On the norm of the hyperinterpolation operator on the unit disc and its use for the solution of the nonlinear Poisson equation

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[Received on 31 August 2006; revised on 29 November 2007]

In this article, we study the properties of the hyperinterpolation operator on the unit disc \( D \) in \( \mathbb{R}^2 \). We show how hyperinterpolation can be used in connection with the Kumar–Sloan method to approximate the solution of a nonlinear Poisson equation on the unit disc (discrete Galerkin method). A bound for the norm of the hyperinterpolation operator in the space \( C(D) \) is derived. Our results prove the convergence of the discrete Galerkin method in the maximum norm if the solution of the Poisson equation is in the class \( C^{1,\delta}(D) \), \( \delta > 0 \). Finally, we present numerical examples which show that the discrete Galerkin method converges faster than \( O(n^{-k}) \), for every \( k \in \mathbb{N} \) if the solution of the nonlinear Poisson equation is in \( C^\infty(D) \).

Keywords: hyperinterpolation operator; discrete Galerkin method; projector norm; nonlinear Poisson equation.

1. Introduction

The most common approach to solving numerically integral and partial differential equations over a planar region is to use piecewise polynomial approximations of the solution. An approach that has been used much less is to use polynomial approximations of the solution. We used this recently in Atkinson & Hansen (2006) to solve the nonlinear Poisson equation \( \Delta u = f(\cdot, u) \) over the closed unit disc \( D \subseteq \mathbb{R}^2 \). (By a simple change of variables, this extends to a wide variety of other planar regions; e.g. see Atkinson & Han (2004) for a conformal transformation of the ellipse onto the disc.) Methods based on polynomial approximations are often more rapidly convergent, and usually they lead to the solution of linear or nonlinear systems that are of much smaller size than those used in numerical methods based on piecewise polynomial approximations.

Numerical methods for approximating differential or integral equations are usually of collocation or Galerkin type. We have been considering methods of Galerkin type because collocation methods require results on multivariate polynomial interpolation theory, and this theory is still being developed (e.g. see Xu, 2004). With Galerkin methods, there are many integrals to be evaluated numerically. In attempting to minimize the order of the quadrature rule being used, one is led to the idea of ‘hyperinterpolation’.

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a concept and term introduced by Sloan (1995). In this paper, we investigate hyperinterpolation in connection with the Galerkin method over the unit disc.

The Galerkin method with polynomial approximations makes use of the orthogonal projection from $L^2(D)$ onto the subspace of polynomials of degree at most $n$. The hyperinterpolation operator is based on approximating the integrations that appear in the formula for the orthogonal projection operator. In Section 2 we introduce the hyperinterpolation operator and discuss bounding its norm as an operator on $C(D)$. The main result of this paper is given at the conclusion of the section, and its proof is given as a series of lemmata in Sections 3 and 4.

The bound that we derive for the hyperinterpolation operator implies that the resulting discrete Galerkin method is convergent in the maximum norm if the solution of the equation has a smoothness of $C^{1, \delta}(D), \delta > 0$. In previous articles, see Atkinson & Hansen (2006), we assumed that certain integrals are known or can be approximated with a sufficiently accurate integration rule. Our analysis here shows that the quadrature rules used in the construction of the hyperinterpolation operator are sufficient to guarantee the convergence and that the resulting convergence rate is at most $O(\log(n))$ slower than the optimal rate that we can expect. In Section 5, we apply the method to nonlinear Poisson equations where the solution is in $C^\infty(D)$. Here, we expect that the convergence is faster than $O(n^{-k})$ for every $k \in \mathbb{N}$. Our numerical results confirm this.

2. The hyperinterpolation operator on the disc

We consider the disc $D := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$ and the space $L^2(D)$ with the scalar product

\[(f, g) := \frac{1}{\pi} \int_D f(x)g(x)dx = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} f(r, \phi)g(r, \phi)r \, d\phi \, dr.\]  

We always identify the two representations of a function $f$ on $D$: $f(x) = f(r, \phi)$. We denote By $\Pi_n$ the space of all polynomials in two variables of degree $n$ or less:

\[\Pi_n := \left\{ \sum_{j=0}^n \sum_{k=0}^j a_{j,k} x_1^j x_2^k \mid a_{j,k} \in \mathbb{R} \right\};\]

note that the dimension of $\Pi_n$ is $\binom{n+2}{2}$.

Let $\{\Psi_{j,k}\}_{j=0,\ldots,n,k=0,\ldots,j}$ be an orthonormal basis for $\Pi_n$. The orthogonal projection of $L^2(D)$ onto $\Pi_n$ is given by

\[(P_n f)(x) := \sum_{j=0}^n \sum_{k=0}^j (f, \Psi_{j,k}) \Psi_{j,k}(x).\]

The property

\[\|P_n\|_{L^2(D) \to L^2(D)} = 1\]

is well known, and Xu (2001) proved the important result

\[\|P_n\|_{C(D) \to C(D)} \sim n;\]

(2.3)

here $A(n) \sim n$ means that there are constants $c_1 > 0$ and $c_2 > 0$ such that $c_1 n \leq A(n) \leq c_2 n$. 


To approximate the projection $P_n$, we can replace the scalar product by a finite sum

$$(f, g)_d := \frac{1}{\pi} \sum_{l=0}^{n} \sum_{m=0}^{2n} f \left( r_l, \frac{2\pi m}{2n+1} \right) g \left( r_l, \frac{2\pi m}{2n+1} \right) \omega_l \frac{2\pi}{2n+1} r_l$$

$$= \sum_{l=0}^{n} \sum_{m=0}^{2n} f \left( r_l, \frac{2\pi m}{2n+1} \right) g \left( r_l, \frac{2\pi m}{2n+1} \right) \omega_l \frac{2}{2n+1} r_l; \quad (2.4)$$

where we use the trapezoidal rule for the azimuthal direction and a Gaussian quadrature rule for the radial direction. This quadrature is exact for all polynomials $f, g \in \Pi_n$. Here, the numbers $\omega_l$ are the weights of Gauss–Legendre quadrature on $[0, 1]$:

$$\int_0^1 p(x) dx = \sum_{l=0}^{n} p(r_l) \omega_l$$

for all single-variable polynomials $p(x)$ with $\deg(p) \leq 2n + 1$. (Another possible choice would be Gauss quadrature on $[0, 1]$ with measure $r \, dr$; the term $r_l$ in formula (2.4) would then disappear.) The discrete semidefinite scalar product $(\cdot, \cdot)_d$ depends on $n$ but we do not indicate this dependency explicitly.

With the help of the discrete scalar product, we can now define an approximation to the orthogonal projection $P_n f$ when $f$ is restricted to being continuous over $D$:

$$(L_n f)(x) := \sum_{j=0}^{n} \sum_{k=0}^{j} (f, \Psi_{j,k})_d \Psi_{j,k}(x).$$

The operator $L_n$ is the ‘hyperinterpolation operator’ of Sloan & Womersley (2000), and it can also be considered as a ‘discrete orthogonal projection operator’, as in Atkinson & Bogomolny (1987). Methods using this approximating operator are sometimes called ‘discrete Galerkin methods’.

### 2.1 Bounds on the projection error

When using the Galerkin method or the discrete Galerkin method for solving integral equations, the rate of convergence is generally related to the error in the orthogonal projection. For a discussion of a general framework for analysing Galerkin methods and discrete Galerkin methods for solving integral equations, see Atkinson (1997, Chapter 3). The error analysis often leads to a formula

$$\| f - f_n \| \leq c \| f - Q_n f \| \quad (2.5)$$

that is shown to hold for all sufficiently large $n$. We consider both the cases $Q_n = P_n$ and $Q_n = L_n$. If the analysis is being done within $L^2(D)$, with the norm being $\| \cdot \|_{L^2}$, then we know that $P_n f \to f$ as $n \to \infty$ for any $f \in L^2(D)$. However, when using $L_n$ we must work in $C(D)$, and we are also often interested in doing so for $P_n$; the function space norm is then $\| \cdot \|_\infty$. In both cases $C(D)$ and $L^2(D)$, we are interested in examining the rate of convergence $Q_n f$ to $f$ as it is affected by the smoothness of the function $f$. 

To examine this question, we can use results on best polynomial approximations, and for this we use results from Ragozin (1971, p. 164) as summarized below. Assume that $f \in C^k(D)$ with $k \geq 0$ an integer. For the norm on $C^k(D)$, we use the standard definition

$$
\|f\|_{C^k} = \sum_{i+j \leq k} \left\| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right\|_{\infty}.
$$

In addition, we define various moduli of continuity by

$$
\omega(f; h) = \sup \{|f(x_1, y_1) - f(x_2, y_2)| : \| (x_1, y_1) - (x_2, y_2) \| \leq h \},
$$

$$
\omega_k(f; h) = \sum_{i+j = k} \omega \left( \frac{\partial^{i+j} f}{\partial x^i \partial y^j}; h \right), \quad k \geq 1.
$$

Then there exists a sequence of polynomials $p_n$ of degree $\leq n$ such that

$$
\| f - p_n \|_{\infty} \leq \frac{B_k}{n^k} \left[ \frac{\| f \|_{C^k}}{n} + \omega_k \left( f; \frac{1}{n} \right) \right], \quad d \geq 1, \quad (2.6)
$$

where each constant $B_k$ depends only on $k \geq 0$.

To bound $\| f - Q_n f \|$, note that with both choices for $Q_n$, we have

$$
Q_n p = p \quad \forall p \in \Pi_n.
$$

Thus,

$$
f - Q_n f = (f - p_n) - Q_n(f - p_n),
$$

$$
\| f - Q_n f \| \leq (1 + \| Q_n \|) \| f - p_n \|. \quad (2.7)
$$

When the function space is $L^2(D)$ and when $Q_n = P_n$, we know that $\| P_n \|_{L^2 \to L^2} = 1$; thus

$$
\| f - P_n f \|_{L^2} \leq 2 \| f - p_n \|_{L^2}
$$

$$
\leq 2\pi \| f - p_n \|_{\infty}. \quad (2.8)
$$

With the space $C(D)$ and either choice for $Q_n$, we have

$$
\| f - Q_n f \|_{\infty} \leq (1 + \| Q_n \|_{C \to C}) \| f - p_n \|_{\infty}. \quad (2.9)
$$

With $Q_n = P_n$, we know that $\| P_n \|_{C \to C} = O(n \to \infty)(n)$ (cf. (2.3)). When this is combined with (2.6), we lose one power of $n$ in the rate of uniform convergence of $P_n f$ to $f$. We need $k > 1$ to ensure uniform convergence—although, using other results from Ragozin (1971)—this can be weakened to requiring $f$ to have first derivatives that are Hölder continuous with some exponent $\alpha > 0$.

If $Q_n = L_n$, we need to know $\| L_n \|_{C \to C}$ in order to bound $\| f - L_n f \|_{\infty}$. Estimating this norm is the focus of the present paper.
2.2 The reproducing kernel for $\Pi_n$

To obtain another useful formula for $L_n f$, we need a result from Xu (2001), where he derived formulas for the reproducing kernel $G_n$ for $\Pi_n$. We specialize his formulas to our case, obtaining the following:

$$G_n(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{j} \Psi_{j,k}(x) \Psi_{j,k}(y)$$

$$= \int_{0}^{\pi} \left[ C_n^{(2)} \left( x \cdot y + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \cos(\psi) \right) + C_{n-1}^{(2)} \left( x \cdot y + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \cos(\psi) \right) \right] d\psi$$

$$= \frac{2 \Gamma \left( \frac{5}{2} \right) \Gamma(n + 3)}{\Gamma(4) \Gamma(n + \frac{3}{2})} \times \int_{0}^{\pi} P_n^{\left( \frac{1}{2}, \frac{1}{2} \right)} \left( x \cdot y + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \cos(\psi) \right) d\psi. \quad (2.10)$$

Here, $P_n^{(\lambda, \mu)}$ denotes a standard Jacobi polynomial of degree $n$ (cf. Abramowitz & Stegun, 1965, p. 774; Andrews et al., 2001, Section 6.3; Szegö, 1975, Chapter 4), and we remark that the multiplying constant in (2.10) satisfies

$$\frac{2 \Gamma \left( \frac{5}{2} \right) \Gamma(n + 3)}{\Gamma(4) \Gamma(n + \frac{3}{2})} \sim n^{\frac{3}{2}}. \quad (2.12)$$

Using $G_n(x, y)$, we have the following reproducing kernel property over $\Pi_n$:

$$\int_{D} G_n(x, y) f(x) dx = f(y), \quad y \in D, \text{ for all } f \in \Pi_n. \quad (2.13)$$

Now we can proceed as in Sloan & Womersley (2000, (4.13)) and derive a representation for $L_n$ with the help of $G_n$:

$$(L_n f)(x) = \sum_{j=0}^{n} \sum_{k=0}^{j} \left( \sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} f(\zeta_{l,m}) \Psi_{j,k}(\zeta_{l,m}) \right) \Psi_{j,k}(x)$$

$$= \sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} f(\zeta_{l,m}) \left( \sum_{j=0}^{n} \sum_{k=0}^{j} \Psi_{j,k}(\zeta_{l,m}) \Psi_{j,k}(x) \right)$$

$$= \sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} f(\zeta_{l,m}) G_n(\zeta_{l,m}, x). \quad (2.14)$$
It is straightforward to show that
\[
\|L_n\|_{C(D)\to C(D)} = \max_{x \in D} \sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} |G_n(\xi_{l,m}, x)|.
\]

Because the kernel function \(G_n\) is continuous, we know that there is a \(\xi_0 = \hat{r} = (\cos(\alpha_0), \sin(\alpha_0)) \in D\) (we omit the dependency on \(n\)) such that
\[
\|L_n\|_{C(D)\to C(D)} = \sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} |G_n(\hat{r}, \xi_0)|
\]
\[
\leq \frac{2\Gamma\left(\frac{5}{2}\right)\Gamma(n+3)}{\Gamma(4)\Gamma(n+\frac{5}{2})} \\
\times \sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} \int_{0}^{\pi} \left|P_n\left(\frac{1}{2}, \frac{1}{2}, \hat{r} \cos(\alpha_0 - \frac{2m\pi}{2n+1}) + \sqrt{1 - \hat{r}^2} \sqrt{1 - r^2} \cos(\psi)\right)\right| d\psi.
\]
(2.15)

2.3 Bounding the hyperinterpolation operator

We can obtain a simple bound for \(\|L_n\|_{C(D)\to C(D)}\) by modifying an argument given in Sloan & Womersley (2000, Theorem 5.5.2). We begin by using the Cauchy–Schwarz inequality
\[
\|L_n\|_{C(D)\to C(D)} = \sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} |G_n(\xi_{l,m}, \xi_0)|
\]
\[
= \sum_{l=0}^{n} \sum_{m=0}^{2n} \sqrt{w_{l,m}} \sqrt{w_{l,m}} |G_n(\xi_{l,m}, \xi_0)|
\]
\[
\leq \left(\sum_{l=0}^{n} \sum_{m=0}^{2n} (\sqrt{w_{l,m}})^2\right)^{\frac{1}{2}} \left(\sum_{l=0}^{n} \sum_{m=0}^{2n} (\sqrt{w_{l,m}} |G_n(\xi_{l,m}, \xi_0)|)^2\right)^{\frac{1}{2}}
\]
\[
= \left(\sum_{l=0}^{n} \sum_{m=0}^{2n} w_{l,m} |G_n(\xi_{l,m}, \xi_0)|^2\right)^{\frac{1}{2}}
\]
\[
= \left(\int_D |G_n(x, \xi_0)|^2 dx\right)^{\frac{1}{2}}.
\]

The last equality follows from the exactness of the quadrature formula for polynomials of degree at most \(2n\) and from \([G_n(x, \xi_0)]^2\) being a polynomial of degree \(2n\) in the integration variable \(x\). Using the
reproducing kernel property (2.13) of $G_n$ with $f(x) = G_n(x, \bar{\zeta}_0)$, the integral term simplifies to

$$\int_D [G_n(x, \bar{\zeta}_0)]^2 \, dx = G_n(\bar{\zeta}_0, \bar{\zeta}_0),$$

and thus

$$\|L_n\|_{C(D) \to C(D)} \leq \sqrt{\pi G_n(\bar{\zeta}_0, \bar{\zeta}_0)}.$$  \hspace{1cm} (2.16)

Next, we use the following bound on Jacobi polynomials, taken from Abramowitz & Stegun (1965, (22.14.1)):

$$|P_n^{(\lambda, \mu)}(t)| \leq \left(\frac{n + q}{n}\right) \approx n^q, \quad q = \max(\lambda, \mu), \quad -1 \leq t \leq 1,$$  \hspace{1cm} (2.17)

provided that $q \geq -\frac{1}{2}$ and $\lambda, \mu > -1$. In the integral of (2.15), we have

$$\left|\int_0^\pi P_n^{(\frac{3}{2}, \frac{3}{2})} \left(x \cdot y + \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \cos(\psi)\right) \, d\psi\right| \leq \pi \left(\frac{n + \frac{3}{2}}{n}\right), \quad x, y \in B.$$

When this is combined with (2.12) and (2.17), we have

$$|G_n(x, y)| = O_{n \to \infty}(n^3)$$

and then from (2.16),

$$\|L_n\|_{C(D) \to C(D)} \leq O_{n \to \infty} \left(n^\frac{3}{2}\right).$$  \hspace{1cm} (2.18)

Can this result be improved to the growth rate given in (2.3) for $\|P_n\|_{C(D) \to C(D)}$? The main result of this paper shows something quite close to this.

**Theorem 2.1** The hyperinterpolation operator $L_n$ satisfies

$$\|L_n\|_{C(D) \to C(D)} = O_{n \to \infty}(n \ln(n)).$$

The proof of this result is given as a series of lemmata; which are given in Sections 3 and 4 of this paper. Before taking up the proof, we note the following:

1. The proof shows not only that our special choice of trapezoidal and Gaussian rules for the hyperinterpolation operator leads to the above norm estimate, but also that every choice of a quadrature rule that is a convergent Riemann sum with maximal stepwidth proportional to $\frac{1}{n}$ will lead to the above estimate. Even so, we still need a quadrature rule in (2.4) that calculates the inner product in (2.1) exactly for functions in $\Pi_n$ in order for $L_n$ to be a projection operator from $C(D)$ onto $\Pi_n$.

2. The proof of Theorem 2.1 includes also a proof that $\|P_n\|_{C(D) \to C(D)} = O_{n \to \infty}(n)$. The proof given here is independent of the proof by Xu (1999).
3. Bounds for the angular quadrature

The proof of Theorem 2.1 is shown through a series of lemmata. We begin by looking at the angular quadrature portion of the formula (2.15):

\[
\frac{2\pi}{2n+1} \sum_{m=0}^{2n} \int_0^\pi \left| P_n^{(3, 1/2)} \left( r \hat{r} \cos \left( \frac{2m\pi}{2n+1} \right) + \sqrt{1 - \hat{r}^2} \sqrt{1 - r^2} \cos(\psi) \right) \right| d\psi.
\]

(3.1)

To bound this sum, we begin with a result from Xu (2001, Lemma 3.2).

**Lemma 3.1** For \( \alpha, \beta > -1 \), \( t \in [0, 1] \), we have

\[
|P_n^{(\alpha, \beta)}(t)| \leq \frac{c_0}{\sqrt{n}} \left( 1 + \frac{1}{n^2} - t \right)^{-\frac{\alpha + 1/2}{2}},
\]

where \( c_0 \) does not depend on \( t \) or \( n \).

Because \( |P_n^{(\alpha, \beta)}(-t)| = |P_n^{(\beta, \alpha)}(t)| \), we get a similar estimate for \( t \in [-1, 0] \), namely

\[
|P_n^{(\alpha, \beta)}(t)| \leq \frac{c_0}{\sqrt{n}} \left( 1 + \frac{1}{n^2} + t \right)^{-\frac{\beta + 1/2}{2}}.
\]

Let \( \alpha = 3/2 \) and \( \beta = 1/2 \); add the two bounds to obtain the overall bound

\[
\left| P_n^{(3/2, 1/2)}(t) \right| \leq \frac{c_0}{\sqrt{n}} \left( \frac{1}{1 + \frac{1}{n^2} - t} + \frac{1}{\sqrt{1 + \frac{1}{n^2} + t}} \right), \quad -1 \leq t \leq 1.
\]

(3.2)

For a general \( \alpha > 0 \), we have

\[
\frac{1}{\sqrt{\alpha}} \leq \frac{c}{\alpha},
\]

provided that \( c \geq \sqrt{\alpha} \). Using this with the final fraction in (3.2), we find that we need \( c \geq \sqrt{3} \) when allowing \(-1 \leq t \leq 1 \) and \( n \geq 1 \). Thus,

\[
\left| P_n^{(3/2, 1/2)}(t) \right| \leq \frac{\sqrt{3}c_0}{\sqrt{n}} \left( \frac{1}{1 + \frac{1}{n^2} - t} + \frac{1}{1 + \frac{1}{n^2} + t} \right).
\]

(3.3)

Using the estimate (3.3) in (2.15) and the definition of \( w_{m,l} \), we get

\[
\|L_n\|_{C(D) \to C(D)} \leq c_1(n) \sum_{l=0}^{n} \sum_{m=0}^{2n} \frac{2\pi}{2n+1} \int_0^\pi \frac{1}{1 + \frac{1}{n^2} - t(n, r, a_0, r_l, m, \psi)} \left[ \frac{1}{1 + \frac{1}{n^2} + t(n, r, a_0, r_l, m, \psi)} \right] d\psi,
\]

(3.4)
where we have introduced
\[ t(n, \hat{r}, a_0, r, m, \psi) := \hat{r} r \cos \left( a_0 - \frac{2m\pi}{2n+1} \right) + \sqrt{1 - \hat{r}^2} \sqrt{1 - r^2} \cos(\psi) \tag{3.5} \]
and
\[ c_1(n) := \frac{8c_0 \Gamma \left( \frac{5}{2} \right) \Gamma(n+3)}{\Gamma(4) \Gamma(n + \frac{5}{2}) \sqrt{n}} = O_{n \to \infty}(n). \tag{3.6} \]

To estimate the trapezoidal rule in the square brackets of (3.4), we start by calculating the integral. For this, use (Gradshteyn & Ryzhik, 2000, 2.553),
\[ \int_0^\pi \frac{1}{a + b \cos(\psi)} d\psi = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad |a| > |b|. \tag{3.7} \]
Apply this to (3.4) with
\[ a = 1 + \frac{1}{n^2} \pm \hat{r} l \cos \left( a_0 - \frac{2m\pi}{2n+1} \right), \]
\[ b = \pm \sqrt{1 - \hat{r}^2} \sqrt{1 - r_l^2}. \]

Note that
\[ a^2 - b^2 \geq \left( 1 + \frac{1}{n^2} - \hat{r} l \right)^2 - (1 - \hat{r}^2)(1 - r_l^2) \]
\[ = (\hat{r} - r_l)^2 + \frac{1}{n^4} + \frac{2}{n^2} (1 - \hat{r} l) \]
\[ \geq \frac{1}{n^4}, \quad \hat{r}, r_l \in [0, 1]. \]

Introduce the notation
\[ H_1^\pm(n, \hat{r}, r, a) := \frac{\pi}{\sqrt{(\hat{r}^2 + r_l^2 - \hat{r}^2 r_l^2 + \frac{1}{n^4} + \frac{2}{n^2}) \pm 2\hat{r} l (1 + \frac{1}{n^2}) \cos(a) + \hat{r}^2 r_l^2 \cos(a)^2}}. \tag{3.8} \]

For (3.4), introduce the notation
\[ T_n(\hat{r}, a_0, r) \]
\[ := \sum_{m=0}^{2n} \frac{2\pi}{2n+1} \int_0^\pi \left( \frac{1}{1 + \frac{1}{n^2} - t(n, \hat{r}, a_0, r_l, m, \psi)} + \frac{1}{1 + \frac{1}{n^2} + t(n, \hat{r}, a_0, r_l, m, \psi)} \right) d\psi. \tag{3.9} \]
Applying (3.7), we have
\[ T_n(\hat{r}, a_0, r) = \sum_{m=0}^{2n} \frac{2\pi}{2n+1} \left[ H_1^- \left( n, \hat{r}, r, a_0 - \frac{2m\pi}{2n+1} \right) + H_1^+ \left( n, \hat{r}, r, a_0 - \frac{2m\pi}{2n+1} \right) \right]. \]
Introduce

\[ I_1(n, \hat{r}, r) := \int_0^{2\pi} [H_1^+(n, \hat{r}, r, \alpha) + H_1^-(n, \hat{r}, r, \alpha)]d\alpha. \]  

(3.10)

Then, (3.9) is nothing more than a trapezoidal rule, shifted by \( a_0 \), for the approximation of \( I_1(n, \hat{r}, r_l) \). To further simplify our analysis, note that the integrand in the definition of \( I_1 \) has period \( \pi \). Also, the simple identity \( \cos(\pi - \alpha) = -\cos(\alpha) \) can be used to show that the integral over \([0, \pi]\) of \( H_1^+ \) equals that of \( H_1^- \). Consequently,

\[ I_1(n, \hat{r}, r) = 4 \int_0^\pi H_1^-(n, \hat{r}, r, \alpha)d\alpha. \]

To estimate \( T_n(\hat{r}, a_0, r) \), we need to find estimates for the integral \( I_1 \) and for the quadrature error of the shifted trapezoidal rule. We use the following result. A proof for the case of the trapezoidal rule is given in Brass (1977), but it works in the same way for every Riemann sum.

**Lemma 3.2** Let \( f: [a, b] \rightarrow \mathbb{R} \) be a continuous function with bounded variation and \( Q_n \) a quadrature rule which is also a Riemann sum:

\[ Q_n(f) := \sum_{i=0}^{n} f(\xi^{[n]}_i)(x^{[n]}_i - x^{[n]}_{i-1}), \]

where \( a = x^{[n]}_0 < x^{[n]}_1 < \ldots < x^{[n]}_n = b, \xi^{[n]}_i \in [x^{[n]}_{i-1}, x^{[n]}_i] \). Then,

\[ \left| \int_a^b f(x)dx - Q_n(f) \right| \leq \text{Var}(f) \max_{i=1}^n (x^{[n]}_i - x^{[n]}_{i-1}), \]

where \( \text{Var}(f) \) is the variation of \( f \) over \([a, b]\). This implies that

\[ |Q_n(f)| \leq \int_a^b |f(x)|dx + \text{Var}(f) \max_{i=1}^n (x^{[n]}_i - x^{[n]}_{i-1}). \]

Because the shifted trapezoidal rule is a Riemann sum, we can use Lemma 3.2 to estimate \( T_n(\hat{r}, a_0, r) \).

**Lemma 3.3** For the sum \( T_n \), defined in (3.9), we get

\[
T_n(\hat{r}, a_0, r) \leq \frac{8\pi}{\sqrt{A(n, \hat{r}, r)}} K \left( \frac{4\hat{r}r\sqrt{1-\hat{r}^2}\sqrt{1-r^2}}{A(n, \hat{r}, r)} \right) + \frac{2\pi^2}{n} \frac{1}{\sqrt{(\hat{r} - r)^2 + \frac{1}{n^2} + \frac{2}{n^2}(1-\hat{r}r)}},
\]

(3.11)

\[ A(n, \hat{r}, r) := \hat{r}^2 + r^2 - 2\hat{r}r^2 + \frac{1}{n^4} + \frac{2}{n^2} + 2\hat{r}r\sqrt{1-\hat{r}^2}\sqrt{1-r^2}. \]

**Remark 3.4** Here, \( K \) denotes the complete elliptic integral of the first kind; see Abramowitz & Stegun (1965, Section 17.3), Andrews et al. (2001, p. 132) and Byrd & Friedman (1971).
Proof. By Lemma 3.2, we have to calculate $\int_0^\pi H_1^{-}(n, \hat{r}, r, \alpha)\,d\alpha$ and estimate $\text{Var}(H_1^{-}(n, \hat{r}, r, \alpha))$. We start with the integral and introduce the abbreviations

$$\kappa = \frac{1}{n^4} + \frac{2}{n^2},$$

$$a = \hat{r}^2 + r^2 - \hat{r}^2 r^2 + \kappa,$$

$$b = -2\hat{r}r \left(1 + \frac{1}{n^2}\right),$$

$$c = \hat{r}^2 r^2.$$  \hfill (3.12)

Now, we can write

$$\int_0^\pi H_1^{-}(n, \hat{r}, r, \alpha)\,d\alpha = \int_0^\pi \frac{\pi}{\sqrt{a + b \cos(\alpha) + c \cos(\alpha)^2}}\,d\alpha$$

$$= 2\pi \int_0^\infty \frac{1}{\sqrt{a + b \frac{1 - \beta^2}{1 + \beta^2} + c \left(\frac{1 - \beta^2}{1 + \beta^2}\right)^2}} \frac{d\beta}{1 + \beta^2}$$

$$= 2\pi \int_0^\infty \frac{1}{\sqrt{c_1 + b_1 \beta^2 + a_1 \beta^4}}\,d\beta.$$  

Here we have used the substitution

$$\alpha = 2 \arctan(\beta)$$

and have introduced the abbreviations

$$a_1 = a - b + c$$

$$= (\hat{r} + r)^2 + \frac{1}{n^4} + \frac{2}{n^2}(1 + \hat{r}r) \geq \kappa,$$

$$b_1 = 2(a - c)$$

$$= 2(\hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \kappa) \geq \kappa,$$

$$c_1 = a + b + c$$

$$= (\hat{r} - r)^2 + \frac{2}{n^2}(1 - \hat{r}r) + \frac{1}{n^4} \geq \frac{1}{n^4}.$$
Finally, we use the substitution
\[ \gamma = \beta^2 \]
and obtain the following formula for \( I_1(n, \hat{r}, r) \):
\[ I_1(n, \hat{r}, r) = 4\pi \int_0^\infty \frac{1}{\sqrt{\gamma} \sqrt{c_1 + b_1 \gamma + a_1 \gamma^2}} d\gamma. \]

The above method of transforming the integral is the common way of transforming these kind of integrals in order to bring them into a standard form connected to elliptic integrals; see Gradshteyn & Ryzhik (2000, 2.580). Finally, we calculate the zeros \( 0 > \gamma_1 > \gamma_2 \) of
\[ 0 = a_1 \gamma^2 + b_1 \gamma + c_1 \]
and get
\[ \gamma_1 = -\frac{(\hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \kappa) + 2\hat{r} r \sqrt{1 - \hat{r}^2} \sqrt{1 - r^2}}{(\hat{r} + r)^2 + \frac{1}{n^2} + \frac{2}{n^2}(1 + \hat{r} r)}, \]
\[ \gamma_2 = -\frac{(\hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \kappa) - 2\hat{r} r \sqrt{1 - \hat{r}^2} \sqrt{1 - r^2}}{(\hat{r} + r)^2 + \frac{1}{n^2} + \frac{2}{n^2}(1 + \hat{r} r)}. \]

This leads us to
\[ I_1(n, \hat{r}, r) = \frac{4\pi}{\sqrt{a_1}} \int_0^\infty \frac{1}{\sqrt{(\gamma - 0)(\gamma - \gamma_1)(\gamma - \gamma_2)}} d\gamma \]
\[ = \frac{4\pi}{\sqrt{a_1}} \sqrt{-\gamma_2} K \left( \frac{\sqrt{\gamma_1 - \gamma_2}}{-\gamma_2} \right) \]
with the complete elliptic integral of first kind \( K \); see Gradshteyn & Ryzhik (2000, 3.131.8). Plugging in the expressions for \( a_1, \gamma_1 \) and \( \gamma_2 \) gives us the first term in our estimate for the trapezoidal rule in (3.11).

Finally, we calculate the total variation of the function \( H_1^-(n, \hat{r}, r, \alpha) \) over \([0, 2\pi]\). We define the function
\[ f(\alpha) := -2\hat{r} r \left( 1 + \frac{1}{n^2} \right) \cos(\alpha) + \hat{r}^2 r^2 \cos(\alpha)^2. \]
Then, \( f'(\alpha) = 0 \) for \( \alpha \in \{0, \pi, 2\pi\} \), and checking these numbers for \( H_1^- \) we find that \( H_1^-(n, \hat{r}, r, 0) = H_1^-(n, \hat{r}, r, 2\pi) \) is the maximum and \( H_1^-(n, \hat{r}, r, \pi) \) is the minimum. In between, the function is monotone, so the variation is bounded by
\[ \text{Var}(H_1^-(n, \hat{r}, r, \cdot)) = 2(H_1^-(n, \hat{r}, r, 0) - H_1^-(n, \hat{r}, r, \pi)) \leq 2H_1^-(n, \hat{r}, r, 0). \]
Together with a stepwidth of \( \frac{2\pi}{2n+1} \) of the trapezoidal rule, Lemma 3.2 gives the second term in formula (3.11). \( \square \)
4. Bounds for the radial quadrature

Combining (3.4), Lemma 3.3 and the notation

\[ J_1(n, \hat{r}, r) := \frac{8\pi}{\sqrt{A(n, \hat{r}, r)}} K \left( \sqrt{\frac{4\hat{r}r \sqrt{1 - \hat{r}^2} \sqrt{1 - r^2}}{A(n, \hat{r}, r)}} \right), \]  

(4.1)

\[ A(n, \hat{r}, r) := \hat{r}^2 + r^2 - 2\hat{r}^2 r^2 + \frac{1}{n^4} + \frac{2}{n^2} + 2\hat{r} r \sqrt{1 - \hat{r}^2} \sqrt{1 - r^2}, \]  

(4.2)

\[ J_2(n, \hat{r}, r) := \frac{2\pi^2}{\sqrt{(\hat{r} - r)^2 + \frac{1}{n^2} + \frac{2}{n^2}(1 - \hat{r} r)}}, \]  

(4.3)

we can estimate

\[ \|L_n\|_{C(D) \to C(D)} \leq c_1(n) \sum_{i=0}^{n} w_i r_i \left( J_1(n, \hat{r}, r_i) + \frac{1}{n} J_2(n, \hat{r}, r_i) \right) \]

\[ = c_1(n) Q_n^G \left[ r \left( J_1(n, \hat{r}, r) + \frac{1}{n} J_2(n, \hat{r}, r) \right) \right], \]  

(4.4)

where \( Q_n^G \) denotes the \((n + 1)\)-point Gaussian quadrature rule on \([0, 1]\). For these quadrature rules, we use the following result.

**Theorem 4.1** The Gaussian quadrature rules are Riemann sums, and if we denote by \( a < x_0^{[n]} < x_1^{[n]} < \cdots < x_n^{[n]} < b \) the knots of the Gaussian quadrature rule on \([a, b]\), then

\[ \max_{i=1}^{n} (x_i^{[n]} - x_{i+1}^{[n]}) \leq c_2 \frac{b - a}{n}, \]

where \( c_2 > 0 \) is independent of \( n \).

**Proof.** See, e.g. Brass (1977, Theorems 53 and 85) or Szegö (1975, 3.41.1 and 6.21.3). \( \square \)

Theorem 4.1 allows us to use Lemma 3.2 to estimate the right-hand side of (4.4) in a similar way to the proof of Lemma 3.3. We prove the required estimates in the next four lemmata.

**Lemma 4.2** For the function \( J_1 \), defined in formula (4.1), we get

\[ \int_0^1 r J_1(n, \hat{r}, r) dr \leq c_3, \]  

(4.5)

where \( c_3 \) does not depend on \( n \) or \( \hat{r} \).

**Proof.** First, we define the angle \( \psi \in \left[0, \frac{\pi}{4}\right] \) by \( \hat{r} = \cos(\psi) \) and then we substitute \( r = \cos(\phi) \) in the integral in (4.5). We remark that

\[ A(n, \cos(\psi), \cos(\phi)) = \kappa + \sin(\phi + \psi)^2. \]
See (4.2) for the definition of $A$ and (3.12) for the definition of $\kappa$. Using these results and the substitution, we get

$$\int_0^1 r J_1(n, \cos(\psi), r) dr = 8\pi \int_0^{\frac{\pi}{2}} \frac{\cos(\phi) \sin(\phi)}{\sqrt{\kappa + \sin(\psi + \phi)^2}} K\left(\frac{\sin(2\psi) \sin(2\phi)}{\kappa + \sin(\psi + \phi)^2}\right) d\phi.$$ 

For $\phi, \psi \in [0, \frac{\pi}{2}]$, we can estimate

$$\frac{\cos(\phi) \sin(\phi)}{\sqrt{\kappa + \sin(\psi + \phi)^2}} \leq \frac{\cos(\phi) \sin(\phi)}{\sin(\psi) \cos(\phi) + \cos(\psi) \sin(\phi)}$$

$$= \frac{1}{\sin(\psi) + \cos(\psi)}$$

and therefore

$$\int_0^1 r J_1(n, \cos(\psi), r) dr \leq 8\pi \int_0^{\frac{\pi}{2}} K\left(\frac{\sin(2\psi) \sin(2\phi)}{\kappa + \sin(\psi + \phi)^2}\right) d\phi.$$ 

To estimate the integral, we first rewrite the complete elliptic integral as a hypergeometric function (see Andrews et al., 2001, (3.2.3))

$$K(z) = \frac{\pi}{2} F_{2,1} \left(\frac{1}{2}, \frac{1}{2}; 1; z^2\right).$$

Furthermore, the function $F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; \cdot)$ is monotone increasing on $[0, 1]$ and (Andrews et al., 2001, Theorem 2.1.3)

$$\lim_{z \to 1^-} \frac{F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; z)}{-\ln(1-z)} = \frac{1}{\pi}.$$ 

This implies that there is a constant $c_4$ such that

$$F_{2,1} \left(\frac{1}{2}, \frac{1}{2}; 1; z\right) \leq c_4 (1 - \ln(1 - z)), \quad 0 \leq z < 1. \quad (4.6)$$

Using this estimate and the fact that

$$F_{2,1} \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\sin(2\psi) \sin(2\phi)}{\kappa + \sin(\psi + \phi)^2}\right) \leq F_{2,1} \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\sin(2\psi) \sin(2\phi)}{\sin(\psi + \phi)^2}\right)$$
shows us that we need only to bound
\[- \int_0^{\pi/2} \ln \left( 1 - \frac{\sin(2\psi) \sin(2\phi)}{\sin(\psi + \phi)^2} \right) \, d\phi\]
independently of \(\psi\) to finish the proof of the lemma. We define
\[f(\psi, \phi) := \frac{\sin(2\psi) \sin(2\phi)}{\sin(\psi + \phi)^2}, \quad 0 \leq \psi, \phi \leq \frac{\pi}{2}.\]

Then
\[f(\psi, \psi) = 1,\]
\[f(\psi, 0) = f(\psi, \frac{\pi}{2}) = 0.\]

We derive some properties of \(f\):
\[
\frac{\partial f}{\partial \phi}(\psi, \phi) = 2 \frac{\sin(2\psi)}{\sin^3(\psi + \phi)} \sin(\psi - \phi)
\]
\[= 0 \quad \iff \quad \psi = \phi,
\]
so all functions \(f(\psi, \cdot)\) are increasing between 0 and \(\psi\) and then decreasing between \(\psi\) and \(\pi/2\). Furthermore,
\[f \left( \frac{\pi}{2} - \psi, \frac{\pi}{2} - \phi \right) = f(\psi, \phi),\]
so we have only to consider \(\psi \in \left[0, \frac{\pi}{4}\right]\). For \(\varepsilon \in [0, \psi]\), we get
\[f(\psi, \psi - \varepsilon) = \frac{\sin(2\psi) \sin(2\psi - 2\varepsilon)}{\sin(2\psi - \varepsilon)^2}\]
and
\[
\frac{\partial f}{\partial \psi}(\psi, \psi - \varepsilon) = \frac{2 \sin(\varepsilon)}{\sin(2\psi - \varepsilon)^3} [\sin(2\psi) - \sin(2\psi - 2\varepsilon)]
\]
\[\geq 0.
\]
This proves that
\[f(\psi, \psi - \varepsilon) \leq f \left( \frac{\pi}{4}, \frac{\pi}{4} - \varepsilon \right)\]
\[= 1 - \tan(\varepsilon)^2, \quad 0 \leq \varepsilon \leq \psi,
\]
and therefore
\[f(\psi, \phi) \leq 1 - \tan(\psi - \phi)^2, \quad 0 \leq \phi \leq \psi \leq \frac{\pi}{4},\quad (4.7)\]
Now, we study \( \epsilon \in \left[0, \frac{\pi}{4}\right] \). Noting that
\[
f(\psi, \psi + \epsilon) = \frac{\sin(2\psi) \sin(2\psi + 2\epsilon)}{\sin(2\psi + \epsilon)^2},
\]
we get
\[
\frac{\partial f(\psi, \psi + \epsilon)}{\partial \psi} = \frac{2\sin(\epsilon)}{\sin(2\psi + \epsilon)^3} (\sin(2\psi + 2\epsilon) - \sin(2\psi))
\]
\[
= 0 \iff \psi = \frac{\pi}{4} - \frac{\epsilon}{2}.
\]
Therefore,
\[
f(\psi, \psi + \epsilon) \leq f \left( \frac{\pi}{4} - \frac{\epsilon}{2}, \frac{\pi}{4} + \frac{\epsilon}{2} \right)
\]
\[
= \cos(\epsilon)^2
\]
and this proves the estimate
\[
f(\psi, \phi) \leq \cos(\phi - \psi)^2, \quad \psi \leq \phi \leq \psi + \frac{\pi}{4} \quad (4.8)
\]
But \( f(\psi, \phi) \) is monotone decreasing for \( \phi > \psi \), so we also get
\[
f(\psi, \phi) \leq f \left( \psi, \psi + \frac{\pi}{4} \right)
\]
\[
= \cos \left( \frac{\pi}{4} \right)^2
\]
\[
= \frac{1}{2}, \quad \psi + \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2} \quad (4.9)
\]
Using the estimates (4.7–4.9), we get
\[
- \int_0^{\frac{\pi}{2}} \ln(1 - f(\psi, \phi)) d\phi \leq - \int_0^{\psi} \ln(\tan(\psi - \phi)^2) d\phi
\]
\[
- \int_{\psi}^{\psi + \frac{\pi}{4}} \ln(1 - \cos(\phi - \psi)^2) d\phi - \int_{\psi + \frac{\pi}{4}}^{\frac{\pi}{2}} \ln \left( \frac{1}{2} \right) d\phi
\]
\[
\leq -2 \int_0^{\psi} \ln(\tan(\tau)) d\tau - 2 \int_0^{\frac{\pi}{2}} \ln(\sin(\tau)) d\tau + \int_{\psi + \frac{\pi}{4}}^{\frac{\pi}{2}} \ln(2) d\phi
\]
\[
\leq -2 \int_0^{\frac{\pi}{2}} \ln(\tan(\tau)) d\tau - 2 \int_0^{\frac{\pi}{4}} \ln(\sin(\tau)) d\tau + \frac{\pi}{4} \ln(2)
\]
\[
< \infty,
\]
which is a bound for the integral, independent of \( \psi \). This proves the lemma. \( \square \)
LEMMA 4.3 The function $J_2$ defined by (4.3) satisfies

$$\int_0^1 r J_2(n, \hat{r}, r)dr \leq c_5 \ln(n), \quad (4.10)$$

where $c_5$ does not depend on $n$ or $\hat{r}$.

**Proof.** First, rewrite the integral in (4.10):

$$\int_0^1 r J_2(n, \hat{r}, r)dr = 2\pi^2 \int_0^1 \frac{r}{\sqrt{(1 - \hat{r}^2)\kappa + (r - \hat{r}(1 + \frac{1}{n^2}))^2}}dr$$

$$= 2\pi^2 \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{\hat{r}(1 + 1/n^2) + u}{\sqrt{(1 - \hat{r}^2)\kappa + u^2}}du$$

$$= 2\pi^2 \hat{r}(1 + 1/n^2) \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{1}{\sqrt{(1 - \hat{r}^2)\kappa + u^2}}du$$

$$+ 2\pi^2 \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{u}{\sqrt{(1 - \hat{r}^2)\kappa + u^2}}du$$

$$=: K_1(n, \hat{r}) + K_2(n, \hat{r}),$$

where we have used the substitution $u = r - \hat{r}(1 + 1/n^2)$ and the definition of $\kappa$ in (3.12). Note that $K_2(n, \hat{r})$ is bounded:

$$K_2(n, \hat{r}) = 2\pi^2 / \sqrt{(1 - \hat{r}^2)\kappa + u^2} \bigg|_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)}$$

$$= 2\pi^2 \left[ \sqrt{(1 - \hat{r}^2)\kappa + (1 - \hat{r}(1 + 1/n^2))^2} - \sqrt{(1 - \hat{r}^2)\kappa + (-\hat{r}(1 + 1/n^2))^2} \right]$$

$$\leq 2\sqrt{2} \pi^2$$

if we assume that $n \geq 2$; which implies that $\kappa < 1$.

Before we start to estimate $K_1(n, \hat{r})$, we calculate $K_1(n, 1)$:

$$K_1(n, 1) = 2\pi^2 \left(1 + \frac{1}{n^2}\right) \int_{-(1+1/n^2)}^{-1/n^2} \frac{1}{|u|}du$$

$$= 2\pi^2 \left(1 + \frac{1}{n^2}\right) \ln(1 + n^2)$$

$$= O_{n \to \infty}(\ln(n)).$$
Thus, we cannot expect to find a finite bound for the function $K_1$. To estimate $K_1$, we consider three cases:

1. $0 \leq \hat{r} \leq \frac{1}{2 + 1/n^2}$. In this case, the upper limit of the integral for $K_1$ is larger than zero, and in particular, it is larger than $\hat{r}(1 + 1/n^2)$.

2. $\frac{1}{2 + 1/n^2} \leq \hat{r} \leq \frac{1}{1 + 1/n^2}$. This implies that the upper limit in the integral for $K_1$ is still positive. The upper limit is also smaller than $\hat{r}(1 + 1/n^2)$.

3. $\frac{1}{1 + 1/n^2} < \hat{r} < 1$. Now zero is no longer in the interval of integration for $K_1$.

In each case, we are able to find a logarithmic bound for $K_1$.

**Case 1.** Here, we use the fact that $\hat{r} \leq 1/2$, $\kappa > 2/n^2$ and $1 + 1/n^2 \leq 2$ to estimate

$$
K_1(n, \hat{r}) \leq 2\pi^2 \cdot 2 \int_{-1}^{1} \frac{1}{\sqrt{\frac{1}{2} \cdot \frac{2}{n^2} + u^2}}
$$

$$
\leq 4\pi^2 \int_{-1}^{1} \frac{1}{\sqrt{\frac{1}{n^2} + u^2}}
$$

$$
= 4\pi^2 \int_{-n}^{n} \frac{1}{\sqrt{1 + v^2}}
$$

$$
\leq 8\pi^2 \left( \int_{0}^{1} dv + \int_{1}^{n} \frac{1}{v} dv \right)
$$

$$
= O(n \ln(n)).
$$

**Case 2.** In addition to the above estimates for $\kappa$ and $(1 + 1/n^2)$, we use $\hat{r} < 1$ and the fact that now $\hat{r}(1 + 1/n^2) \geq 1 - \hat{r}(1 + 1/n^2)$. It then follows that

$$
K_1(n, \hat{r}) \leq 4\pi^2 \int_{-\hat{r}(1 + 1/n^2)}^{\hat{r}(1 + 1/n^2)} \frac{1}{\sqrt{\frac{1 - \hat{r}^2}{n^2} + u^2}}
$$

$$
\leq 4\pi^2 \int_{-\hat{r}}^{\hat{r}} \frac{1}{\sqrt{1 - \hat{r}^2} + u^2}
$$

$$
\leq 8\pi^2 \int_{0}^{\hat{r}} \frac{1}{\sqrt{1 - \hat{r}^2} + u^2}
$$

where we used the substitution $v = nu/\sqrt{1 - \hat{r}^2}$. It is easy to see that the function $\hat{r} \mapsto \hat{r}/\sqrt{1 - \hat{r}^2}$ is monotone increasing on $[0, 1)$, so the upper limit of the above integral is (in case 2)
smaller than
\[ \frac{2n \hat{r}}{\sqrt{1 - \hat{r}^2}}\bigg|_{\hat{r} = \frac{1}{1 + 1/n^2}} = \frac{2n}{\sqrt{1 - \left(\frac{1}{1 + 1/n^2}\right)^2}} \]
\[ = \frac{2n}{\sqrt{(1 + 1/n^2)^2 - 1}} \]
\[ \leq \frac{2n}{\sqrt{1/n^2}} \]
\[ = 2n^2. \]

Similar to Case 1, we estimate that
\[ K_1(n, \hat{r}) \leq 8\pi^2 \left( \int_0^1 1 \, dv + \int_1^{2n^2} \frac{1}{\nu} \, dv \right) \]
\[ = O(\ln(n)). \]

Case 3. Here, the integral is only over negative numbers. We again use the above-mentioned estimates for \( \kappa \) and \( (1 + 1/n^2) \), and we use \( \hat{r} < 1 \) to derive
\[ K_1(n, \hat{r}) \leq 4\pi^2 \int_{-\hat{r}(1+1/n^2)}^{1-\hat{r}(1+1/n^2)} \frac{1}{\sqrt{1 - \hat{r}^2 + u^2}} \, du \]
\[ = 4\pi^2 \int_{\frac{\hat{r}^2(1+1/n^2)}{1 - \hat{r}^2}}^{\frac{n\hat{r}^2(1+1/n^2)}{1 - \hat{r}^2}} \frac{1}{\sqrt{1 + \nu^2}} \, d\nu. \]

We have again used the substitution \( \nu = nu/\sqrt{1 - \hat{r}^2} \). This time we calculate the integral in order to get our estimate. First, we remember that
\[ \int \frac{1}{\sqrt{1 + \nu^2}} \, d\nu = \ln \left( \nu + \sqrt{1 + \nu^2} \right). \]

Then we get
\[ K_1(n, \hat{r}) \leq 4\pi^2 \ln \left( \frac{f_n(\hat{r})}{g_n(\hat{r})} \right), \]
where
\[ g_n(\hat{r}) := n \left( \hat{r} \left( 1 + \frac{1}{n^2} \right) - 1 \right) + \sqrt{(1 - \hat{r})^2 + n^2 \left( \hat{r} \left( 1 + \frac{1}{n^2} \right) - 1 \right)^2}, \]
\[ f_n(\hat{r}) := n\hat{r} \left( 1 + \frac{1}{n^2} \right) + \sqrt{(1 - \hat{r})^2 + n^2 \hat{r}^2 \left( 1 + \frac{1}{n^2} \right)^2}. \]
The function
\[
\psi(n, \hat{r}) := (1 - \hat{r})^2 + n^2 \left( \hat{r} \left( 1 + \frac{1}{n^2} \right) - 1 \right)^2
\]
has its minimum at
\[
r^* := \frac{n^2 + 1}{n^2 + 1 + \frac{1}{n^2}} \in \left[ \frac{1}{1 + \frac{1}{n^2}}, 1 \right]
\]
and one can derive
\[
\psi(n, r^*) \geq \frac{1}{2n^2}.
\]
This implies that
\[
g_n(\hat{r}) \geq \frac{1}{\sqrt{2n}}.
\]
For \(f_n(r)\), it is easy to see that
\[
1 \leq f_n(\hat{r}) \leq 4n.
\]
So, we finally get
\[
K_1(n, \hat{r}) \leq 4\pi^2 \ln(8n^2).
\]

Using the results from Cases 1–3 together with the estimate for \(K_2\) proves the existence of a constant \(c_5 > 0\) in the statement of the lemma. \(\square\)

In the next two lemmata, we study the total variation of \(r J_1(n, \hat{r}, r)\) and \(r J_2(n, \hat{r}, r)\) on \([0, 1]\).

**Lemma 4.4** There is a constant \(c_6\) independent of \(n\) and \(\hat{r}\) such that
\[
\text{Var}(r J_1(n, \hat{r}, r)) \leq c_6 n \ln(n).
\]

**Proof.** As in the proof of Lemma 4.2, we introduce the notation \(\hat{r} = \cos(\psi), \, \psi \in [0, \pi/2]\); and substitute \(r = \cos(\phi)\); this will not change the maximum values of our function and the monotonicity is just reversed, but that does not change the total variation. So we study the function

\[
f(n, \psi, \phi) := f_1(n, \psi, \phi) f_2(n, \psi, \phi),
\]

\[
f_1(n, \psi, \phi) := \frac{\cos(\phi)}{\sqrt{\kappa + \sin(\psi + \phi)^2}},
\]

\[
f_2(n, \psi, \phi) := K \left( \frac{\sin(2\psi) \sin(2\phi)}{\kappa + \sin(\phi + \psi)^2} \right) = \pi \frac{F_{2,1}}{2} \left( \frac{1}{2}, \frac{1}{2}, 1; \frac{\sin(2\psi) \sin(2\phi)}{\kappa + \sin(\phi + \psi)^2} \right).
\]
See Lemma 4.2 and formula (3.12) for the definition of $\kappa$. Note that we also neglected the constant in the function $r J_1(n, \hat{r}, r)$. The following observation allows us to treat $f_1$ and $f_2$ separately:

$$\text{Var}(f_1 f_2) \leq \| f_1 \|_\infty \text{Var}(f_2) + \| f_2 \|_\infty \text{Var}(f_1).$$

First, we study $f_1$:

$$f_1(n, \psi, 0) = \frac{1}{\sqrt{\kappa + \sin(\psi)^2}},$$

$$f_1\left(n, \psi, \frac{\pi}{2}\right) = 0,$$

$$\frac{\partial f_1(n, \psi, \phi)}{\partial \phi} = -\frac{\kappa \sin(\phi) + \sin(\psi + \phi) \cos(\psi)}{(\kappa + \sin(\psi + \phi))^{3/2}} \leq 0, \quad \psi, \phi \in \left[0, \frac{\pi}{2}\right].$$

This implies that

$$\| f_1 \|_\infty = \text{Var}(f_1) = \frac{1}{\sqrt{\kappa + \sin(\psi)^2}} \leq \frac{1}{\sqrt{\kappa}} \leq n.$$

Now, we turn to $f_2$, and remember that $F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; \cdot)$ is a monotone increasing function on $[0, 1]$, $F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; 0) = 1$ and (see (4.6))

$$F_{2,1}\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) \leq c_4(1 - \ln(1 - z)), \quad 0 \leq z < 1.$$

So, we first have to understand the behaviour of the function inside the logarithmic term:

$$f_3(n, \psi, \phi) := 1 - \frac{\sin(2\psi) \sin(2\phi)}{\kappa + \sin(\phi + \psi)^2},$$

$$f_3(n, \psi, 0) = 1,$$

$$f_3\left(n, \psi, \frac{\pi}{2}\right) = 1,$$

$$\frac{\partial f_3(n, \psi, \phi)}{\partial \phi} = -\frac{2 \sin(2\psi)}{(\kappa + \sin(\psi + \phi)^2)^2} \left[\kappa \cos(2\psi) + \sin(\psi + \phi) \sin(\psi - \phi)\right].$$

It is easy to see that both terms in the square brackets are decreasing. The values range from $\kappa + \sin(\psi)^2 > 0$ to $-\kappa - \cos(\psi)^2 < 0$, so $f_3$ is first decreasing and then increasing. We estimate the
minimum value of $f_3$ (see also the proof of Lemma 4.2):

\[
f_3(n, \psi, \phi) = \frac{\kappa + \sin(\psi + \phi)^2 - \sin(2\psi) \sin(2\phi)}{\kappa + \sin(\psi + \phi)^2} \geq \frac{\kappa}{\kappa + \sin(\psi + \phi)^2} \geq \frac{\kappa}{\kappa + 1} \geq \frac{\kappa}{2} \geq \frac{1}{n^2}
\]

if $n \geq 2$. Together with the monotonicity of $F_{2,1}$, this proves that

\[
\|f_2\|_\infty \leq c_7 \ln(1/\kappa) \leq c_7 \ln(n^2),
\]

\[
\text{Var}(f_2) \leq c_7 \ln(1/\kappa) \leq c_7 \ln(n^2)
\]

with a suitable constant $c_7$ independent of $n$ and $\hat{r} = \cos(\psi)$. This finishes the proof of the lemma. □

**Lemma 4.5** The function $r J_2(n, \hat{r}, r)$, defined in (4.3), satisfies

\[
\text{Var}(r J_2(n, \hat{r}, r)) \leq c_8 n^2,
\]

where $c_8 > 0$ is independent of $n$ and $\hat{r}$.

**Proof.** We define

\[
f(n, \hat{r}, r) := \frac{r}{\sqrt{(\hat{r} - r)^2 + \frac{1}{n^2} + \frac{2}{n^2} (1 - \hat{r} r)}}
\]

to estimate the total variation of $r J_2(n, \hat{r}, r)$, where we neglect the constant in (4.3). We have

\[
f(n, \hat{r}, 0) = 0,
\]

\[
\max_{r \in [0,1]} f(n, \hat{r}, r) \leq n^2,
\]

\[
\frac{\partial f(n, \hat{r}, r)}{\partial r} = \frac{-\hat{r} r (1 + \frac{1}{n^2}) + \hat{r}^2 + \frac{1}{n^2} + \frac{2}{n^2}}{((\hat{r} - r)^2 + \frac{1}{n^2} + \frac{2}{n^2} (1 - \hat{r} r))^{3/2}}.
\]

So we have

\[
\frac{\partial f(n, \hat{r}, r^*)}{\partial r} = 0 \iff r^* = \frac{\hat{r}^2 + \frac{1}{n^2} + \frac{2}{n^2}}{\hat{r} (1 + \frac{1}{n^2})}.
\]
This implies that \( f(n, \hat{r}, \cdot) \) is either increasing and then decreasing, or only increasing on \([0, 1]\). This allows us to conclude that

\[
\Var(f(n, \hat{r}, \cdot)) \leq 2n^2,
\]

which proves the lemma. \(\square\)

**Proof of Theorem 2.1.** Theorem 4.1 shows that we can use Lemma 3.2 to estimate the sum (4.4). This gives us

\[
\|L_n\|_{C(D) \to C(D)} \leq c_1(n) \left[ \int_0^1 r J_1(n, \hat{r}, r) dr + \frac{1}{n} \int_0^1 r J_2(n, \hat{r}, r) dr 
\right.
\]

\[
\left. + \frac{c_2}{n} \Var(r J_1(n, \hat{r}, r)) + \frac{c_2}{n^2} \Var(r J_2(n, \hat{r}, r)) \right]
\]

\[
\leq c_1(n) \left[ c_3 + c_5 \ln(n) + c_6 c_7 \ln(n) + c_8 \right]
\]

\[
= O_{n \to \infty}(n \ln(n)),
\]

where we have used Lemmas 4.2–4.5 and the fact that \( c_1(n) = O_{n \to \infty}(n) \).

**Remark 4.6** The proof shows that \( \|P_n\|_{C(D) \to C(D)} \leq c_3 c_1(n) \) because this is the estimate for the iterated integral.

**5. Numerical examples**

We solve a semilinear Poisson problem of the form

\[
-\Delta u(x) = f(x, u(x)), \quad x \in \overset{\circ}{D},
\]

\[
u(x) = 0, \quad x \in \partial D.
\]  

Let \( G(x; y) \) be the Green’s function where \( x, y \in D \). The solution \( u \) to (5.1) satisfies

\[
u(x) = \int_D G(x, y) f(y, u(y)) dy, \quad x \in D.
\]  

As in Kumar & Sloan (1987), introduce \( v(x) = f(x, u(x)) \). The function \( v \) is a solution of

\[
v(x) = f \left( x, \int_D G(x, y)v(y) dy \right), \quad x \in D.
\]  

This is the equation that we solve using the Galerkin method. After finding \( v \), we calculate

\[
u(x) = \int_D G(x, y)v(y) dy, \quad x \in D.
\]  

Let \( \Pi_n \) denote the space of all polynomials in two variables of degree \( n \) or less, as described in Section 2, and let \( \{A_n: 1 \leq n \leq N := \frac{(n+1)(n+2)}{2} \} \) be a basis for \( \Pi_n \). We choose \( A_n \) to be the 'ridge
polynomials’ introduced by Logan & Shepp (1975). We approximate \( v \) by \( v_n \):

\[
v(x) \approx v_n(x) = \sum_{m=1}^{N} a_m A_m(x).
\]

The Galerkin method for solving (5.3) consists of determining the coefficients \( \{a_m\} \) by solving the nonlinear system

\[
\sum_{m=1}^{N} a_m (A_m, A_n) - \left( f \left( x, \int_D G(x, y)v_n(y)dy \right), A_n \right) = 0
\]

for \( n = 1, \ldots, N \). Since the \( A_n \) are the ridge polynomials, we have

\[
(A_m, A_n) = \delta_{mn}
\]

and

\[
\int_D G(x, y) A_m(y)dy = \Psi_m(x),
\]

where \( \Psi_m \) are polynomials defined in Atkinson & Hansen (2006). The term

\[
\left( f \left( x, \int_D G(x, y)v_n(y)dy \right), A_n \right) = \left( f \left( x, \sum_{m=1}^{N} a_m \Psi_m \right), A_n \right)
\]

is also approximated by

\[
\left( f \left( x, \sum_{m=1}^{N} a_m \Psi_m \right), A_n \right)
\]

as defined by (2.4). Thus, the nonlinear system (5.5) is simplified as

\[
a_n - \left( f \left( x, \sum_{m=1}^{N} a_m \Psi_m \right), A_n \right) = 0 \quad \text{for} \quad n = 1, \ldots, N.
\]

Newton’s method was used to solve the nonlinear system (5.6). For the solution of (5.1),

\[
u_n(x) = \sum_{m=1}^{N} a_m \Psi_m(x).
\]

The first numerical example that we solve is the problem as seen in Atkinson & Hansen (2006). Note that \( D \) is the unit disc in \( \mathbb{R}^2 \):

\[-\Delta u(x) = e^{u(x,y)} + \beta(x, y), \quad x \in \overset{\circ}{D},
\]

\[u(x) = 0, \quad x \in \partial D,
\]
with $\beta(x, y)$ chosen such that the true solution is
\[ u(x, y) = (1 - x^2 - y^2)e^{x \cos y}, \quad (x, y) \in D. \]

In Table 1, we give numerical results for $n = 1, \ldots, 20$. The error was evaluated using a polar coordinate mesh of approximately 2500 points. The linearity of the semilog graph in Fig. 1 shows that the convergence is exponential in $n$. From (2.6) and (2.9), we expect the convergent rate to be faster than $O(n^{-k})$, for every $k \in \mathbb{N}$, if the solution is in $C^\infty(D)$.

The second numerical example that we solve is the Debye–Hückel equation (see Deryagin & Landau (1967))
\[
\begin{align*}
-\Delta \tilde{u}(x, y) &= -\sinh(\bar{u}(x, y)), \quad (x, y) \in \mathring{D}, \\
\tilde{u}(x, y) &= \bar{g}(x, y), \quad (x, y) \in \partial D.
\end{align*}
\] (5.7)

### Table 1 Maximum error in $u_n$

<table>
<thead>
<tr>
<th>Degree</th>
<th>$N$</th>
<th>$|u - u_n|_\infty$</th>
<th>$|\tilde{u} - \tilde{u}<em>n|</em>\infty$</th>
<th>Degree</th>
<th>$N$</th>
<th>$|u - u_n|_\infty$</th>
<th>$|\tilde{u} - \tilde{u}<em>n|</em>\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>$7.33 \times 10^{-1}$</td>
<td>$7.84 \times 10^{-2}$</td>
<td>11</td>
<td>78</td>
<td>$5.93 \times 10^{-7}$</td>
<td>$1.13 \times 10^{-6}$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>$7.61 \times 10^{-2}$</td>
<td>$2.74 \times 10^{-2}$</td>
<td>12</td>
<td>91</td>
<td>$1.42 \times 10^{-7}$</td>
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<tr>
<td>3</td>
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<td>$2.10 \times 10^{-2}$</td>
<td>$7.19 \times 10^{-3}$</td>
<td>13</td>
<td>105</td>
<td>$3.67 \times 10^{-8}$</td>
<td>$1.42 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
<td>$4.92 \times 10^{-3}$</td>
<td>$2.15 \times 10^{-3}$</td>
<td>14</td>
<td>120</td>
<td>$9.53 \times 10^{-9}$</td>
<td>$5.08 \times 10^{-8}$</td>
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<tr>
<td>5</td>
<td>21</td>
<td>$1.44 \times 10^{-3}$</td>
<td>$7.18 \times 10^{-4}$</td>
<td>15</td>
<td>136</td>
<td>$2.26 \times 10^{-9}$</td>
<td>$1.81 \times 10^{-8}$</td>
</tr>
<tr>
<td>6</td>
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<td>$4.04 \times 10^{-4}$</td>
<td>$2.32 \times 10^{-4}$</td>
<td>16</td>
<td>153</td>
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<tr>
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<td>$9.28 \times 10^{-5}$</td>
<td>$7.84 \times 10^{-5}$</td>
<td>17</td>
<td>171</td>
<td>$1.36 \times 10^{-10}$</td>
<td>$2.33 \times 10^{-9}$</td>
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<tr>
<td>8</td>
<td>45</td>
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<td>$2.66 \times 10^{-5}$</td>
<td>18</td>
<td>190</td>
<td>$3.20 \times 10^{-11}$</td>
<td>$8.46 \times 10^{-10}$</td>
</tr>
<tr>
<td>9</td>
<td>55</td>
<td>$8.03 \times 10^{-6}$</td>
<td>$9.30 \times 10^{-6}$</td>
<td>19</td>
<td>210</td>
<td>$7.75 \times 10^{-12}$</td>
<td>$3.06 \times 10^{-10}$</td>
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<tr>
<td>10</td>
<td>66</td>
<td>$2.05 \times 10^{-6}$</td>
<td>$3.24 \times 10^{-6}$</td>
<td>20</td>
<td>231</td>
<td>$1.80 \times 10^{-12}$</td>
<td>$1.10 \times 10^{-10}$</td>
</tr>
</tbody>
</table>

Fig. 1. $\log_{10}(\text{Error})$ versus $n.$
We assume that $\tilde{g}$ is given as a function on $D$. Define

$$u(x, y) = \tilde{u}(x, y) - \tilde{g}(x, y).$$

Then, $u(x, y) = 0$ for $(x, y) \in \partial D$ and

$$-\Delta u = -\Delta (\tilde{u} - \tilde{g}) = -\Delta \tilde{u} + \Delta \tilde{g} = - \sinh(\tilde{u}) + \Delta \tilde{g}$$

$$= - \sinh(\tilde{u}(x, y) - \tilde{g}(x, y) + \tilde{g}(x, y)) + \Delta \tilde{g}(x, y)$$

$$= f(x, y, u(x, y)).$$

Thus, instead of solving the Debye–Hückel equation, we solve (5.1). Then the approximated solution $\tilde{u}_n$ of (5.7) is

$$\tilde{u}_n(x, y) = u_n(x, y) + \tilde{g}(x, y), \quad (x, y) \in D.$$  

As a test case, we choose

$$\tilde{g}(x, y) = \exp \left( x + \frac{y}{\pi} \right).$$

The true solution of $\tilde{u}$ is unknown, so we use $\tilde{u}_{25}$ as our true solution; this is illustrated in Fig. 2.

As in Example 1, we give numerical results for $n = 1, \ldots, 20$ in Table 1. The error was evaluated using a polar coordinate mesh of approximately 2500 points. The linearity of the semilog graph in Fig. 1 shows that the convergence is exponential in $n$, as in Example 1.
REFERENCES


