NUMERICAL METHODS FOR ODEs

Consider the initial value problem

$$y' = f(x, y), \quad x_0 \le x \le b, \quad y(x_0) = Y_0$$

and denote its solution by $Y(x)$. Most numerical methods solve this by finding values at a set of *node points*:

$$x_0 < x_1 < \cdots < x_N \le b$$

The approximating values are denoted in this book in various ways. Most simply, we have

$$y_1 \approx Y(x_1), \; \cdots, \; y_N \approx Y(x_N)$$

We also use

$$y(x_i) \equiv y_i, \quad i = 0, 1, ..., N$$

To begin with, and for much of our work, we use a fixed *stepsize* h, and we generate the node points by

$$x_i = x_0 + i\,h, \quad i = 0, 1, ..., N$$

Then we also write

$$y_h(x_i) \equiv y_i \approx Y(x_i), \quad i = 0, 1, ..., N$$

EULER'S METHOD

Euler's method is defined by

$$y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, ..., N-1$$

with $y_0 = Y_0$. Where does this method come from?

There are various perspectives from which we can derive numerical methods for solving

$$y' = f(x, y), \quad x_0 \le x \le b, \quad y(x_0) = Y_0$$

and Euler's method is simplest example of most such perspectives. Moreover, the error analysis for Euler's method is introduction to the error analysis of most more rapidly convergent (and more practical) numerical methods.

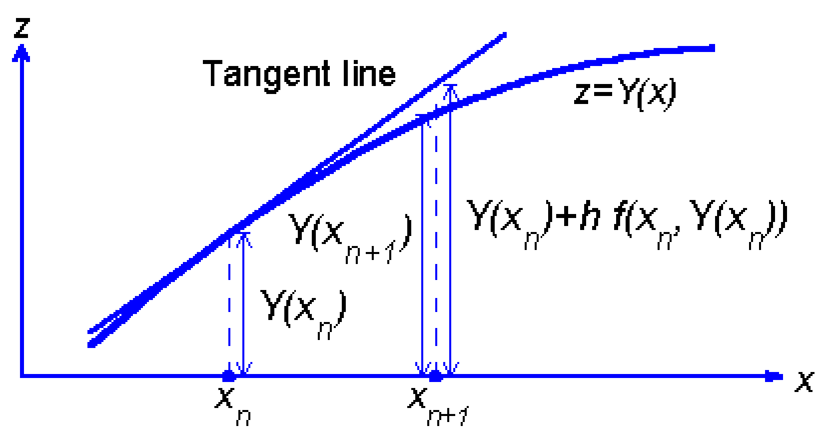
A GEOMETRIC PERSPECTIVE

Look at the graph of $y = Y(x)$, beginning at $x = x_0$. Approximate this graph by the line tangent at $(x_0, Y(x_0))$:

$$\begin{aligned} y &= Y(x_0) + (x - x_0)Y'(x_0) \\ &= Y(x_0) + (x - x_0)f(x_0, Y_0) \end{aligned}$$

Evaluate this tangent line at x_1 and use this value to approximate $Y(x_1)$. This yields Euler's approximation.

We could generalize this by looking for more accurate means of approximating a function, e.g. by using a higher degree Taylor approximation.



An illustration of Euler's method derivation

TAYLOR'S SERIES

Approximate $Y(x)$ about x_0 by a Taylor polynomial approximation of some degree:

$$Y(x_0 + h) \approx Y(x_0) + h\,Y'(x_0) + \frac{h^2}{2!}Y''(x_0) + \cdots + \frac{h^p}{p!}Y^{(p)}(x_0)$$

Euler's method is the case $p = 1$:

$$\begin{aligned} Y(x_0 + h) &\approx Y(x_0) + h\,Y'(x_0) \\ &= y_0 + h\,f(x_0, y_0) \equiv y_1 \end{aligned}$$

We have an error formula for Taylor polynomial approximations; and in this case,

$$Y(x_1) - y_1 = \frac{h^2}{2}Y''(\xi_0)$$

with some $x_0 \le \xi_0 \le x_1$.

GENERAL ERROR FORMULA

In general,

$$y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, ..., N-1$$

$$\begin{aligned} Y(x_{n+1}) &= Y(x_n) + h\, Y'(x_n) + \frac{h^2}{2} Y''(\xi_n) \\ &= Y(x_n) + h\, f(x_n, Y(x_n)) + \frac{h^2}{2} Y''(\xi_n) \end{aligned}$$

with some $x_n \le \xi_n \le x_{n+1}$.

We will use this as the starting point of our error analyses of Euler's method. In particular,

$$\begin{aligned} Y(x_{n+1}) - y_{n+1} &= Y(x_n) - y_n \\ &\quad + h\, [\, f(x_n, Y(x_n)) - f(x_n, y_n)] \\ &\quad + \frac{h^2}{2} Y''(\xi_n) \end{aligned}$$

NUMERICAL DIFFERENTIATION

From beginning calculus,

$$Y'(x_n) \approx \frac{Y(x_n + h) - Y(x_n)}{h}$$

This leads to

$$\begin{aligned} Y(x_n + h) &\approx Y(x_n) + h\,Y'(x_n) \\ &= Y(x_n) + h\ f(x_n, Y(x_n)) \\ &\approx y_n + h\,f(x_n, y_n) \end{aligned}$$

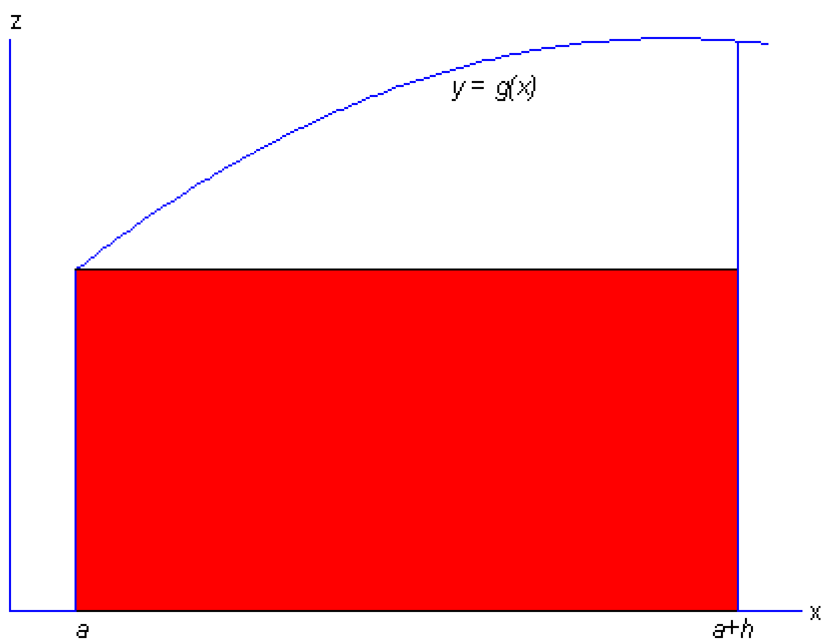
Most numerical differentiation approximations can be used to obtain numerical methods for solving the initial value problem. **However**, most such formulas turn out to be poor methods for solving differential equations. We will see an example of this in the one of the following sections of the book.

NUMERICAL INTEGRATION

Consider the numerical approximation

$$\int_a^{a+h} g(x)\,dx \approx hg(a)$$

which is called the *left-hand rectangle rule*. It is the area of the rectangle with base $[a, a+h]$ and height $g(a)$.



An illustration of the left-hand rectangle rule: $\int_a^{a+h} g(x)\,dx \approx h\,g(a)$

EULER'S METHOD VIA NUMERICAL INTEGRATION

Return to the differential equation $y' = f(x, y)$ and substitute the solution $Y(x)$ for y:

$$Y'(x) = f(x, Y(x))$$

Integrate this over the interval $[x_n, x_{n+1}]$,

$$\int_{x_n}^{x_{n+1}} Y'(x)\,dx = \int_{x_n}^{x_{n+1}} f(x, Y(x))\,dx$$

$$Y(x_{n+1}) = Y(x_n) + \int_{x_n}^{x_{n+1}} f(x, Y(x))\,dx$$

Approximate this with the *left-hand rectangle rule*,

$$Y(x_{n+1}) \approx Y(x_n) + h\,f(x_n, Y(x_n))$$

Again this leads to Euler's method.

EXAMPLE. Solve

$$Y'(x) = \frac{Y(x) + x^2 - 2}{x+1}, \qquad Y(0) = 1$$

The solution is

$$Y(x) = x^2 + 2x + 2 - 2\,(x+1)\log(x+1)$$

We give selected results for three values of h. Note the behaviour of the error as h is halved.

h	x	$y_h(x)$	*Error*	*Relative Error*
0.2	1.0	2.1592	$6.82E-2$	0.0306
	3.0	5.4332	$4.76E-1$	0.0805
	5.0	14.406	1.09	0.0703
0.1	1.0	2.1912	$3.63E-2$	0.0163
	3.0	5.6636	$2.46E-1$	0.0416
	5.0	14.939	$5.60E-1$	0.0361
0.05	1.0	2.2087	$1.87E-2$	0.00840
	3.0	5.7845	$1.25E-1$	0.0212
	5.0	15.214	$2.84E-1$	0.0183