ONE-DIMENSIONAL WAVE EQUATION

The model initial boundary value problem consists of the PDE

$$\frac{\partial^2 u}{\partial t^2} = a\,\frac{\partial^2 u}{\partial x^2} + f, \quad 0 < x < L,\ t > 0$$

the boundary conditions

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad t \geq 0$$

and the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad 0 \leq x \leq L$$

We will solve the initial boundary value problem for $0 \leq t \leq T$.

The given data are: coefficient $a > 0$, interval lengths $L > 0$ and $T > 0$, function $f(x,t)$ for $0 \leq x \leq L$ and $0 \leq t \leq T$, functions $u_0(x)$ and $v_0(x)$ for $0 \leq x \leq L$, functions $g_1(t)$ and $g_2(t)$ for $0 \leq t \leq T$.

Like for the one-dimensional heat equation, we can consider semi-discrete and fully discrete methods. Here, we focus on a standard fully discrete scheme for the initial value problem of the one-dimensional wave equation.

We use the same notations for the partitions of the spatial and time intervals introduced in solving the one-dimensional heat equation:

$$h_x = L/n_x, \quad x_i = (i-1)\,h_x,\ 1 \le i \le n_x + 1$$
$$h_t = T/n_t, \quad t_k = (k-1)\,h_t,\ 1 \le k \le n_t + 1$$

Denote by u_i^k the finite difference approximation value of $u(x_i, t_k)$.

We use the three point central difference approximations for the second-order partial derivatives:

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_k) \approx \frac{u_{i+1}^k - 2\,u_i^k + u_{i-1}^k}{h_x^2}$$
$$\frac{\partial^2 u}{\partial t^2}(x_i, t_k) \approx \frac{u_i^{k+1} - 2\,u_i^k + u_i^{k-1}}{h_t^2}$$

Then for $2 \le i \le n_x$, $2 \le k \le n_t$, we obtain the difference equation

$$\frac{u_i^{k+1} - 2\,u_i^k + u_i^{k-1}}{h_t^2} = a\,\frac{u_{i+1}^k - 2\,u_i^k + u_{i-1}^k}{h_x^2} + f_i^k$$

These difference equations are supplemented by numerical boundary values

$$u_0^k = g_1(t_k), \quad u_{n_x+1}^k = g_2(t_k)$$

for $1 \le k \le n_t + 1$, and by numerical initial values.

Discretization of the first initial condition

$$u(x,0) = u_0(x), \quad 0 \leq x \leq L$$

is straightforward:

$$u_i^1 = u_0(x_i), \quad 1 \leq i \leq n_x + 1$$

For the second initial condition

$$u_t(x,0) = v_0(x), \quad 0 \leq x \leq L$$

both the forward difference and backward difference lead to first-order accuracy in time stepsize, $O(h_t)$. So we introduce artificial variables u_i^0, $1 \leq i \leq n_x + 1$, intended as approximations of $u(x_i, -h_t)$ when the true solution u is suitably extended for negative t. Then we use

$$\frac{u(x_i, h_t) - u(x_i, -h_t)}{2\, h_t}$$

as an $O(h_t^2)$ approximation of $u_t(x_i, 0)$. So the discretization of the second initial condition is

$$\frac{u_i^2 - u_i^0}{2\, h_t} = v_0(x_i), \quad 1 \leq i \leq n_x + 1$$

With the use of the artificial variables u_i^0, $1 \le i \le n_x + 1$, we need difference equations at $(x_i, 0)$ for $2 \le i \le n_x$. So we require the difference equation

$$\frac{u_i^{k+1} - 2\,u_i^k + u_i^{k-1}}{h_t^2} = a\,\frac{u_{i+1}^k - 2\,u_i^k + u_{i-1}^k}{h_x^2} + f_i^k$$

to be valid also for $k = 1$ (i.e. $t = 0$).

Denote the ratio $\gamma = a\,h_t^2/h_x^2$. Then from the difference equations for $k = 1$:

$$u_i^2 = \gamma\,u_{i-1}^1 + 2\,(1-\gamma)\,u_i^1 + \gamma\,u_{i+1}^1 - u_i^0 + h_t^2 f_i^1$$

and from the discretization of the second initial condition:

$$u_i^2 = u_i^0 + 2\,h_t v_0(x_i)$$

Adding the two equations, we can eliminate u_i^0 to get

$$u_i^2 = \frac{\gamma}{2}\,u_{i-1}^1 + (1-\gamma)\,u_i^1 + \frac{\gamma}{2}\,u_{i+1}^1 + h_t v_0(x_i) + \frac{h_t^2}{2} f_i^1$$

where $u_i^1 = u_0(x_i)$.

Summarizing, we use the following steps to determine the numerical solution:

First,

$$u_i^1 = u_0(x_i), \quad 1 \le i \le n_x + 1$$

Second,

$$\begin{aligned} u_1^2 &= g_1(h_t), \quad u_{n_x+1}^2 = g_2(h_t) \\ u_i^2 &= \frac{\gamma}{2} u_0(x_{i-1}) + (1-\gamma)\, u_0(x_i) \\ &\quad + \frac{\gamma}{2} u_0(x_{i+1}) + h_t v_0(x_i) + \frac{h_t^2}{2} f_i^1 \\ &\qquad 2 \le i \le n_x \end{aligned}$$

Finally, for $k = 2, \cdots, n_t$,

$$\begin{aligned} u_1^{k+1} &= g_1(k\, h_t), \quad u_{n_x+1}^{k+1} = g_2(k\, h_t) \\ u_i^{k+1} &= \gamma u_{i-1}^k + 2\,(1-\gamma)\, u_i^k + \gamma u_{i+1}^k \\ &\quad - u_i^{k-1} + h_t^2 f_i^k \\ &\qquad 2 \le i \le n_x \end{aligned}$$

Stability and Convergence

It can be shown that the stability condition is $\gamma \le 1$, i.e.,

$$\sqrt{a}\, h_t \le h_x$$

This condition is not as restrictive as that for the case of solving the one-dimensional heat equation ($a\, h_t \le h_x^2/2$).

Under the stability condition, when the true solution u has several continuous partial derivatives, a theoretical result on the error bound is

$$\max_{\substack{1\le i\le n_x+1\\ 1\le k\le n_t+1}} |u(x_i, t_k) - u_i^k| = O(h_t^2 + h_x^2)$$

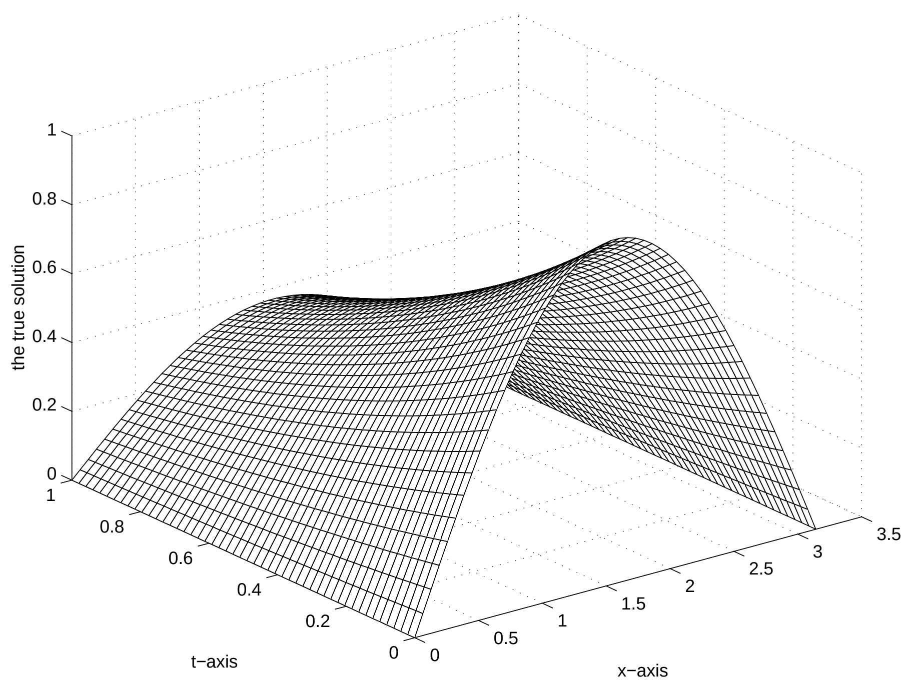
i.e., the scheme is of second order in both x and t stepsizes.

Numerical Example

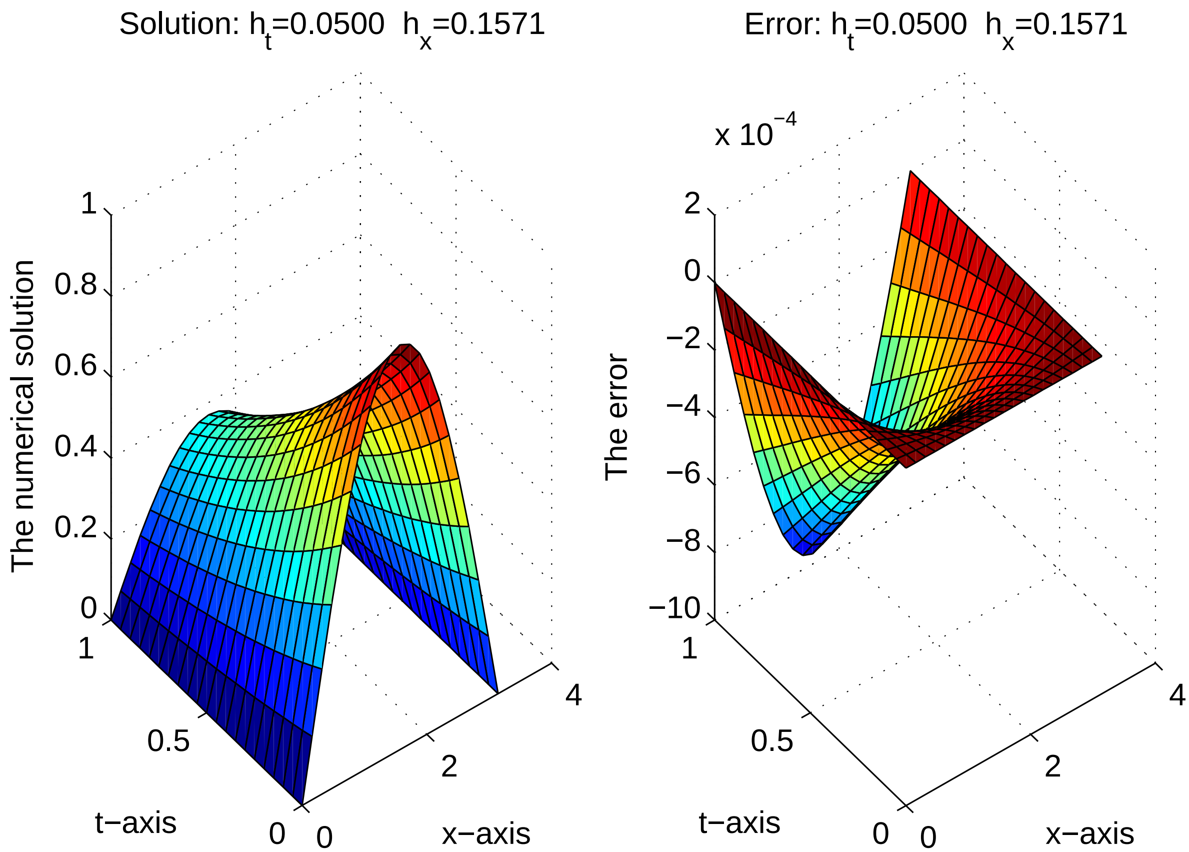
Consider the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + 2\,e^{-t}\sin x, \quad x \in (0,\pi),\ t \in (0,1), \\ u(x,0) = \sin x,\ u_t(x,0) = -\sin x, \quad x \in [0,\pi], \\ u(0,t) = u(\pi,t) = 0, \quad t \in [0,1]. \end{cases}$$

The true solution is $u(x,t) = e^{-t}\sin x$, and is shown in the following figure.



We solve the problem with $(n_x, n_t) = (5, 5)$, $(10, 10)$ and $(20, 20)$. The numerical results with $(n_x, n_t) = (20, 20)$ are shown in the figure.



To see more clearly the error behavior, we provide in the following table the maximum numerical solution errors

$$\max_{1 \le i \le n_x+1} |u(x_i, t_k) - u_i^k|$$

at $t_k = 0.2, 0.4, 0.6, 0.8, 1$. We observe that the ratios are all close to 4, indicating a convergence order of two for the method.

t	$n = 5$	$n = 10$	Ratio	$n = 20$	Ratio
0.2	1.82E−3	4.72E−4	3.87	1.17E−4	4.02
0.4	4.49E−3	1.17E−3	3.85	2.91E−4	4.01
0.6	7.72E−3	2.01E−3	3.84	5.02E−4	4.01
0.8	1.13E−2	2.94E−3	3.83	7.34E−4	4.01
1.0	1.49E−3	3.89E−3	3.83	9.69E−4	4.01