

Defn: A set  $V$  together with two operations, called addition and scalar multiplication is a **vector space** if the following vector space axioms are satisfied for all vectors  $\mathbf{u}, \mathbf{v}$ , and  $\mathbf{w}$  in  $V$  and all scalars,  $c, d$  in  $R$ .

Vector space axioms:

- a.)  $\mathbf{u} + \mathbf{v}$  is in  $V$
- b.)  $c\mathbf{u}$  is in  $V$
- c.)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- d.)  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- e.) There is a vector, denoted by  $\mathbf{0}$ , in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u}$  in  $V$
- f.) For each  $\mathbf{u}$  in  $V$ , there is an element, denoted by  $-\mathbf{u}$ , in  $V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- g.)  $(cd)\mathbf{u} = c(d\mathbf{u})$
- h.)  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- i.)  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- j.)  $1\mathbf{u} = \mathbf{u}$

Examples:

- 1.)  $R^k$  with the usual operations of addition and scalar multiplication is a vector space.
- 2.) The set  $M^{k,n}$ , the set of all  $k \times n$  matrices with the usual operations of addition and scalar multiplication is a vector

Linear Algebra Review: Eigenvalues and Eigenvectors

Defn:  $\lambda$  is an **eigenvalue** of the linear transformation  $T : V \rightarrow V$  if there exists a nonzero vector  $\mathbf{x}$  in  $V$  such that  $T(\mathbf{x}) = \lambda\mathbf{x}$ . The vector  $\mathbf{x}$  is said to be an **eigenvector** corresponding to the eigenvalue  $\lambda$ .

Example: Let  $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$ .

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Note  $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of  $A$  and  $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$  is a corresponding eigenvector of  $A$ .

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Note  $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of  $A$  and  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  is a corresponding eigenvector of  $A$ .

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Note  $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$  for any  $k$ .

Thus  $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$  is NOT an eigenvector of  $A$ .

MOTIVATION:

$$\text{Note } \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Thus } A \begin{bmatrix} 2 \\ 8 \end{bmatrix} &= A\left(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \end{aligned}$$

Finding eigenvalues:

Suppose  $A\mathbf{x} = \lambda\mathbf{x}$  (Note  $A$  is a SQUARE matrix).

Then  $A\mathbf{x} = \lambda I\mathbf{x}$  where  $I$  is the identity matrix.

$$\text{Thus } \lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$$

Thus if  $A\mathbf{x} = \lambda\mathbf{x}$  for a nonzero  $\mathbf{x}$ , then  $(\lambda I - A)\mathbf{x} = \mathbf{0}$  has a nonzero solution.

$$\text{Thus } \det(\lambda I - A)\mathbf{x} = 0.$$

Note that the eigenvectors corresponding to  $\lambda$  are the nonzero solutions of  $(\lambda I - A)\mathbf{x} = \mathbf{0}$ .

Thus to find the eigenvalues of  $A$  and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$\det(\lambda I - A) = 0 \text{ for } \lambda.$$

Step 2: For each eigenvalue  $\lambda_0$ , find its corresponding eigenvectors by solving the homogeneous system of equations

$$(\lambda_0 I - A)\mathbf{x} = \mathbf{0} \text{ for } \mathbf{x}.$$

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Defn:  $\det(\lambda I - A) = 0$  is the **characteristic equation** of  $A$ .

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

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Defn: The **eigenspace** corresponding to an eigenvalue  $\lambda_0$  of a matrix  $A$  is the set of all solutions of  $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$ .

Note: An eigenspace is a vector space

The vector  $\mathbf{0}$  is always in the eigenspace.

The vector  $\mathbf{0}$  is never an eigenvector.

The number 0 can be an eigenvalue.

Thm: A square matrix is invertible if and only if  $\lambda = 0$  is not an eigenvalue of  $A$ .