Math 34 Differential Equations Exam #2
October 29, 2010 SHOW ALL WORK

[4] 1a.) $\mathcal{L}(0) = \underline{\ 0\ }$

[10] 1b.) $\mathcal{L}^{-1}(\frac{2}{(s-4)^2+5}) = \underline{\ \frac{2}{\sqrt{5}}e^{4t}sin(t\sqrt{5})\ }$

$$\mathcal{L}^{-1}(\frac{2}{(s-4)^2+5}) = \frac{2}{\sqrt{5}}\mathcal{L}^{-1}(\frac{\sqrt{5}}{(s-4)^2+5}) = \frac{2}{\sqrt{5}}e^{4t}sin(t\sqrt{5})$$

[4] 2.) Circle T for True or F for False:

Suppose $y = f(t)$ is a solution to $3y'' + 10y = cos(t)$, $y(0) = 0$, $y'(0) = 0$ and suppose $y = g(t)$ is a solution to $3y'' + 10y = cos(t)$, $y(0) = 100$, $y'(0) = -200$. For large values of t, $f(t) - g(t)$ is very small. **Note: no damping** F

[4] 3a.) Given $2y''+5y = cos(wt)$, determine the value w for which undamped resonance occurs:

$2r^2+5 = 0$. Thus $r = i\sqrt{\frac{5}{2}}$. Hence homogeneous soln is $y(t) = c_1cos(t\sqrt{\frac{5}{2}})+c_2sin(t\sqrt{\frac{5}{2}})$. Hence a potential solution for the non-homogeneous equation $2y''+5y = cos(t\sqrt{\frac{5}{2}})$ would be of the form: $t[Acos(t\sqrt{\frac{5}{2}}) + Bsin(t\sqrt{\frac{5}{2}})]$.

Answer $w = \underline{\sqrt{\frac{5}{2}}}$

[3] 3b.) Briefly describe in words the long-term behaviour of a solution to $2y'' + 5y = cos(wt)$ for this value of w.

The solution oscillates and the pseudo-amplitude gets increasingly larger, approaching infinity.

[15] 4.) A mass of 4 kg stretches a spring 5m. The mass is acted on by an external force of $6e^t$ N (newtons) and moves in a medium that imparts a viscous force of 8 N when the speed of the mass is 15 m/sec. The mass is pulled downward 1m below its equilibrium position, and then set in motion in the upward direction with a velocity of 10 m/sec. Formulate the initial value problem describing the motion of the mass.

$m = 4$. $F_{viscous}(t) = -\gamma v(t)$, where v =velocity. Hence $8 = 15\gamma$ implies $\gamma = \frac{8}{15}$. Also, $mg = kL$. Hence $k = \frac{mg}{L} = \frac{4(9.8)}{5}$.

Answer $\underline{\ 4u''(t) + \frac{8}{15}u'(t) + \frac{4(9.8)}{5}u(t) = 6e^t\ }$

[20] 5.) Use ch 3 methods to solve the given initial value problem.

$$y'' + 4y = sin(t), \quad y(0) = 0, \ \ y'(0) = 0$$

Step 1.) Find the general solution to $y'' + 4y = 0$:

Guess $y = e^{rt}$. Then $r^2 e^{rt} + 4e^{rt} = 0$ implies $\mathrm{r}^2 + 4 = 0$ which implies $r = \pm 2i$.

homogeneous solution: $y(t) = c_1 cos(2t) + c_2 sin(2t)$

Step 2.) Find ONE solution to $y'' + 4y = sin(t)$:

Educated guess: $y = Asin(t)$ (since no y' term).

$y = Asin(t)$ $\qquad$ $y' = Acos(t)$ $\qquad$ $y'' = -Asin(t)$

$-Asin(t) + 4Asin(t) = sin(t)$.

$3Asin(t) = sin(t)$. Hence $3A = 1$ and $A = \frac{1}{3}$.

The general solution to NON-homogeneous equation is

$$c_1 cos(2t) + c_2 sin(2t) + \frac{1}{3} sin(t)$$

Step 3.) Initial value problem:

Once general solution to problem is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

$y(t) = c_1 cos(2t) + c_2 sin(2t) + \frac{1}{3} sin(t)$

$y'(t) = -2c_1 sin(2t) + 2c_2 cos(2t) + \frac{1}{3} cos(t)$

$y(0) = 0$: $0 = c_1$

$y'(0) = 0$: $0 = 2c_2 + \frac{1}{3}$. Hence $2c_2 = -\frac{1}{3}$ and $c_2 = -\frac{1}{6}$

Answer $\underline{\ y(t) = -\frac{1}{6} sin(2t) + \frac{1}{3} sin(t)\ }$

[25] 6.) Use the LaPlace transform to solve the given initial value problem.

$$y'' + 4y = sin(t), \quad y(0) = 0, \;\; y'(0) = 0$$

$\mathcal{L}(y'' + 4y) = \mathcal{L}(sin(t))$

$\mathcal{L}(y'') + 4\mathcal{L}(y) = \frac{1}{s^2+1}$

$s^2\mathcal{L}(y) - sy(0) - y'(0) + 4\mathcal{L}(y) = \frac{1}{s^2+1}$

$s^2\mathcal{L}(y) + 4\mathcal{L}(y) = \frac{1}{s^2+1}$

$\mathcal{L}(y)[s^2 + 4] = \frac{1}{s^2+1}$

$\mathcal{L}(y) = \frac{1}{(s^2+1)(s^2+4)}$. Hence $y = \mathcal{L}^{-1}(\frac{1}{(s^2+1)(s^2+4)})$

Partial Fractions:

$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$

$1 = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 1)$

$1 = As^3 + Bs^2 + 4As + 4B + Cs^3 + Ds^2 + Cs + D$

$1 = (A + C)s^3 + (B + D)s^2 + (4A + C)s + 4B + D$

$A + C = 0$ and $4A + C = 0$.

Hence $C = -A$ and $4A - A = 0$. Hence $3A = 0$ and $A = 0$, $C = 0$.

Alternatively note $A = 0$, $C = 0$ is "obviously a solution" and you only need one (plus it is "obvious" that there is only one solution). Note how "obvious" this is depends on your linear algebra background.

$B + D = 0$ and $4B + D = 1$

Hence $D = -B$ and $4B - B = 1$. Hence $3B = 1$ and $B = \frac{1}{3}$, $D = -\frac{1}{3}$

$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3(s^2+1)} + \frac{-1}{3(s^2+4)}$

$y = \mathcal{L}^{-1}(\frac{1}{(s^2+1)(s^2+4)}) = \mathcal{L}^{-1}(\frac{1}{3(s^2+1)} + \frac{-1}{3(s^2+4)}) = \frac{1}{3}\mathcal{L}^{-1}(\frac{1}{s^2+1}) - \frac{1}{3}\mathcal{L}^{-1}(\frac{1}{s^2+4})$

$= \frac{1}{3}sin(t) - \frac{1}{6}\mathcal{L}^{-1}(\frac{2}{s^2+4}) = \frac{1}{3}sin(t) - \frac{1}{6}sin(2t)$

Answer $\underline{\frac{1}{3}sin(t) - \frac{1}{6}sin(2t)}$

[15] 7.) Prove that if $F(s) = \mathcal{L}(f(t))$ exists for $s > a \geq 0$, and if c is a positive constant, then $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$ with domain $s > a$.

Hint: $\int_0^\infty h(t)dt = \int_0^c h(t)dt + \int_c^\infty h(t)dt$ and use u-substitution (let $u = t - c$).

Proof: If the integral $\int_0^\infty e^{-st}u_c(t)f(t-c)dt$ exists, then

$\mathcal{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-st}u_c(t)f(t-c)dt$

$= \int_0^c e^{-st}u_c(t)f(t-c)dt + \int_c^\infty e^{-st}u_c(t)f(t-c)dt$

$= \int_0^c e^{-st} \cdot 0 \cdot f(t-c)dt + \int_c^\infty e^{-st} \cdot 1(t-c)dt$

$= 0 + \int_c^\infty e^{-st}f(t-c)dt$

Let $u = t - c$, then $du = dt$ and $t = u + c$. When $t = c$, $u = c - c = 0$

$\int_c^\infty e^{-st}f(t-c)dt$

$= \int_0^\infty e^{-s(u+c)}f(u)du$

$= \int_0^\infty e^{-su}e^{-sc}f(u)du$

$= e^{-sc}\int_0^\infty e^{-su}f(u)du$ since e^{-sc} is a constant with respect to u.

$= e^{-sc}\mathcal{L}(f(u))$

$= e^{-sc}\mathcal{L}(f(t))$

Note $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-su}f(u)du$ exists for $s > a$.

Hence $\mathcal{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-st}u_c(t)f(t-c)dt$ exists for $s > a$ and $\mathcal{L}(u_c(t)f(t-c)) = e^{-sc}\mathcal{L}(f(t))$.