integral equation eigenvalue problem.

We can use (3.7) to solve (1.3) when $S = U$. Write $\rho$ in its Laplace expansion, with unknown coefficients. Substituting into (1.3), expanding $f$ in its Laplace expansion, and matching coefficients, we are led to the solution $\rho$ of (1.3):

$$\rho(\phi, \theta) = \sum_{n=0}^{\infty} \left\{ A_n P_n(\cos \theta) + \sum_{m=1}^{n} \left[ A_n^m \cos (m\phi) + B_n^m \sin (m\phi) \right] P_n^m(\cos \theta) \right\},$$

$$A_n = \frac{2n+1}{4\pi(n+1)} \alpha_n, \quad A_n^m = \frac{2n+1}{4\pi(n+1)} \alpha_n^m, \quad B_n^m = \frac{2n+1}{4\pi(n+1)} \beta_n^m,$$

where $\{\alpha_n, \alpha_n^m, \beta_n^m\}$ are the Laplace expansion coefficients of $f$. The corresponding solution of (1.1) is given by

$$u(A) = \sum_{n=0}^{\infty} \frac{4\pi(n+1)}{2n+1} r^n \left\{ A_n P_n(\cos \theta) + \sum_{m=1}^{n} \left[ A_n^m \cos (m\phi) + B_n^m \sin (m\phi) \right] P_n^m(\cos \theta) \right\},$$

where $A = (r \cos \phi \sin \theta, r \sin \phi \sin \theta, r \cos \theta)$ in spherical coordinates.

4. The Galerkin method. We change the variable of integration in (1.3), converting it to a new integral equation defined on $U$. The Galerkin method is applied to this new equation, using spherical polynomials to define the approximating subspaces.

We assume there is a mapping $\mathcal{M} : U \overset{1-1}{\longrightarrow} S$ for which the following property is satisfied:

$$f \in H_{k,\lambda}(S), \quad S \in L_{k+1,\lambda} \quad \text{implies} \quad \mathcal{f} \in H_{k,\lambda}(U),$$

where

$$\mathcal{f}(Q) = f(\mathcal{M}(Q)), \quad Q \in U.$$
All of our numerical examples have been for regions $D$ starlike with respect to the origin, but the numerical method is not restricted to such regions. For starlike regions, we assume that a general point, $P = \mathcal{M}(Q)$, of $S$ is given by

$$P = R(Q) \cdot (A\xi, B\eta, C\zeta), \quad Q = (\xi, \eta, \zeta) \in U.$$  

The constants $A, B, C$ are positive, and the function $R$ is a continuous positive function on $U$. If $R \in H_{k+1,\lambda}(U)$, then (4.1) is satisfied.

Change the variable of integration on (1.3), to obtain the new equation over $U$,

$$(2\pi + \hat{R})\hat{\rho} = \hat{f}, \quad \hat{f} \in C(U).$$

The notation $\hat{\cdot}$ will denote the change of variable from $S$ to $U$, as in (4.2). The operator $(2\pi + \hat{R})^{-1}$ exists and is bounded on $C(U)$ and $L^2(U)$.

The theoretical framework for the Galerkin method is that given in [3, p. 62]. Let $\mathcal{X} = L^2(U)$, and let the approximating subspace of spherical polynomials of degree $\leq N$ be denoted by $\mathcal{X}_N$. The dimension of $\mathcal{X}_N$ is $d_N = (N + 1)^2$, and we will let $\{h_1, \ldots, h_d\}$ denote the basis of spherical harmonics given in (3.2).

Galerkin’s method for solving (4.4) is given by

$$(2\pi + \mathcal{P}_N \hat{R})\hat{\rho}_N = \mathcal{P}_N \hat{f}.$$\n
The solution is given by

$$\hat{\rho}_N = \sum_{i=1}^{d} \alpha_i h_i,$$

$$2\pi \alpha_i(h_i, h_i) + \sum_{j=1}^{d} \alpha_j (h_i, \mathcal{H} h_j) = (y, h_i), \quad i = 1, \ldots, d.$$\n
A convergence theorem follows immediately from [3, Thm. 2, p. 51] and the results in §3.

**Theorem 1.** Assume that $f \in H_{k,\lambda}(S)$, $S \in L_{k+1,\lambda}$, and that the mapping $\mathcal{M}$ satisfies (4.1) for some $k \geq 0$. Then for all sufficiently large $N$, the inverses $(2\pi + \mathcal{P}_N \hat{R})^{-1}$ exist and are uniformly bounded, and

$$\|\rho - \rho_N\|_2 \leq \frac{c}{N^{k+\lambda}},$$

where $0 < \lambda' < \lambda$ is arbitrary. The constant $c$ depends on $k$, $\rho$, and $\lambda'$.

Convergence in $C(U)$. To prove uniform convergence of $\hat{\rho}_N$ to $\hat{\rho}$ is slightly more difficult. The main problem is that there are $\hat{\rho}$ in $C(U)$ for which $\mathcal{P}_N \hat{\rho}$ does not converge to $\hat{\rho}$; convergence for all $\hat{\rho}$ would imply uniform boundedness of $\|\mathcal{P}_N\|$, contradicting (3.5).

**Theorem 2.** Assume that $S \in L_{1,\lambda}$ with $\lambda > \frac{1}{2}$, and that $\mathcal{M}$ satisfies (4.1) with $k = 0$. Then considering $\hat{\mathcal{X}}$ as an operator on $C(U)$,

$$\|\hat{\mathcal{X}} - \mathcal{P}_N \hat{\mathcal{X}}\| \to 0 \quad \text{as} \quad N \to \infty.$$\n
This implies the existence and uniform boundedness on $C(U)$ of $(2\pi + \mathcal{P}_N \hat{R})^{-1}$ for all sufficiently large $N$.

If $(2\pi + \hat{R})\hat{\rho} = \hat{f}$ and $(2\pi + \mathcal{P}_N \hat{R})\hat{\rho}_N = \mathcal{P}_N \hat{f}$, and if $f \in H_{0,\lambda}(S)$, then $\hat{\rho}_N$ converges uniformly to $\hat{\rho}$. Moreover, if $S \in L_{k+1,\lambda}$ and $f \in H_{k,\lambda}(S)$, $k + \lambda > 1/2$, then

$$\|\rho - \rho_N\|_\infty \leq \frac{c}{N^{k+\lambda-1/2}},$$

with $0 < \lambda' < \lambda$ arbitrary. The constant $c$ depends on $f$, $k$, and $\lambda'$. 

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Proof. To prove (4.8), we must show $\mathcal{P}_N \to I$ uniformly on the set

$$\mathcal{F} = \{ \hat{\mathcal{H}} \hat{\rho} : \| \hat{\rho} \|_\infty \leq 1, \hat{\rho} \in C(U) \}.$$

Looking at $\hat{\mathcal{H}} \hat{\rho}$ on $U$ is equivalent to looking at $\mathcal{H} \rho$ with $\rho \in C(S)$ and $\| \rho \|_\infty \leq 1$. Pick $\frac{1}{2} < \lambda' < \lambda$. It is shown in [9, Thm. 1, p. 49] that the set of all such $\mathcal{H} \rho$ have a uniform Hölder constant.

For each $\psi \in \mathcal{F}$, let $T_N(\psi)$ be the spherical polynomial approximation of degree $\leq N$ which is referred to in (3.4) with $l = 0$. Then

$$\| \psi - T_N(\psi) \|_\infty \leq \frac{K_0}{N^{1/2}}, \quad N \geq 1, \quad \psi \in \mathcal{F}. \quad (4.10)$$

From the derivation in [8], the constant $K_0$ is a simple multiple of the Hölder constant of $\psi$, and is otherwise independent of $\psi$. Since there is a uniform Hölder constant for elements of $\mathcal{F}$, this shows the convergence of $T_N(\psi)$ to $\psi$ is uniform on $\mathcal{F}$.

Using $\mathcal{P}_N T_N(\psi) = T_N(\psi)$, we have

$$\| \psi - \mathcal{P}_N \psi \|_\infty = \| [\psi - T_N(\psi)] + \mathcal{P}_N [T_N(\psi) - \psi] \|_\infty \leq (1 + \| \mathcal{P}_N \|) \| \psi - T_N(\psi) \|_\infty.$$

Combined with (4.10),

$$\| \psi - \mathcal{P}_N \psi \|_\infty \leq \frac{c}{N^{1/2}}, \quad N \geq 1, \quad \psi \in \mathcal{F}, \quad (4.11)$$

for some $c$ independent of $\psi$. Thus

$$\| \hat{\mathcal{H}} - \mathcal{P}_N \hat{\mathcal{H}} \| \leq \frac{c}{N^{1/2}}. \quad (4.12)$$

The remainder of the proof is straightforward from (3.4) and [3, Thm. 2, p. 51].

From the identity

$$(2\pi + \mathcal{P}_N \hat{\mathcal{H}})(\hat{\rho} - \hat{\rho}_N) = 2\pi (\hat{\rho} - \mathcal{P}_N \hat{\rho}),$$

it is easy to see that $\rho_N \to \rho$ if and only if $\mathcal{P}_N \rho \to \rho$. Thus the existence of $\hat{\rho} \in C(U)$ for which $\mathcal{P}_N \hat{\rho}$ does not converge to $\hat{\rho}$ will also furnish an example in which $\hat{\rho}_N$ does not converge uniformly to $\hat{\rho}$. But Theorem 1 still assures us that we will have convergence in $L^2(U)$ of $\hat{\rho}_N$ to $\hat{\rho}$, for any $\hat{\rho} \in C(U)$.

The approximate harmonic function. For $\rho_N$, define

$$u_N(A) = \int_S \rho_N(Q) \frac{\partial}{\partial v_Q} \left( \frac{1}{r_{QA}} \right) d\sigma, \quad A \in D. \quad (4.13)$$

Then $u_N(A)$ is harmonic in $D$. Using the maximum principle, we have

$$\max_{A \in D} | u(A) - u_N(A) | = \max_{P \in S} | u(P) - u_N(P) |,$$

with respect to the true solution $u$ of (1.2). Let $A \to P \in S$ in (4.2), obtaining

$$u_N(P) = 2\pi \rho_N(P) + \mathcal{K} \rho_N(P), \quad P \in S. \quad (4.14)$$

Subtracting from (1.3) gives

$$| u(P) - u_N(P) | \leq (2\pi + \| \mathcal{K} \|) \| \rho - \rho_N \|_\infty, \quad P \in S.$$

For convex regions $D$, the kernel of $\mathcal{K}$ is positive, and then $\| \mathcal{K} \| = 2\pi$. In any case, the convergence results for $\rho_N$ lead directly to similar uniform bounds for $u_N(A)$. 

\[ \hat{\mathcal{H}} \hat{\rho} \]
For an example of the present method on which all calculations can be carried out explicitly, refer to (3.8) and (3.9) for formulas on the sphere. In that case, \( \rho_N \) and \( u_N \) are just the appropriate partial sums in (3.8) and (3.9), respectively, chosen to lie in the subspace \( \mathcal{H}_N \).

5. Implementation of Galerkin's method. Most of the work of this method is in the setup of the linear system (4.6) and the evaluation of \( u_N \). And in both cases, the most costly step is the numerical integration of surface integrals over \( U \). We have used the product Gaussian quadrature method with fairly good success; it will be discussed below. Once the linear system has been formed, it is relatively inexpensive to solve. The parameter \( N \) in (4.6) can usually be kept fairly small, say \( N \leq 7 \), and the approximations \( \rho_N \) and \( u_N \) will be sufficiently accurate. Since \( d = (N + 1)^2 \) is the order of the system, it is a relatively small linear system and ordinary elimination can be used.

Throughout the calculations, it is very important to arrange the calculations quite carefully so as to avoid repeating calculations unnecessarily. By doing so, we have obtained computer programs which are much faster than might have been expected. Because the Galerkin coefficients \( (h_i, \tilde{h}_i) \) depend only on the surface \( S \), we are able to save time by calculating them separately, say for \( N \leq N_{\text{max}} \), and they are stored on disk, in a form for rapid retrieval by the main program used in solving (1.1).

Numerical integration on the unit sphere. Let

\[
I(f) = \int_U f(Q) \, d\sigma = \int_0^{2\pi} \int_0^\pi f(\phi, \theta) \sin \theta \, d\theta \, d\phi.
\]

Approximate it by the product Gaussian formula

\[
I_M(f) = \delta \sum_{i=1}^{2M} \sum_{j=1}^M w_{ij} f(\phi_i, \theta_i)
\]

or its midpoint variant

\[
\tilde{I}_M(f) = \delta \sum_{i=1}^{2M} \sum_{j=1}^M w_{ij} f(\tilde{\phi}_i, \theta_i).
\]

Here \( \delta = \pi/M, \phi_i = i\delta, \tilde{\phi}_i = (i-.5)\delta \); and \( \{w_{ij}\}, \{\cos \theta_i\} \) are the Gauss–Legendre weights and nodes on \([-1, 1]\). The degree of precision of both formulas is \( 2M - 1 \); a proof for (5.1) is given in Stroud [16, p. 40], and a similar one can be given for (5.2).

THEOREM 3. For any \( f \in C(U) \),

\[
I_M(f) \to I(f) \quad \text{as} \quad M \to \infty.
\]

If \( f \in H_{1,\lambda}(U) \), then

\[
|I(f) - I_M(f)| \leq \frac{c}{(2M - 1)^{1+\lambda}}, \quad M \geq 1,
\]

for some \( c > 0 \). The same results are also true for \( \tilde{I}_M(f) \).

Proof. As a linear functional on \( C(U) \),

\[
\|I_M\| = 4\pi,
\]

and all the weights in (5.1) are positive. Since the spherical polynomials are dense in \( C(U) \), it is straightforward to show \( I_M(f) \) converges to \( I(f) \), for any \( f \in C(U) \).
Let $E_M(f) = I(f) - I_M(f)$, and let $T_{2M-1}$ denote any spherical polynomial of degree $\leq 2M - 1$. Since the degree of precision of $I_M$ is $2M - 1$,

\[ E_M(f) = E_M(f - T_{2M-1}), \]

\[ |E_M(f)| \leq \|E_M\| \|f - T_{2M-1}\|_\infty \leq 8\pi \|f - T_{2M-1}\|_\infty. \]

To complete the proof, let $T_{2M-1}$ be the polynomial in (3.4). []

It is possible to prove convergence for other integrands. We will consider integrands which are bounded and which are continuous except at a finite number of points on $U$. Convergence can be shown; but we omit the proof since we cannot obtain any usable result on the rate of convergence.

The integral for $u_N(A)$ is evaluated using $I_M$. In (4.12), the integrand is increasingly peaked as $A$ approaches the boundary. Let $P \in S$ be chosen near to $A$. Use the identity

\[ A D, \]

\[ (5.4) \]

\[ u_N(A) = 4\pi \rho(P) + \int_S [\rho_N(Q) - \rho_N(P)] \frac{\partial}{\partial \nu_Q} \left( \frac{1}{\rho_{OA}} \right) d\sigma. \]

This will result in a substantial increase in the accuracy of the numerical integration, without any significant increase in computation time.

**Calculation of the Galerkin coefficients.** To simplify the writing of the formulas, let $\hat{K}(P, Q)$ denote the kernel of $\hat{K}$. The coefficients $(h_i, \hat{K}h_i)$ are fourfold integrals with a singular integrand. To decrease the effect of the singularity in computing $\hat{K}h_i(P)$, use the identity

\[ (5.5) \]

\[ \int K(P, Q) d\sigma = 2\pi, \quad P \in U, \]

\[ \hat{K}h_i(P) = 2\pi h_i(P) + \int_U [h_i(Q) - h_i(P)] \hat{K}(P, Q) d\sigma. \]

The new integrand contains a bounded discontinuity at $Q = P$. We numerically evaluate the integral in (5.5) by applying $I_M$ to it, with respect to the spherical coordinates representation $Q = Q(\phi', \theta')$, $\phi \leq \phi' \leq 2\pi + \phi$, $0 \leq \theta' \leq \pi$, where $P = P(\phi, \theta)$. This results in some decrease in the integration error, since the point of discontinuity occurs at a boundary point of the integration region.

Calculate the entire coefficient by

\[ (5.6) \]

\[ (h_i, \hat{K}h_i) = I_M(h_i \cdot \hat{K}h_i), \]

and approximate $\hat{K}h_i$ as above. Denote the complete integration rule by $\mathcal{I}_M(h_i, \hat{K}h_i)$. Because $\hat{K}h_i$ should be quite smooth, by (2.4), the integration in (5.6) should be quite accurate with low $M$. Thus the major error will be in integrating (5.5). Empirically,

\[ (5.7) \]

\[ (h_i, \hat{K}h_i) - \mathcal{I}_M(h_i, \hat{K}h_i) = \mathcal{O}\left(\frac{1}{M^3}\right), \]

and this has been used for extrapolation. The behaviour of the integration rule is still not as regular as one would like, and we are continuing to look for better methods for calculating these coefficients.
The cost of evaluating all of the Galerkin coefficients of degree $\leq N$ is proportional to $M^4$. Empirically, we have found the total cost in time is about

$$c \left( \frac{4}{3} \right)^N M^4,$$

although the fraction $\frac{4}{3}$ should be replaced by a slowly decreasing function of $N$. On a CYBER 72, with any ellipsoidal region $S$, $c \approx 0.012$ seconds. Also note that we usually need only $N \leq 7$ for quite adequate accuracy in solving (1.1).

6. Numerical examples. We begin with two calculations involving the sphere $S = U$, and then we give examples for three other surfaces. The sphere is a good case in which to test the approximations at different stages of the method because we know the exact answers in most cases, based on (3.7)–(3.9).

Example 1. We consider the calculation of the Galerkin coefficients $(h_i, \mathcal{H} h_i)$. Using (3.7), $(h_i, \mathcal{H} h_i) = 0$ for $i \neq j$, and we give some of the cases

$$k_{ii} = \frac{(h_i, \mathcal{H} h_i)}{(h_i, h_i)}.$$

In Table 1, if $m = 0$, then the coefficient $k_{ii}$ is for $h_i = P_n (\cos \theta)$, and if $m > 0$, it is for $h_i = P_n^m (\cos \theta) \cos (m\phi)$. $\mathcal{S}_{14}$ and $\mathcal{S}_{16}$ are defined following (5.6), and $\mathcal{S}_{16}^*$ denotes the Richardson extrapolate of $\mathcal{S}_{14}$ and $\mathcal{S}_{16}$, based on (5.7). The results show the potential value of using extrapolation.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$k_{ii}$</th>
<th>$\mathcal{S}_{14}$</th>
<th>$\mathcal{S}_{16}$</th>
<th>$\mathcal{S}_{16}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
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<td>$-6.47E-4$</td>
<td>$-4.39E-4$</td>
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</tr>
<tr>
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<td>$-1.90E-3$</td>
<td>$-1.29E-3$</td>
<td>$-4.40E-5$</td>
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<tr>
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<td>$4.16E-4$</td>
<td>$2.77E-4$</td>
<td>$-6.07E-6$</td>
</tr>
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</table>

In Table 1, if $m = 0$, then the coefficient $k_{ii}$ is for $h_i = P_n (\cos \theta)$, and if $m > 0$, it is for $h_i = P_n^m (\cos \theta) \cos (m\phi)$. $\mathcal{S}_{14}$ and $\mathcal{S}_{16}$ are defined following (5.6), and $\mathcal{S}_{16}^*$ denotes the Richardson extrapolate of $\mathcal{S}_{14}$ and $\mathcal{S}_{16}$, based on (5.7). The results show the potential value of using extrapolation.

Example 2. We compare the calculation of $u_N(A)$ for $S = U$ using different amounts of approximation in the solution process. The four options given in Table 2 are as follows; in all cases, $N = 5$.

1. Evaluate $u_N(A)$ using the exact formula (3.9), with the coefficients $(f, h_i)$ evaluated using $I_{16}$.
2. Use (3.8) to evaluate $\rho_N$, but evaluate $u_N(A)$ using (5.4) to cancel partially the peaked behavior of the integrand when $A$ is near to $U$. Use $I_{16}$ to evaluate $u_N(A)$ and the coefficients $(f, h_i)$.
3. Use the linear system (4.6) to solve for $\rho_N$, with the Galerkin coefficients obtained using $\mathcal{S}_{16}^*$. Use $I_8$ to evaluate $u_N(A)$ (as in (5.4)) and $(f, h_i)$.
4. The same as (3), but replace $I_8$ with $I_{16}$.

The true solution is

$$(6.1) \quad u = e^x \cos (y) + e^z \sin (x).$$
TABLE 2

<table>
<thead>
<tr>
<th>Option</th>
<th>Error in ( u_3(A) ) for given ( A = (x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
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<tr>
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<td>-1.11E-8</td>
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<tr>
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<tr>
<td>4</td>
<td>-1.08E-7</td>
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</tbody>
</table>

It will also be used in the remaining examples. It is fairly illustrative of the behavior of the method, being neither the best nor worst of the examples computed.

The additional inaccuracy of options 3 and 4, in comparison with option 1, is due to two causes. For inner points \( A \), the Galerkin coefficients have not been computed with sufficient accuracy; and for the outer points \( A \), the integration of \( u_N(A) \) is not sufficiently accurate. In all cases, if we want more accuracy near the boundary, we will have to increase the degree \( N \) of the approximation. The series (3.9) converges more rapidly for smaller values of \( r = |A| \), and thus the worst error in \( u_N(A) \) is near the boundary.

Example 3. Let \( S \) be the ellipsoidal surface given by

\[
x^2 + \left( \frac{y}{1.5} \right)^2 + \left( \frac{z}{2} \right)^2 = 1.
\]

The solution \( u \) is given by (6.1). We solve (4.6) for \( \rho_N \), the Galerkin coefficients are calculated using \( \mathcal{F}_6 \), and both \( u_N(A) \) and the coefficients \( f, h_i \) are calculated using \( I_{16} \). In Table 3, \( \alpha \) is the number for which \( 1/\alpha(x, y, z) \) is on the boundary \( S \); it is a measure of the closeness of \((x, y, z)\) to the boundary.

TABLE 3

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( \alpha )</th>
<th>( u )</th>
<th>( u - u_4 )</th>
<th>( u - u_5 )</th>
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</thead>
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<tr>
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<td>.1</td>
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<td>.5</td>
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<td>.65</td>
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<td>.765039</td>
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<tr>
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<tr>
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<td>.56</td>
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<td>3.8E-3</td>
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</tbody>
</table>

Increasing the accuracy of the integration of \( u_N(A) \) by changing to \( I_{32} \) does not decrease the error in the calculated \( u \). Note again that fairly reasonable accuracy has been obtained by solving a linear system of order 36, which is quite small for a three-dimensional problem.

Example 4. We use another ellipsoidal surface,

\[
x^2 + \left( \frac{y}{2} \right)^2 + \left( \frac{z}{5} \right)^2 = 1,
\]

which is more ill-behaved in the sense of being thin and narrow in shape. We solve (1.1) for two cases,

\[
u^{(1)} = [(x - 5)^2 + (y - 4)^2 + (z - 3)^2]^{-1/2}
\]
and $u^{(2)}$ given by (6.1). Table 4 contains the results of solving for $u_7$, with the Galerkin coefficients computed using $\mathcal{S}_{16}$. The coefficients $(f, h_i)$ and $u_N$ were computed with $I_{32}$.

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
</tr>
<tr>
<td>-------</td>
</tr>
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</tr>
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<tr>
<td>0</td>
</tr>
</tbody>
</table>

The function $u^{(2)}$ varies greatly and rapidly over some sections of this region; and that leads to the need for a higher degree $N$ in order to accurately represent the double layer density $\rho^{(2)}$ associated with $u^{(2)}$. The function $u^{(1)}$ is a better behaved function.

**Example 5.** Our final surface $S$ is defined by

$$\begin{align*}
(x, y, z) &= R \left( \sin \theta \cos \phi, 2 \sin \theta \sin \phi, \cos \theta \right), \\
R &= \sqrt{\cos(2\theta) + \sqrt{1.1 - \sin^2(2\theta)}}.
\end{align*}$$

Loosely speaking, it is a "peanut-shaped" region, based on the "ovals of Cassini". The cross-section in the $x, z$-plane is shown in Fig. 1.

![Fig. 1](image)

We will solve for the solutions $u^{(1)}$ and $u^{(2)}$ used with Example 4. The Galerkin coefficients were calculated with $\mathcal{S}_{16}$, and the coefficients $(f, h_i)$ and the solution $u_N$ were computed with $I_{32}$. Table 5 contains the results for $N = 7$. The results are acceptable in both cases; but they are much better for $u^{(1)}$ than for $u^{(2)}$, just as was true in the previous example.

**Concluding remarks.** The method of this paper results in linear systems that are quite small for solving the three-dimensional Laplace equation. It is also a fairly rapid method, especially after the Galerkin coefficients have been calculated and stored on disk. The method is not suitable for regions which do not have a smooth boundary, and...
these probably form the large majority of potential theory problems. But for that smaller class of problems which do have smooth boundaries, the present method should be superior to the use of a finite element method for solving the integral equations, and to most direct methods of solution of the Laplace equation. In the sequel [4] to this paper, we consider the extension of the present work to the exterior Dirichlet problem and to the interior and exterior Neumann problems.

As an addendum at the time of this paper’s acceptance, we have developed a better method of calculation for the Galerkin coefficients. This work will appear in [5].

Acknowledgments. I wish to thank Vladimir Oliker, University of Iowa, for helpful discussions, and I thank Dick Askey, University of Wisconsin, for references to the literature on the convergence of the Laplace series.

REFERENCES