INTRODUCTION TO GEOMETRY OF REPRESENTATIONS OF ALGEBRAS

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Contents

Comments on the lectures and these notes

These notes are provided in support of the author's 5 hour mini-course at the CIMPA research school "Geometric and Homological Methods in the Representation Theory of Associative Algebras and Their Applications", held in Medellín, Colombia during June 2018. The author is very grateful to the organizers for the opportunity to give the course, and meet many talented and motivated students in the process.

The material does not contain any new results and does not attempt to broadly cover the entire field of geometry of representations of finite-dimensional algebras. Quite the opposite, the intention was to choose one theme (representation type of algebras) and present introductory examples and classical results which might inspire students to pursue further studies in geometry of representations of algebras. Only the final lecture arrives at some open questions related to the author's research interests, about extending classical work on general representations of quivers to quivers with relations. Corrections to historical inaccuracies or missed citations are welcome.

The background necessary to get started is just basic linear algebra, including the correspondence between matrices and linear maps, and Jordan canonical form. Some facts and intuitions from algebraic geometry are recalled as needed. An introduction to the theory of quiver representations can be found in many accessible textbooks at this point, for example [ASS06, Sch14, DW17]. The notes of Michel Brion [Bri12] provide a excellent companion to these notes, including a complete proof of Gabriel's theorem, and are freely available at the URL linked in the references. More comprehensive treatments of advanced topics can be found in survey articles such as [Bon98, Rei08, Zwa11, HZ14].

1. Representations of quivers

A "quiver" is just another name for a directed graph, when used in the context of representation theory. Here is the formal definition.

Definition 1.1. A *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 is a set of *vertices*, Q_1 a set of *arrows*, and $s, t\colon Q_1 \to Q_0$ are functions give the *source* and *target* of each arrow, respectively. For an arrow $a \in Q_1$, we can visualize the relation as: $s(a) \xrightarrow{a} t(a)$.

Example 1.2. Here is a quiver with $Q_0 = \{0, 1, 2\}$ and $Q_1 = \{a, b, c\}$. Notice a and b have the same source and target. That is allowed, as are loops (arrows with source equal to their target).

$$\begin{tikzcd} 0 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 1 & 2 \arrow[l, "c"'] \end{tikzcd}$$

□

1.1. **Quivers.** We fix a base field $\Bbbk$ throughout the notes. We assume it is algebraically closed for simplicity though this is not always required. All vector spaces and algebras are over $\Bbbk$, so the field is typically omitted from the notation.

Definition 1.3. A *representation* V of a quiver Q consists of a collection of finite dimensional vector spaces $(V_i)_{i\in Q_0}$, one for each vertex of Q, and a collection of linear maps

$$V_a \colon V_{s(a)} \to V_{t(a)}, \qquad a \in Q_1,$$

one for each arrow of Q. The collection of all representations of Q is denoted $\operatorname{rep}(Q)$.

An element $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ is called a *dimension vector* for Q, and given a representation V, its dimension vector is $\mathbf{d}(V) = (\dim V_i)_{i\in Q_0}$.

Many definitions associated to modules over rings have analogs for quiver representations. We will see why below. The following is the analogue of "submodule".

Definition 1.4. A *subrepresentation* $V' \subseteq V$ is a collection of vector subspaces $(V'_i \subseteq V_i)_{i\in Q_0}$ such that for all $a \in Q_1$, we have $V_a(V_{s(a)}) \subseteq V_{t(a)}$, with each map V'_a being just the restriction of V_a to $V'_{s(a)}$.

The *quotient* of V by a subrepresentation V' is the collection of vector spaces $(V_i/V'_i)_{i\in Q_0}$ with the linear maps induced by those in V (check that these are well-defined).

The analogue of a homomorphism between two representations (of the same quiver) is a collection of maps associated to the vertices that respect the given maps over the arrows, in the following sense.

Definition 1.5. Given two quiver representations $V, W \in \operatorname{rep}(Q)$, a *morphism*

$$\varphi = (\varphi_i)_{i\in Q_0} \colon V \to W$$

in $\operatorname{rep}(Q)$ is a collection of linear maps $\varphi_i \colon V_i \to W_i$ for each $i \in Q_0$ which respect the maps over the arrows in V and W, meaning that

$$\varphi_{t(a)} V_a = W_a \varphi_{s(a)} \qquad \text{for all } a \in Q_1.$$

In other words, the data of V, W and φ gives a square for each arrow of Q, and each of these squares must commute for φ to be a morphism.

The *kernel, image*, and *cokernel* of a morphism $\varphi \colon V \to W$ are taken at each vertex. Check that they are well-defined representations. For example, the kernel of φ is the subrepresentation $\ker\varphi = (\ker\varphi_i)_{i\in Q_0}$.

A morphism is said to be a *monomorphism, epimorphism, isomorphism*, etc. if every φ_i has the corresponding property as a linear map.

Definition 1.6. The collection of all morphism from V to W is denoted $\operatorname{Hom}_Q(V, W)$, and is a subspace of the vector space $\bigoplus_{i\in Q_0} \operatorname{Hom}_{\Bbbk}(V_i, W_i)$. The *endomorphism ring* of a representation V is denoted by $\operatorname{End}_Q(V) := \operatorname{Hom}_Q(V, V)$, and is a finite-dimensional $\Bbbk$-algebra.

We will now briefly explain why quiver representations behave like modules over a ring, although this isn't strictly necessary for the remainder of the course. A *path* in Q has the obvious meaning of a sequence of arrows $a_\ell \cdots a_2 a_1$ such that $s(a_i) = t(a_{i-1})$ for all i, and we extend s, t to paths. Note that the path is read right to left in the same order that functions are evaluated when taking the input on the right. We also include a *trivial path* e_i of length 0 at each vertex $i \in Q_0$ with $s(e_i) = t(e_i) = i$.

Definition 1.7. Given a quiver Q, the *path algebra of* Q, written $\Bbbk Q$, is a vector space with basis consisting of all paths in Q. Multiplication pq of two paths p and q is the concatenation of the paths, whenever that makes sense (i.e. $s(p) = t(q)$), and 0 otherwise. This extends by linearity to make $\Bbbk Q$ an associative $\Bbbk$-algebra.

Exercise 1. *Give a definition of the path algebra* $\Bbbk Q$ *in terms of* $\Bbbk$*-algebra generators and relations. Hint: you only need paths of length 0 and 1 to generate* $\Bbbk Q$*.*

An *oriented cycle* in Q is a path p of length greater than zero in Q such that $s(p) = t(p)$.

Exercise 2. *Prove that* $\Bbbk Q$ *is finite-dimensional if and only if* Q *has no oriented cycles.*

Every representation of Q determines a left $\Bbbk Q$-module and vice versa. This correspondence translates morphisms of representations of Q into $\Bbbk Q$-module homomorphisms, and vice versa. This can be formalized as an equivalence of categories. We mainly stick with the language of representations throughout these notes.

Theorem 1.8. *There is an equivalence of categories* $\mathrm{rep}(Q) \simeq \Bbbk Q\text{-mod}$.

Notice that an arbitrary representation of Q can be explicitly written down in terms of elementary linear algebra, which is not always the case when studying modules over rings in general.

Definition 1.9. Let $V, W \in \mathrm{rep}(Q)$. We define the *direct sum* $V \oplus W$ as the representation with vector space $V_i \oplus W_i$ at each vertex $i \in Q_0$, and direct sum of maps $V_a \oplus W_a$ over each arrow.

$$(V \oplus W)_a : \qquad V_{s(a)} \oplus W_{s(a)} \xrightarrow{\begin{bmatrix} V_a & 0 \\ 0 & W_a \end{bmatrix}} V_{t(a)} \oplus W_{t(a)}$$

We now introduce two kinds of "building blocks" of representations. We use the symbol $0 \in \mathrm{rep}(Q)$ to denote the unique representation of a given Q of dimension vector $(0, 0, ..., 0)$.

Definition 1.10. Let $V \in \mathrm{rep}(Q)$. Then V is called *irreducible*, or *simple*, if it has exactly two subrepresentations, 0 and V itself (in particular, $V \neq 0$). We say V is *indecomposable* if, whenever $V = V_1 \oplus V_2$ for subrepresentations $V_1, V_2 \subseteq V$, either $V_1 = 0$ or $V_2 = 0$ (or both).

Example 1.11. (Take it as an **Exercise** to work out the details using linear algebra and the definitions above.)

(a) Let $Q = \bullet \to \bullet$. There are 3 isomorphism classes of indecomposable representations, represented by:

$$\Bbbk \to 0, \qquad 0 \to \Bbbk, \qquad \Bbbk \xrightarrow{[1]} \Bbbk. \tag{1.12}$$

The first 2 are simple. Note that we don't write a matrix over an arrow when the 0 map is the only possibility. In general, two representations are isomorphic if and only if they have the same dimension vector and their maps have the same rank.

(b) Let $Q = \bullet \to \bullet \to \bullet$. There are 6 isomorphism classes of indecomposable representations, represented by:

$$\begin{gathered} \Bbbk \to 0 \to 0, \qquad 0 \to \Bbbk \to 0, \qquad 0 \to 0 \to \Bbbk \\ \Bbbk \xrightarrow{[1]} \Bbbk \to 0, \qquad 0 \to \Bbbk \xrightarrow{[1]} \Bbbk, \qquad \Bbbk \xrightarrow{[1]} \Bbbk \xrightarrow{[1]} \Bbbk. \end{gathered} \tag{1.13}$$

The first 3 are simple.

(c) Let Q be the quiver with one vertex and one loop at that vertex, hereafter known as *the loop quiver*. A representation consists of a vector space $\Bbbk^n$ and an $n \times n$ matrix. The representation is indecomposable if and only if the matrix is similar (equal under conjugation action of $GL(n)$)

to a single Jordan block, since we assume $\Bbbk$ is algebraically closed. The simple representations are exactly the representations with $n = 1$.

You can also note that the data of the quiver representation is the same data as a module M over the polynomial ring $\Bbbk[x]$ such that $\dim_{\Bbbk} M < \infty$. Since $\Bbbk[x]$ is a principal ideal domain, you can use the classification of finite length modules over PIDs if you know this. $\square$

There is a more computable criterion for indecomposability than the definition.

Theorem 1.14. *A representation is indecomposable if and only if* $\operatorname{End}_Q(V)$ *is a* local ring*, meaning that it has a unique maximal left (equivalently, right) ideal.*

Representations of quivers have the following very nice property. It does not hold for modules over rings in general.

Theorem 1.15. *(Krull-Schmidt Theorem). Each* $V \in \operatorname{rep}(Q)$ *admits a decomposition* $V \simeq V_1 \oplus \cdots \oplus V_r$ *with each* V_i *indecomposable. Furthermore, if*

$$V \simeq V_1' \oplus \cdots \oplus V_s'$$

is another decomposition with each V_j' *also indecomposable, then* $r = s$ *and there is a permutation* σ *of* $\{1, \ldots, r\}$ *such that* $V_i \simeq V_{\sigma(i)}'$ *for all* i*.*

1.2. **Quivers with relations.**

Definition 1.16. Fix a quiver Q. A *relation* r is a $\Bbbk$-linear combination of paths in Q

$$r = \sum_i \lambda_i p_i, \qquad \lambda_i \in \Bbbk, \tag{1.17}$$

such that $s(p_i) = s(p_j)$ and $t(p_i) = t(p_j)$ for all i, j. A pair (Q, R) as above is called a *quiver with relations.*

Let $I = \langle R \rangle$ be the two-sided ideal generated by a set of relations R. Then R (or I) is called *admissible* if the following hold:

(a) for all $r \in R$ written as in (1.17), every p_i has length at least 2;
(b) there exists $N \in \mathbb{Z}_{\geq 0}$ such that I contains all paths of length N.

A pair (Q, R) as above is called a *quiver with admissible relations* or a *bound quiver.*

The following exercise is useful for understanding the significance of conditions (a) and (b) in the previous definition, but note that we do not assume relations are admissible unless we say so.

Exercise 3. *Let* Q *be a quiver,* R *a set of relations (not necessarily admissible), and* $A = \Bbbk Q/\langle R \rangle$ *be the quotient algebra. Prove that if (a) fails to hold, then there is a quiver* Q' *properly contained in* Q*, and set of relations* R' *on* Q' *such that* $A \simeq \Bbbk Q'/\langle R' \rangle$*. Also prove that (b) is equivalent to* A *being finite-dimensional over* $\Bbbk$*.*

Given a finite dimensional algebra A, it may not be isomorphic to one given by a quiver with relations. Even when it is, it may be challenging to find a quiver with *admissible* relations such that $A \simeq \Bbbk Q/\langle R \rangle$.

Exercise 4. *Fix a positive integer* n *and assume* $\Bbbk$ *contains* ζ *a primitive* n^{th} *root of unity (*$\zeta^n = 1$ *in* $\Bbbk$ *and* $\zeta^m \neq 1$ *for* $1 \leq m < n$*). Consider the algebra* $T(n)$ *with generators* g, x *satisfying relations*

$$g^n = 1, \quad x^n = 0, \quad xg = \zeta g x \tag{1.18}$$

(a) Write down a quiver with any relations (not admissible) (Q, R) *such that* $\Bbbk Q/\langle R \rangle \simeq T(n)$*. You can do it with 1 vertex and 2 arrows.*
(b) Do the same but with admissible relations*. The quiver will be much more complicated than in part (a)–it requires* n *vertices.*

Warning: this exercise is probably difficult for most students who have not seen quivers before. Everyone should try it with $n = 2$ (so $\zeta = -1$) using elementary methods. It may be quite difficult for general n for students who have not studied group algebras.

Hint for $n = 2$: find two elements $e_1, e_2 \in T(n)$ satisfying $e_i^2 = e_i$ and $e_1e_2 = e_2e_1 = 0$. These correspond to your vertices and can be written as a linear combination of powers of g. Then for each ordered pair of vertices (i, j), compute e_ixe_j to figure out how many paths, up to relations, go from vertex i to j. From this, maybe you can reverse engineer the number of arrows between any pair of vertices and the relations.

Hint for arbitrary n: find a collection of n orthogonal idempotents by using that $T(n)/\langle x\rangle$ is isomorphic to a group algebra. Then follow the hint above. Start with $n = 2, 3, 4, ...$

We use the following shorthand for composing maps along arrows: let $p = a_\ell \cdots a_1$ be a path in Q and $V \in \operatorname{rep}(Q)$. We write

$$V_p := V_{a_\ell} \circ \cdots \circ V_{a_1}. \tag{1.19}$$

Definition 1.20. We say $V \in \operatorname{rep}(Q)$ *satisfies the relation* $r = \sum_i \lambda_i p_i$ if

$$\sum_i \lambda_i V_{p_i} = 0. \tag{1.21}$$

Let R be a set of relations on Q. We write $\operatorname{rep}(Q, R)$ for the full subcategory of $\operatorname{rep}(Q)$ consisting of representations satisfying all the relations in R, or $\operatorname{rep}(Q, I)$ where $I = \langle R\rangle \subseteq \Bbbk Q$.

Note that a representation of (Q, R) is also a representation of Q. Morphisms in $\operatorname{rep}(Q, R)$ are the same as morphisms in $\operatorname{rep}(Q)$. In the next lemma, we observe some properties which don't depend on which context we use.

Lemma 1.22. *Suppose $V, W \in \operatorname{rep}(Q, R)$. Then every subrepresentation and quotient representation of V satisfies the set of relations R.*

Now we see that representations of quivers with relations can be used to study representations of a quite general class of algebras.

Theorem 1.23. *Suppose $\Bbbk$ is algebraically closed, and let A be an arbitrary finite-dimensional, associative $\Bbbk$-algebra. Then there exists a bound quiver (Q, R) such that A-mod $\simeq \operatorname{rep}(Q, R)$*

Example 1.24. (a) If A is the algebra of upper triangular $n \times n$ matrices, then A is isomorphic to the path algebra of the quiver $\bullet \to \bullet \to \cdots \to \bullet$ with n vertices.
(b) If A is the algebra $\Bbbk[t]/\langle t^n\rangle$, then A is isomorphic to the path algebra of the loop quiver (say the loop is labeled α) with the relation $\alpha^n = 0$.
(c) If A is the full algebra of $n \times n$ matrices, then A-mod $\simeq \Bbbk$-mod (the fundamental example of a Morita equivalence), so the corresponding quiver has one vertex and no arrows. □

With this in mind, we can use quivers with relations to study the following major problem when A is finite-dimensional and associative (and $\Bbbk$ algebraically closed, still).

Central Problem of representation theory of algebras: Classify the (finite-dimensional) representations of a given algebra A (or equivalently, given quiver with relations), up to isomorphism.

We remark that infinite-dimensional representations are also very interesting but not within the scope of these lectures. I recommend the following resources to begin exploring this topic [Rin79, CB98, Rin00]. The Krull-Schmidt theorem reduces classification of (finite-dimensional) quiver representations to classification of indecomposables (all up to isomorphism, of course). For most algebras, no one has any idea how to do this.

2. Representation varieties

In this section we define (affine) representation varieties.

2.1. Quivers without relations.

For positive integers k, l, we let $\mathrm{Mat}(k \times l)$ denote the space of $k \times l$ matrices over $\Bbbk$.

Definition 2.1. Fix a quiver Q and dimension vector $\mathbf{d}$. The *representation space* of (framed) representations of Q of dimension vector $\mathbf{d}$ is

$$\mathrm{rep}(Q, \mathbf{d}) := \bigoplus_{a \in Q_1} \mathrm{Mat}(\mathbf{d}(t(a)) \times \mathbf{d}(s(a))). \tag{2.2}$$

We denote a typical element of $\mathrm{rep}(Q, \mathbf{d})$ by $V = (V_a)_{a \in Q_1}$. Since $\mathbf{d}$ determines a vector space $\Bbbk^{\mathbf{d}(i)}$ associated to each vertex i, and for each arrow the matrix V_a determines a linear map between these spaces, each point of $\mathrm{rep}(Q, \mathbf{d})$ determines a representation of Q with a fixed basis (sometimes called a *framed representation*). Many points of $\mathrm{rep}(Q, \mathbf{d})$ correspond to isomorphic representations of Q.

Example 2.3. (a) Let $Q = \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ and $\mathbf{d} = 2 \to 3 \to 1$. Then we get the space of pairs of matrices

$$\mathrm{rep}(Q, \mathbf{d}) = \left\{ (V_\alpha, V_\beta) = \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \right) \right\} \tag{2.4}$$

where the sizes are fixed, while each entry varies over all elements of $\Bbbk$. Note that $\mathrm{rep}(Q, \mathbf{d}) \simeq \Bbbk^9$ as a vector space.

(b) If Q is the loop quiver, then $\mathrm{rep}(Q, d)$ just the space of $d \times d$ matrices, and isomorphic to $\Bbbk^{d^2}$ as a vector space. □

Note that $\mathrm{rep}(Q, \mathbf{d})$ always has the structure of a vector space since each matrix space has the natural structure of a vector space.

Exercise 5. *Choose any quiver and dimension vector $(Q, \mathbf{d})$ and calculate the dimension of* $\mathrm{rep}(Q, \mathbf{d})$. *Then find a general formula for* $\dim_\Bbbk \mathrm{rep}(Q, \mathbf{d})$ *for any* $(Q, \mathbf{d})$.

Definition 2.5. The *base change group* for $(Q, \mathbf{d})$ is

$$\mathrm{GL}(\mathbf{d}) := \prod_{i \in Q_0} \mathrm{GL}(\mathbf{d}(i)) \tag{2.6}$$

where for any positive integer n, we let $\mathrm{GL}(n)$ denote the general linear group of invertible $n \times n$ matrices over $\Bbbk$.

We denote a typical element of $\mathrm{GL}(\mathbf{d})$ by $g = (g_i)_{i \in Q_0}$. There is a natural left action of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{rep}(Q, \mathbf{d})$ by

$$g \cdot V = (g_i)_{i \in Q_0} \cdot (V_a)_{a \in Q_1} = (g_{t(a)}\, V_a\, g_{s(a)}^{-1})_{a \in Q_1}. \tag{2.7}$$

Proposition 2.8. *Two points $V, V' \in \mathrm{rep}(Q, \mathbf{d})$ are in the same* $\mathrm{GL}(\mathbf{d})$*-orbit if and only if $V \simeq V'$ as representations of Q.*

Proof. For any $g = (g_i)_{i \in Q_0} \in \mathrm{GL}(\mathbf{d})$, we have

$$(g_{t(a)}\, V_a\, g_{s(a)}^{-1})_{a \in Q_1} = (V'_a)_{a \in Q_1} \quad \Longleftrightarrow \quad g_{t(a)}\, V_a = V'_a g_{s(a)} \quad \text{for all } a \in Q_1. \tag{2.9}$$

The existence of g satisfying the equation on the left is the definition of V, V' in the same orbit, while the existence of g satisfying the equations on the right is the definition of $V \simeq V'$ in $\mathrm{rep}(Q)$. □

Therefore, the set of isomorphism classes of representations of Q is in bijection with the union of the sets of orbits in $\operatorname{rep}(Q, \mathbf{d})$, as $\mathbf{d}$ varies over all dimension vectors.

Example 2.10. We continue Example 2.3.
(a) The base change group is $\mathrm{GL}(\mathbf{d}) = \mathrm{GL}(2) \times \mathrm{GL}(3) \times \mathrm{GL}(1)$. A typical element is denoted $g = (g_1, g_2, g_3)$ where g_1 is 2×2, g_2 is 3×3, and g_1 is 1×1, and we have

$$g \cdot V = (g_2 V_\alpha g_1^{-1},\ g_3 V_\beta g_2^{-1}). \tag{2.11}$$

Let us consider in very concrete matrix terms what each factor does to V_α and V_β. Each g_i can be factored into elementary matrices as in a standard linear algebra class, meaning the matrices that represent individual row and column operations (depending which side they act one). The factor $g_1 \in \mathrm{GL}(2)$ acts by a series of column operations on V_α only, since the corresponding vertex only touches the arrow α as the source. The inverse must be taken into account. An elementary matrix in the factor $g_2 \in \mathrm{GL}(3)$ is the most interesting: since the corresponding vertex is both the target of α and sources of β, this factor simultaneously does a row operation on V_α and the inverse column operation on V_β. Then the factor $g_3 \in \mathrm{GL}(1)$ acts just by row operations on V_β.
(b) The base change group is just $\mathrm{GL}(d)$. Since the loop α has the single vertex as both the source and target, the action is $g \cdot V_\alpha = g V_\alpha g^{-1}$. □

Exercise 6. *For any $(Q, \mathbf{d})$, the 1-dimensional normal subgroup of scalar multiples of the identity $\{\lambda 1\}_{\lambda \in \Bbbk} \leq \mathrm{GL}(\mathbf{d})$ acts trivially on $\operatorname{rep}(Q, \mathbf{d})$. Therefore the group action factors through $PGL(\mathbf{d}) := \mathrm{GL}(\mathbf{d})/\{\lambda 1\}_{\lambda \in \Bbbk}$.*

2.2. **Quivers with relations.** Now we consider the effect of adding relations, which requires the language of affine varieties. We only formally introduce the bare minimum of definitions from algebraic geometry. Other facts will be recalled as necessary.

Definition 2.12. A *closed subvariety* of $\Bbbk^n$ is a subset $X \subseteq \Bbbk^n$ for which there exist polynomials $f_1, \dots, f_r \in \Bbbk[x_1, \dots, x_n]$ such that

$$X = \{(c_1, \dots, c_n) \in \Bbbk^n \mid \forall 1 \leq i \leq r : \quad f_i(c_1, \dots, c_n) = 0\}. \tag{2.13}$$

The *Zariski topology* on $\Bbbk^n$ is the topology where the closed sets are exactly the closed subvarieties. The *Zariski topology* on a closed subvariety $X \subseteq \Bbbk^n$ is the subspace topology inherited from $\Bbbk^n$.

Exercise 7. *Show that the axioms of a topology are satisfied in the definition of Zariski topology.*

We will refer to closed subvarieties of $\Bbbk^n$ as simply "varieties" in these notes/lectures. But we warn the reader that there is a much more general meaning of this word. For intuition, we note that if $\Bbbk = \mathbb{C}$, Zariski-closed sets are also closed in the usual topology where we identify $\mathbb{C}^n \approx \mathbb{R}^{2n}$. But the converse is not true.

Exercise 8. *Prove that the only Zariski closed sets of $\Bbbk^1$ are finite sets of points, and all of $\Bbbk^1$. In particular, if $\Bbbk = \mathbb{C}$, the unit disc $\{z \in \mathbb{C} \mid |z| \leq 1\}$ is not closed in the Zariski topology. Its closure is all of $\mathbb{C}$.*

We also remark that the list of polynomials $f_1, \dots, f_r$ does not usually tell much about the geometry of X without more work, because solving systems of nonlinear equations is hard.

Example 2.14. Note that $\Bbbk^n$ is always a closed subvariety of itself by taking an empty list of polynomials. Therefore, $\operatorname{rep}(Q, \mathbf{d}) \simeq \Bbbk^n$ is a variety where $n = \sum_{a \in Q_1} \mathbf{d}(s(a))\mathbf{d}(t(a))$. □

Definition 2.15. Let (Q, R) be quiver with relations and $\mathbf{d}$ a dimension vector for Q. The *representation variety* $\operatorname{rep}(Q, R, \mathbf{d})$ is defined as the set

$$\operatorname{rep}(Q, R, \mathbf{d}) := \{M \in \operatorname{rep}(Q, \mathbf{d}) \mid M \text{ satisfies all relations in } R\}. \tag{2.16}$$

Example 2.17. We continue Examples 2.3 and 2.10.
(a) Keep $(Q, \mathbf{d})$ as before and suppose we put the single relation $R = \{\beta\alpha\}$. Then $\operatorname{rep}(Q, R, \mathbf{d}) = \{(V_\alpha, V_\beta) \in \operatorname{rep}(Q, \mathbf{d}) \mid V_\beta V_\alpha = 0\}$. To see this as a variety, we use the coordinates $a_1, \dots, a_6, b_1, \dots, b_3$ on $\operatorname{rep}(Q, \mathbf{d}) \simeq \mathbb{k}^9$ and multiply out:

$$0 = V_\beta V_\alpha = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} = \begin{bmatrix} b_1a_1 + b_2a_3 + b_3a_5 \\ b_1a_2 + b_2a_4 + b_3a_6 \end{bmatrix}. \tag{2.18}$$

So $\operatorname{rep}(Q, R, \mathbf{d})$ is defined inside $\operatorname{rep}(Q, \mathbf{d})$ as the solution set of the system of two quadratic equations given by setting the polynomials in the two matrix entries at right both equal to 0. We can see that as $\mathbf{d}$ grows, the number of equations will grow.

(b) Consider the relation $R = \{\alpha^2\}$. Then $\operatorname{rep}(Q, R, d) = \{V_\alpha \in \operatorname{Mat}(d \times d) \mid V_\alpha^2 = 0\}$. Writing this out as in part (a), we see it is defined as the solution set of d^2 quadratic equations inside $\operatorname{Mat}(d \times d)$. But we can think of it more conceptually using Jordan canonical form: it is the subset of $\operatorname{Mat}(d \times d)$ consisting of matrices whose Jordan canonical form has all blocks with eigenvalue 0 and no blocks of size larger than 2.

Now consider specifically the case $d = 2$. We get 4 quadratic equations defining $\operatorname{rep}(Q, R, 2)$. If we instead took the relation $R = \{\alpha^3\}$ and $d = 2$, we have $\operatorname{rep}(Q, R, d) = \{V_\alpha \in \operatorname{Mat}(d \times d) \mid V_\alpha^3 = 0\}$, defined by 4 cubic equations. But a 2×2 matrix with cube 0 automatically has square 0, so these different equations actually have the same solution set. In other words $\operatorname{rep}(Q, \{\alpha^2\}, 2) = \operatorname{rep}(Q, \{\alpha^3\}, 2) \subset \operatorname{rep}(Q, 2)$, even though they have different defining equations. □

We refer to the variables in the matrix entries as coordinates on $\operatorname{rep}(Q, R, \mathbf{d})$. We see in the previous example that different collections of equations can be used to define the same variety. It is generally very difficult to find a minimal set of equations of smallest degree defining a given variety.

Exercise 9. *Let (Q, R) be the square quiver with commuting relation. Find equations defining* $\operatorname{rep}(Q, R, \mathbf{d})$ *for* $\mathbf{d} = (1, 2, 3, 1)$ *and* $\mathbf{d} = (2, 2, 2, 2)$.

Exercise 10. *Let R, R' be two sets of relations on a quiver Q which generate the same ideal in $\mathbb{k}Q$. Show that* $\operatorname{rep}(Q, R, \mathbf{d}) = \operatorname{rep}(Q, R', \mathbf{d})$ *for every* $\mathbf{d}$. *If $I = \langle R \rangle$, this justifies the alternate notation* $\operatorname{rep}(Q, I, \mathbf{d})$ *sometimes used.*

From what we have seen above, we get the following.

Proposition 2.19. *For any $(Q, R, \mathbf{d})$, the set* $\operatorname{rep}(Q, R, \mathbf{d})$ *is a closed subvariety of* $\operatorname{rep}(Q, \mathbf{d})$ *which is sent to itself by the action of* $\operatorname{GL}(\mathbf{d})$. *Furthermore, its* $\operatorname{GL}(\mathbf{d})$*-orbits are in bijection with isomorphism classes of representations of (Q, R) of dimension vector* $\mathbf{d}$.

Geometric reformulation of the Central Problem: Classify $\operatorname{GL}(\mathbf{d})$-orbits on the representation varieties $\operatorname{rep}(Q, R, \mathbf{d})$ of a given quiver with relations (Q, R).

Of course, we don't get anything for free by translating a hard problem in terms of new definitions. Indeed, classifying orbits of a group G acting on a variety V is in general a very difficult problem, even when $V = \mathbb{k}^n$ (i.e. the case $R = 0$). What we get from the translation is the opportunity to use new tools. At the 1980 Workshop on Representations of Algebras in Puebla, Mexico [Kra82], Kraft proposed following philosophy to divide the study representation varieties into two subproblems.

Horizontal problem: Find nice $\operatorname{GL}(\mathbf{d})$-stable subsets $U \subset \operatorname{rep}(Q, R, \mathbf{d})$ in which the orbits can be parametrized by continuous data.

For example, "nice" subsets can be representations with a fixed dimension of endomorphism ring. "Parametrize" can mean we associate to U another algebraic variety (even $\mathbb{k}^n$) whose points are in bijection with $\operatorname{GL}(\mathbf{d})$-orbits in U.

Vertical problem: Describe containment of orbit closures by discrete or combinatorial data.

Kraft's key example is a classical one: representations of the loop quiver, Q or the polynomial ring $\Bbbk Q = \mathbb{C}[x]$ (infinite dimensional!). Here, we already know the classification of representations. The example is only meant to illustrate the philosophy.

Fix a dimension n and we find $\operatorname{rep}(Q, n) = \operatorname{Mat}(n \times n)$. We think of $M \in \operatorname{rep}(Q, n)$ both as a matrix and as $\mathbb{C}[x]$-module. Thinking of points as modules, let

$$\operatorname{rep}(Q, n)_d = \left\{ M \mid \dim \operatorname{End}_{\mathbb{C}[x]}(M) = d \right\}, \qquad d \geq 1.$$

so $\operatorname{rep}(Q, n)$ is a disjoint union of these. It can be shown that each of these decomposes into connected components indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ of n. These connected components are called *sheets* $\operatorname{rep}(Q, n)_\lambda$.In the horizontal problem, we fix a sheet and consider the orbits just in this sheet. Kraft showed that the *orbits* (so, isomorphism classes) can be naturally parametrized by $\Bbbk^{\lambda_1}$.

The vertical part asks when one orbit is in the closure of another. This can be roughly thought of as how the sheets glue together. This reduces to the case of nilpotent linear operators. Let M, N be two nilpotent $n \times n$ matrices. Then N is in the closure of the orbit of M if and only if $\operatorname{rank} N^i \leq \operatorname{rank} M^i$ for all i. This can be encoded in purely combinatorial terms.

3. Geometry and representation type

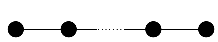
We now survey some notable results on geometry and indecomposable representations which can be stated (though not necessarily proven) using only the elementary notions introduced in the first two sections.

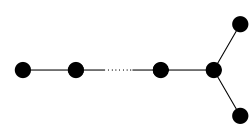
3.1. Quivers without relations.

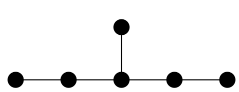
Definition 3.1. A quiver Q is of *finite representation type* if $\operatorname{rep}(Q)$ has only finitely many isomorphism classes of indecomposable objects.

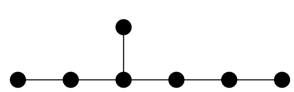
We have seen in previous examples that the quivers $\bullet \to \bullet$ and $\bullet \to \bullet \to \bullet$ are finite representation type, while the one vertex and one loop quiver is not. A quiver which is not of finite representation type is of *infinite representation type.*

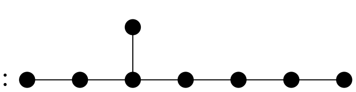
Theorem 3.2 (Gabriel [Gab72]). *A connected quiver Q is of finite representation type if and only if its underlying graph (i.e. ignore the directions of the arrows) is one of the following:*

A_n : 

D_n : 

E_6 : 

E_7 : 

E_8 : 

Note that the subscript indicates the number of vertices, with the first two diagrams indicating infinite families of graphs.

Definition 3.3. The graphs appearing in the theorem are known as the *ADE Dynkin diagrams.* A quiver whose underlying graph is an ADE Dynkin diagram will be called a *Dynkin quiver* for short.

There are a number of very different proofs of Gabriel's theorem. We sketch a proof of the easy direction here using geometry, due to Jacques Tits. First, we make the following elementary observation.

Observation 3.4. *If Q admits a dimension vector $\mathbf{d}$ such that* $\operatorname{rep}(Q, \mathbf{d})$ *has infinitely many orbits, then Q is not of finite representation type. Equivalently, if Q is of finite representation type, then* $\operatorname{rep}(Q, \mathbf{d})$ *has finitely many orbits for any dimension vector* $\mathbf{d}$.

Executive summary of dimension of varieties.

- Every variety has a well defined dimension.
- When the defining equations are linear, it coincides with vector space dimension.
- If $\Bbbk = \mathbb{R}$, then dimension of a variety (solution set of system of polynomials) is what you picture from the "real world". If $\Bbbk = \mathbb{C}$, it is half what you picture, i.e. complex plane or Riemann sphere are 1-d over $\mathbb{C}$. If the solution set is smooth, i.e. a manifold, the dimension as a variety is the same as the manifold dimension.
- For arbitrary variety over arbitrary $\Bbbk$, the dimension can roughly be described as follows. At each point, consider the largest number of parameters from $\Bbbk$ needed to "describe" the variety in a neighborhood of the given point. The dimension is the largest number of such parameters occurring amongst all points of the variety.

We require the following two standard facts.

Fact 1: If $\operatorname{rep}(Q, \mathbf{d})$ has finitely many $\operatorname{GL}(\mathbf{d})$-orbits, one of the orbits must have dimension equal to the dimension of $\operatorname{rep}(Q, \mathbf{d})$. That is, $\operatorname{rep}(Q, \mathbf{d})$ cannot be written as a disjoint union of finitely many closed subvarieties of strictly smaller dimension. Note that here we are using that $\Bbbk$ is infinite, and one can picture the analogous statement for subspaces of $\mathbb{R}^n$ for intuition.

Fact 2: The dimension of any $\operatorname{GL}(\mathbf{d})$-orbit in $\operatorname{rep}(Q, \mathbf{d})$ is $\leq \dim \operatorname{GL}(\mathbf{d}) - 1$. This is because the dimension of an orbit cannot be greater than the dimension of the group that sweeps it out, and the -1 comes from a one dimensional normal subgroup of $\operatorname{GL}(\mathbf{d})$ acting trivially (Exercise 6). The analogous statement for finite groups, where dimension is replaced by cardinality, is well known.

Proof of $\Rightarrow$ direction of Gabriel's Theorem, with some ***details left as exercises****.* Combining Facts 1 and 2, we get

$$\begin{aligned} &Q \text{ of finite representation type} \\ &\Rightarrow \operatorname{rep}(Q, \mathbf{d}) \text{ has finitely many orbits } \forall \mathbf{d} \\ &\Rightarrow \dim \operatorname{GL}(\mathbf{d}) - 1 \geq \dim \operatorname{rep}(Q, \mathbf{d}) \quad \forall \mathbf{d}. \end{aligned} \tag{3.5}$$

With these facts in mind, we consider the following quadratic form on $\mathbb{Z}^{Q_0}$:

$$q_Q(\mathbf{d}) = \dim \operatorname{GL}(\mathbf{d}) - \dim \operatorname{rep}(Q, \mathbf{d}) = \sum_{i \in Q_0} \mathbf{d}(i)^2 - \sum_{a \in Q_1} \mathbf{d}(s(a))\mathbf{d}(t(a)). \tag{3.6}$$

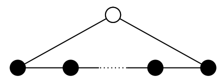
(Note that it does not depend on the directions of the arrows, only the underlying graph of Q.) So (3.5) says:

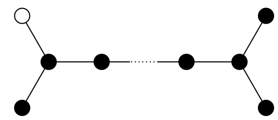
$$\text{If } Q \text{ is of finite representation type, then } q_Q(\mathbf{d}) \geq 1 \text{ for all } \mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}. \tag{3.7}$$

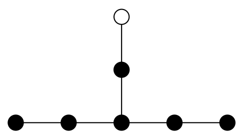
Note this is just a combinatorial statement! We need the results of the following combinatorial exercises.

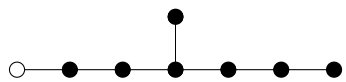
Exercise 11. *Let Q be a quiver and Q' a* subquiver *of Q (i.e. Q' is obtained from Q by deleting some arrows and vertices). Given a dimension vector $\mathbf{d}'$ for Q', let $\mathbf{d}$ be the dimension vector for Q obtained by using 0 at the deleted vertices. Prove that $q_Q(\mathbf{d}) \leq q_{Q'}(\mathbf{d}')$.*

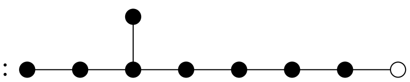
Exercise 12. *Suppose Q is not a Dynkin quiver. Prove that Q contains a subquiver Q' whose underlying graph is one of the following. Hint: you may find it easier to prove the contrapositive,* *using case-by-case analysis.*

$\widetilde{A}_n$: 

$\widetilde{D}_n$: 

$\widetilde{E}_6$: 

$\widetilde{E}_7$: 

$\widetilde{E}_8$: 

These graphs have various names including (ADE) extended Dynkin, affine Dynkin, and Euclidean diagrams. The marked vertex shows how they differ from the corresponding Dynkin diagram by one vertex.

Exercise 13. *For each extended Dynkin quiver/diagram Q', explicitly exhibit a dimension vector $\mathbf{d}'$ such that $q_{Q'}(\mathbf{d}') = 0$.*

Now we prove the half of Gabriel's theorem by contradiction. Suppose Q is of finite representation type, by not a Dynkin quiver. Then by the previous three exercises, Q contains an extended Dynkin quiver Q', which gives a dimension vector $\mathbf{d}$ for Q such that $q_Q(\mathbf{d}) = 0$. This contradicts (3.7). $\square$

Remark 3.8. It is a classical result from Lie theory/combinatorics that a graph giving rise to a quadratic form with this property is an ADE Dynkin diagram. If one knows this, it can be quoted to make the proof much shorter. $\square$

This illustrates how to use only geometry and combinatorics to give a short, conceptual proof of one direction. We never had to compute with explicit representations (in particular, our proof is independent of $\Bbbk$). One very good feature is that the ideas have potential to generalize to quivers with relations.

Remark 3.9. The reader familiar with tame algebras will note that the quivers appearing in Exercise 12 are precisely the tame quivers which are not finite type. In Exercise 13, there is in fact a unique smallest such dimension vector δ_Q for each such quiver Q, and there are infinitely many orbits of indecomposable representations of dimension vector δ_Q, each having endomorphism ring $\Bbbk$. Every dimension vector in which there are infinitely many orbits of indecomposables is a multiple of δ_Q. So the infinite families of indecomposables of the same dimension vector are well understood in this tame case. $\square$

The other direction of Gabriel's theorem requires more work. A (mostly) geometric proof can be found in the summer school notes of Michel Brion [Bri12]. Other methods include elementary linear algebra (Gabriel's original idea [Gab72]), and a categorification of Coxetor combinatorics (Bernstein-Gelfand-Ponomarev reflection functors [BGP73]).

We should mention in passing that V. Kac gave a characterization for arbitrary Q of which dimension vectors $\mathbf{d}$ admit an indecomposable representation, and how many there are. His proof used powerful methods of geometry (in particular, he works over many finite $\Bbbk$ simultaneously, so they are not algebraically closed) and uses invariant theory which are beyond the scope of this mini-course, and even stating his result precisely would take us too far out of the way.

3.2. **Quivers with relations.** We define finite representation type for a quiver with relations (Q, R) in the same way. We can have Q of infinite type but (Q, R) of finite representation type since we restrict the representations of Q we consider. In fact, there is no easy characterization which (Q, R) are representation finite, nor a list of smallest (Q, R) which are representation infinite. But geometric methods (e.g. dimension count as above) can be useful in cases where it is hard to produce explicit infinite families of nonisomorphic representations. Recommended reading includes

Let $A = \Bbbk Q/\langle R\rangle$ and for the rest of this subsection **assume Q has no oriented cycles**. Choose (Q, R) admissible and with a minimal number of elements (not unique!). Let R_{ij} be the set of relations from i to j. It is a fact that $\#R_{ij}$ is uniquely determined by Q. Define a quadratic form on $\mathbb{Z}^{Q_0}_{\geq 0}$ by

$$q_A(\mathbf{d}) = \sum_{i\in Q_0} \mathbf{d}(i)^2 - \sum_{a\in Q_1} \mathbf{d}(s(a))\mathbf{d}(t(a)) + \sum_{i,j\in Q_0} \#R_{ij}\mathbf{d}(i)\mathbf{d}(j). \tag{3.10}$$

Exercise 14. *(For those who know* Ext *now–we will give an elementary characterization of this later, so others can come back and do this later.)*
Prove that if (Q, R) is admissible, then

$$\#\{a \in Q_1 \mid s(a) = i,\ t(a) = j\} = \dim \operatorname{Ext}^1_A(S(i), S(j)). \tag{3.11}$$

It is also true that, when (Q, R) is admissible and R is chosen minimally, we have

$$\#R_{ij} = \dim \operatorname{Ext}^2_A(S(i), S(j)). \tag{3.12}$$

(In particular, $\#R_{ij}$ independent of R as long as $\#R$ minimal.) So $q_A(\mathbf{d})$ can be written in homological terms, kind of a truncated Euler form.

Another interpretation of the new term: a relation from i to j corresponds to a $\mathbf{d}(j) \times \mathbf{d}(i)$ matrix of polynomial entries being set to 0, so each relation gives rise to $\mathbf{d}(j) \times \mathbf{d}(i)$ polynomial equations which must be satisfied by points of $\operatorname{rep}(Q, R, \mathbf{d})$ inside $\operatorname{rep}(Q, \mathbf{d})$. See Example 2.17.

Therefore, the new term in q_A counts the number of equations being used to define $\operatorname{rep}(Q, R, \mathbf{d})$ as a closed subvariety of $\operatorname{rep}(Q, \mathbf{d})$ when R is chosen minimally. (This does not necessarily give the minimal number of equations needed to define $\operatorname{rep}(Q, R, \mathbf{d})$ in $\operatorname{rep}(Q, \mathbf{d})$, that is generally a very hard problem in commutative algebra. A good general reference is the freely available book [AK13], linked in the references.) Krull's height theorem from commutative algebra implies that

$$\dim \operatorname{rep}(Q, R, \mathbf{d}) \geq \sum_{a\in Q_1} \mathbf{d}(s(a))\mathbf{d}(t(a)) - \sum_{i,j\in Q_0} \#R_{ij}\mathbf{d}(i)\mathbf{d}(j). \tag{3.13}$$

Remark 3.14. Intuitively, it says that the drop of dimension from $\operatorname{rep}(Q, \mathbf{d})$ to $\operatorname{rep}(Q, R, \mathbf{d})$ is at most the number of equations. By induction, this reduces to Krull's Principle Ideal theorem, which says that each equation cuts the dimension down by at most 1. □

So we have

$$q_A(\mathbf{d}) \geq \dim \operatorname{GL}(\mathbf{d}) - \dim \operatorname{rep}(Q, R, \mathbf{d}) \tag{3.15}$$

which gives the following purely combinatorial necessary condition for (Q, R) to be finite representation type.

Proposition 3.16. *(a) If A is of finite representation type, then $q_A(\mathbf{d}) \geq 1$ for all $\mathbf{d} \in \mathbb{Z}^{Q_0}_{\geq 0}$.*
(b) If A is of tame representation type, then $q_A(\mathbf{d}) \geq 0$ for all $\mathbf{d} \in \mathbb{Z}^{Q_0}_{\geq 0}$.

Proof. Part (a) follows from the same reasoning as when $R = \varnothing$. Part (b) is straightforward but not within the scope of these notes since we didn't discuss tameness in detail (see [dlPn96]). □

Example 3.17. In these examples we see how geometry leads to purely combinatorial (elementary) check for infinite and even wild type. They are taken from a paper of Sheila Brenner [Bre74].
(a) Consider the quiver with relations and dimension vector shown below.

$$\begin{tikzcd} & & & 2 \arrow[dl] \arrow[dr] \arrow[dd, dashed, no head] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[dr] & & 3 \arrow[dl] & 2 \arrow[l] & 1 \arrow[l] \\ & & & 2 & & & \end{tikzcd}$$

The dashed edge - - - is standard notation for a commuting relation (i.e. the relation that going around the square either way is the same). It can be directly checked from the definition that $q_A(\mathbf{d}) = 0$. Therefore, A is not of finite representation type.
(b) Consider the quiver with relations and dimension vector shown below.

$$\begin{tikzcd} & & & & 2 \arrow[dl] \arrow[dr] \arrow[dd, dashed, no head] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[dr] & & 3 \arrow[dl] & 2 \arrow[l] & 1 \arrow[l] \\ & & & & 2 & & & \end{tikzcd}$$

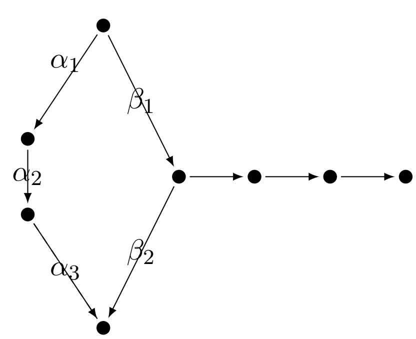
It can be directly checked from the definition that $q_A(\mathbf{d}) = -4$. Therefore, A is of wild representation type. □

There are other ways to find the conclusions of this example, but the point is that this method just uses combinatorics of the graph.

Exercise 15. *Produce your own examples of quivers with relations which you are not sure if they are representation finite or not, then try to prove they are not by finding a* $\mathbf{d}$ *such that* $q_A(\mathbf{d}) \leq 0$*.*

Just as in the case without relations, this is the "easy" direction and only eliminates candidates for finite (or tame) representation type. It cannot be used to prove a quiver is finite or tame representation type. In fact, the converse is not true, in contrast to the case of quivers without relations.

Example 3.18. This example was taken from unpublished lecture notes of de la Peña. Consider the quiver



and two sets of relations:

$$R_1 = \{\alpha_3\alpha_2\alpha_1 - \beta_2\beta_1\}, \qquad R_2 = \{\alpha_3\alpha_2\alpha_1\}.$$

Write $A_i = \Bbbk Q/\langle R_i\rangle$. Since there is one relation between the same two vertices in each case, it is immediate that $q_{A_1} = q_{A_2}$ as quadratic forms. We a lot more work, this form can be factored into a sum of squares even $q_{A_i}(\mathbf{d}) > 0$ for all $\mathbf{d} \in \mathbb{Z}^8_{\geq 0}$. But it turns out that A_1 is finite representation type while A_2 is not. □

The converse can be true if we have some additional assumptions on A. The Auslander-Reiten quiver of an algebra A is said to have a *preprojective component* if it has a connected component in which every indecomposable can be taken to a projective by repeated application of the Auslander-Reiten translation, and furthermore the component has no oriented cycles.

Theorem 3.19 (Bongartz [Bon83]). *Let $A = \Bbbk Q/\langle R\rangle$ as above and assume Q has no oriented cycles. Furthermore assume that the Auslander-Reiten quiver has a preprojective component (e.g. $R = \varnothing$). Then A is representation finite if and only if $q_A(\mathbf{d}) \geq 1$ for all $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$. Furthermore, the map $M \mapsto \underline{dim}(M)$ gives a bijection between indecomposables and roots of $q_A(\mathbf{d})$ (i.e. $\mathbf{d}$ such that $q_A(\mathbf{d}) = 1$).*

The following is an analogue of the above result for tame representation type. We refer the reader to the referenced article for the relevant definitions.

Theorem 3.20 (Brüstle-de la Peña-Skowroński [BdlPS11]). *If A is strongly simply connected, then A is tame if and only if $q_A(\mathbf{d}) \geq 0$ for all $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$.*

3.3. **Finite representation type versus finite orbit type.** Here we make some brief remarks on geometry and a difficult problem in representation theory. The following definition will be temporarily useful.

Definition 3.21. Say a quiver with relations (Q, R) is of *finite orbit type* if $\operatorname{rep}(Q, R, \mathbf{d})$ has finitely many orbits for all dimension vectors $\mathbf{d}$.

Observation 3.4 holds for quivers with relations for the same reason, and says finite representation type implies finite orbit type. A nontrivial geometric consequence of Gabriel's theorem is the converse for quivers without relations:

$$\begin{aligned} & Q \text{ is of finite orbit type} \\ & \Rightarrow Q \text{ is a Dynkin quiver} \\ \text{(harder direction)} \quad & \Rightarrow Q \text{ of finite representation type} \end{aligned}$$

This is not at all obvious since, *a priori*, the dimensions of indecomposables could grow without bound, but each $\operatorname{rep}(Q, \mathbf{d})$ still have only finitely many orbits. But even the harder direction of Gabriel's Theorem could be covered in complete detail in a few lectures, so this is not overwhelmingly difficult.

Could this possibly be true for all (Q, R)? The question becomes *much* more difficult! Again we must worry about the dimensions of indecomposables growing without bound, as in the following example.

Example 3.22. Let $A = \Bbbk[[t]]$ be the algebra of formal power series over $\Bbbk$. Then the finite-dimensional indecomposable representations of A are exactly $A/\langle t^i\rangle$ for $i \geq 1$. Thus there are infinitely many isomorphism classes, but only finitely many (just one) in each dimension. □

But this algebra is infinite-dimensional, and its module category is not equivalent to that of any quiver with relations. The following problem appeared in the work of Jans in the 1950s [Jan57]. It was only resolved in the 1980s (see [NRt75, Bau85, Fis85, BT86]). I am not aware of any proof that could be presented in detail in a few lectures.

Theorem 3.23 (2nd Brauer-Thrall Conjecture). *Recall that we assume $\Bbbk$ is algebraically closed. If a quiver with relations (Q, R) is of infinite representation type, then there exists a dimension vector $\mathbf{d}$ such that (Q, R) has infinitely many indecomposables of dimension vector $\mathbf{d}$, or equivalently, (Q, R) is not of finite orbit type. In fact, there exists infinitely many such $\mathbf{d}$.*

(The statement is actually "if and only if" because the converse is true by Observation 3.4. But we state it this way to emphasize the nontrivial direction.)

4. Relaxations of finite representation type inspired by geometry

Since classifying all representations of a given quiver with relations is generally out of reach of current technology, an alternate approach is to restrict the kind of representations we consider, and try to classify only those. One approach is to work with *general representations*, meaning to restrict to representations in a dense subset of each $\operatorname{rep}(Q, R, \mathbf{d})$ (exactly which subset depends on context). Classical work of Kac [Kac80] and Schofield [Sch92] addresses this problem in the case of quivers without relations. Inspired by the work cited above, the following three finiteness properties for quivers with relations were introduced in joint work with Chindris and Weyman [CKW15]. They depend only on $A = \Bbbk Q/\langle R\rangle$ and not the specific (Q, R), so we can also say an algebra has these properties.

Definition 4.1. A quiver with relations (Q, R) has the *dense orbit property* if, for all $\mathbf{d}$, the variety $\operatorname{rep}(Q, R, \mathbf{d})$ has an dense subset with finitely many orbits.

Informally, we can think of such an algebra as representation finite for general representations. For each $\mathbf{d}$, all infinite families of nonisomorphic representations occur in the "boundary" of $\operatorname{rep}(Q, R, \mathbf{d})$ (i.e. the complement of a finite set of orbits).

Recall that $M \in \operatorname{rep}(Q, R)$ is called *Schur* or a *brick* if $\operatorname{End}_Q(M) = \Bbbk$.

Definition 4.2. We say (Q, R) is *Schur representation finite* if for all $\mathbf{d}$, there are finitely many Schur representations of dimension vector $\mathbf{d}$.

The following problem is open, to my knowledge. If you solve it, please let me know! It is an analogue of the 2nd Brauer-Thrall Conjecture for Schur representations.

Problem 4.3. *Suppose (Q, R) is Schur representation finite in the above sense. Must (Q, R) have only finitely many Schur representations overall? Or can (Q, R) have Schur representations of arbitrarily large dimension?*

The next definition will only be mentioned a few times in these notes for completeness. It is not essential for the reader unfamiliar with Geometric Invariant Theory (GIT). We are in the general situation of wanting to describe the orbits of a group G acting on a variety X (where our group and the action have some nice properties). The most straightforward way of constructing a quotient topological space "X/G" will not have a natural structure of a variety.

Exercise 16. *Let $Q = \bullet \to \bullet$ and $\mathbf{d} = (1, 1)$. Describe the topological quotient of* $\operatorname{rep}(Q, \mathbf{d})$ *by* $\operatorname{GL}(\mathbf{d})$ *and why it is not homeomorphic to a variety.*

GIT is a general technique from algebraic geometry to make the best approximation to a quotient. The survey of Reineke [Rei08] provides an excellent introduction to the subject in the case of quivers without relations. A.D. King's article [Kin94] treating the general case is the original source.

Definition 4.4. We say (Q, R) is *GIT finite* if for all $\mathbf{d}$, each GIT moduli spaces of representations of (Q, R) of dimension vector $\mathbf{d}$ is a finite sets of points.

This is equivalent to the property that is called *multiplicity free* in [CKW15]. The present terminology gives a more intuitive connection to representation theory. The following lemma relates these properties and requires only a little more knowledge of representation varieties beyond what is presented in this course.

Lemma 4.5. *For any quiver with relations (Q, R), we have:*

(4.6)

$$\textit{finite representations type} \Rightarrow \textit{dense orbit property} \Rightarrow \textit{Schur representations finite} \Rightarrow \textit{GIT finite.}$$

Furthermore, by combining classical results it can be seen that these are all equivalent for quivers without relations. This fact generalizes to the following class of (Q, R), since a quiver without relations always has a preprojective component.

Theorem 4.7 ([CKW15])**.** *If the Auslander-Reiten quiver of (Q, R) admits a preprojective component, then (Q, R) has all of the 4 conditions above, or has none of them.*

We conjecture the last 2 are equivalent, and verified this for tame algebras.

Conjecture 4.8. *A quiver with relations is Schur representation finite if and only if it is GIT finite.*

Theorem 4.9 ([CKW15])**.** *Conjecture 4.8 is true for the class of tame algebras.*

The other converses do not hold. We will see this by explicit examples in what follows.

Example 4.10. Consider the quiver with relations

$$Q = \begin{tikzcd} 1\ \bullet \arrow[dd, "c"'] \arrow[dr, "a"] & & \bullet\ 2 \arrow[dl, "b"'] \arrow[dd, "d"] \\ & \overset{3}{\bullet} \arrow[dl, "e"] \arrow[dr, "f"'] & \\ 4\ \bullet & & \bullet\ 5 \end{tikzcd} \tag{4.11}$$

R the set of all length 2 paths: ea, eb, fa, and fb. This example is Schur representation finite but does not have the dense orbit property. Indeed, every infinite family of isomorphism classes of the same dimension vector occurs in a multiple of $\mathbf{d} = (1, 1, 2, 1, 1)$. It can be directly checked that every member of these infinite families admits a nilpotent endomorphism, and thus the algebra is Schur representation finite. But it can also be shown that $\operatorname{rep}(Q, R, \mathbf{d})$ is an irreducible variety with a 1-parameter family of maximal orbits, thus no dense orbit. □

We consider the remaining converse statement in the next section.

4.1. **The dense orbit property.** Further discussion of the dense orbit property requires introducing another fundamental concept of algebraic geometry.

Definition 4.12. A nonempty variety X is said to be *irreducible* if it cannot be written in the form $X = X_1 \cup X_2$ where $X_1, X_2 \subsetneq X$ are proper closed subsets. A maximal irreducible closed subvariety of a given variety Y is an irreducible component of Y.

A basic result in algebraic geometry is that every variety has a well-defined, finite set of irreducible components.

Example 4.13. Continuing earlier examples, we take $Q = \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$ and $R = \{\beta\alpha\}$ and consider $\operatorname{rep}(Q, R, \mathbf{d})$ for various dimension vectors below. The reader can verify the statement in (a) easily, while the statements of the other two are not elementary.
(a) For $\mathbf{d} = (1, 1, 1)$, the variety has 2 irreducible components. They are the closed subvarieties where one of the 1×1 matrices is set to 0 and the other is arbitrary.
(b) For $\mathbf{d} = (2, 3, 1)$ it can be show that the representation variety is irreducible.
(c) For $\mathbf{d} = (n, n, n)$, the representation variety has $n + 1$ irreducible components $C_0, C_1, \ldots, C_n$ where

$$C_r = \{(V_\alpha, V_\beta) \mid \operatorname{rank} V_\alpha \leq r,\ \operatorname{rank} V_\beta \leq n - r\}.$$

□

Generally speaking, it can be very difficult to find the irreducible components of a given variety. The following lemma says that irreducible components are at least preserved by the base change group.

Lemma 4.14. *If $C \subseteq \operatorname{rep}(Q, R, \mathbf{d})$ is an irreducible component, then $\operatorname{GL}(\mathbf{d}) \cdot C = C$.*

Each orbit and each orbit closure of $\mathrm{GL}(\mathbf{d})$ in $\mathrm{rep}(Q, R, \mathbf{d})$ is irreducible. If (Q, R) has the dense orbit property, then every irreducible component of every $\mathrm{rep}(Q, R, \mathbf{d})$ has a unique dense orbit. This associates a well-defined isomorphism class of representations (not necessarily indecomposable) to each of the finitely many irreducible components of each $\mathrm{rep}(Q, R, \mathbf{d})$ when (Q, R) has the dense orbit property. This is why

A very useful result is the geometric Krull-Schmidt decomposition due to de la Peña [dlP91] and Crawley-Boevey–Schröer [CBS02]. It greatly restricts the number of irreducible components of representation varieties which must be studied.

Proposition 4.15. *To check that (Q, R) has the dense orbit property, it is enough to check only that every irreducible component $C \subseteq \mathrm{rep}(Q, R, \mathbf{d})$ which has a dense subset of indecomposable representations has a single dense orbit.*

Remark 4.16. The Artin-Voigt lemma [Voi77] says that if $M \in \mathrm{rep}(Q, R, \mathbf{d})$ has the property that $\mathrm{Ext}^1_{(Q,R)}(M, M) = 0$ (i.e. has no self extensions), then the orbit of M is open in $\mathrm{rep}(Q, R, \mathbf{d})$, and thus dense in the irreducible component of $\mathrm{rep}(Q, R, \mathbf{d})$ in which it lies. The converse is false, however. I do not know any such concise, purely algebraic characterization of when a representation has a dense orbit in an irreducible component. □

The following theorem shows that there are nontrivial examples of dense orbit algebras, meaning ones which are not of finite representation type. It shows that it is in principle possible to classify indecomposable representations with dense orbits even for a wild algebra.

Theorem 4.17 ([CKW15])**.** *Consider the quiver*

$$Q = \begin{tikzcd} \underset{1}{\bullet} \arrow[r, "a"] & \underset{2}{\bullet} \arrow[loop right, "b"] \end{tikzcd} \tag{4.18}$$

and for positive integers $k \leq n$, let $R(k, n) = \{b^n, b^k a\}$. Then $(Q, R(2, n))$ has the dense orbit property for any n, but is wild for $n \geq 7$.

Furthermore, the (finitely many) indecomposables with dense orbits appear precisely with dimension vectors

$$(1, 0),\ (0, m),\ (1, 1),\ (1, m),\ (2, m),\ (1, 1 + n),\ (2, m + n)$$

where $2 \leq m \leq n$ except in the last case, where $2 \leq m \leq n - 2$, and for each dimension vector there is exactly one indecomposable with a dense orbit.

The proof technique is (mostly) elementary but tedious. Here are the main ideas:

(1) By Proposition 4.15, we can reduce to studying $\mathbf{d}$ such that there exists an indecomposable representation of dimension vector $\mathbf{d}$. For example, if $\mathbf{d} \neq (1, 0)$, this forces $\mathbf{d}(1) \leq \mathbf{d}(2)$, since otherwise a simple direct summand supported at vertex 1 splits off.
(2) We show that if $\mathrm{rep}(Q, R, \mathbf{d})$ has an indecomposable representation, then $\mathrm{rep}(Q, R, \mathbf{d})$ is irreducible (this uses slightly more advanced geometry: vector bundles on flag varieties).
(3) Finally we use matrix methods to find explicitly the dense orbit in each such $\mathrm{rep}(Q, R, \mathbf{d})$.

There are many natural questions that arise. The first is a natural place to start before attempting to develop deeper theory.

Problem 4.19. *Construct more examples of dense orbit algebras which are not representation finite.*

It would be nice to classify the dense orbit algebras which are not representation finite. A natural way to go about this is to describe minimal ones. But in contrast to the finite representation type property, we do not know if the dense orbit property is preserved by quotients.

Question 4.20. *Let A be an algebra with the dense orbit property. Does every quotient of A have the dense orbit property?*

Every example or suspected example known to me has at least one loop in the quiver.

Question 4.21. *Does there exist (Q, R) which is representation infinite but has the dense orbit property and Q does not have any oriented cycle?*

We can also take any classical theorem for representations of algebras and ask if it hold for general representations. We note that there is some flexibility in formulating the relevant properties for general representations, so we don't suggest precise versions here. Certainly there should be more examples found before seriously attempting these.

Question 4.22. *Which of the following have true analogues for general relations?*

(a) Tame-wild dichotomy
(b) Brauer-Thrall conjectures

Note that in Theorem 4.17, there are not just finitely many general representations for each $\mathbf{d}$, but finitely many overall. So the analogue of Brauer-Thrall 2 for general representations holds in this example.

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