Math 34 Differential Equations Exam #1
September 24, 2010 SHOW ALL WORK

[5] 1a.) Define: A function f is linear if

$f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$ where a, b are scalars (or real numbers or complex numbers for this class).

Circle T for True or F for False:

[3] 1b.) If $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to a second order homogeneous differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution.

F

[3] 1c.) If $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to a second order linear homogeneous differential equation, then $c_1\phi_1 + c_2\phi_2$ is also a solution.

T

[3] 1d.) $ln(t)y'' - \frac{y'}{t} + y\sqrt{t} = e^t cos(t)$ is a second order linear differential equation.

T

[3] 1e.) If p, and g are continuous, then there exists a unique solution to

$$y' + p(t)y = g(t),\ y(0) = 2.$$

T

[3] 1f.) A first order linear differential equation has a unique solution such that $y(0) = 2$.

F

Choose 4 problems from problems 2 - 6. You may do all the problems for up to 4 pts extra credit. If you do not choose your best 4 problems, I will substitute your extra problem for your lowest scoring problem, but with a 3 point penalty (if it improves your grade).

Circle the numbers corresponding to your 4 chosen problems: 2 3 4 5 6

Extra credit problem (choose 1 from problems 2 - 6): ______________

[16] 2a.) Match the following differential equation to its direction field. Indicate all equilibrium solutions (if any) and state whether **stable, unstable or semi-stable.** If a differential equation has no equilibrium solutions, state so.

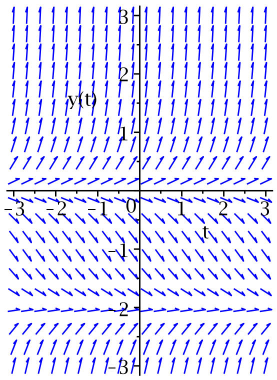
E = I.) $y' = 1 - y$

D = II.) $y' = -1 + y$

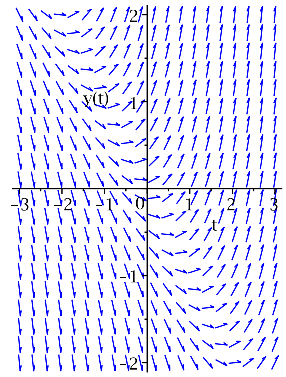
A = III.) $y' = y(y + 2)$

C = IV.) $y' = y^2(2 - y)$

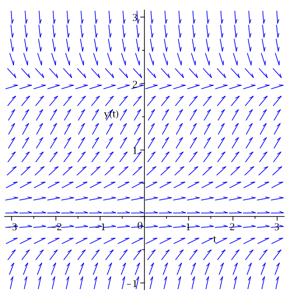
B = V.) $y' = t + y$



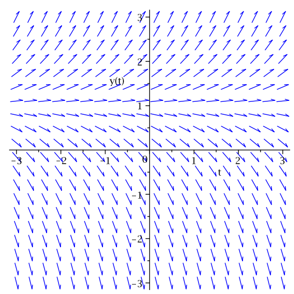
III = A.)
stable: y = -2; unstable: y = 0



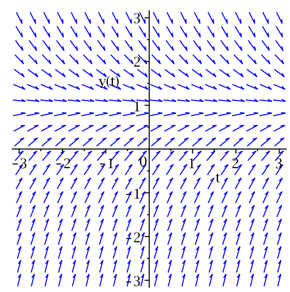
V = B.)
no equilibrium soln.



IV = C.)
semi-stable: y = 0; stable: y = 2



II = D.)
unstable: y = 1



I = E.)
stable y = 1

[4] 2b.) Match the following differential equation initial value problem to its graph:

C = I.) $y'' + y' + 49y = 0$, $y(0) = 0$, $y'(0) = 5$.

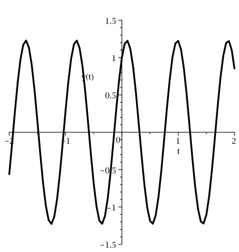
D = II.) $y'' + y' + 49y = 0$, $y(0) = 1$, $y'(0) = 5$

$r^2 + r + 49 = 0$ implies $r = a \pm bi$
General solution:
$y = e^{at}[c_1 cos(bt) + c_2 sin(bt)]$
Note a is negative

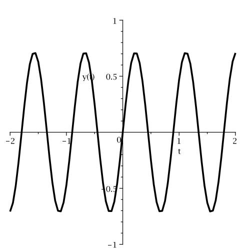
B = III.) $y'' + 49y = 0$, $y(0) = 0$, $y'(0) = 5$

A = IV.) $y'' + 49y = 0$, $y(0) = 1$, $y'(0) = 5$

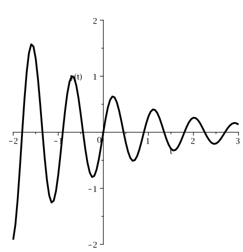
$r^2 + 49 = 0$ implies $r = \pm 7i$
(NOTE: no damping)
General solution:
$y = c_1 cos(bt) + c_2 sin(bt)$



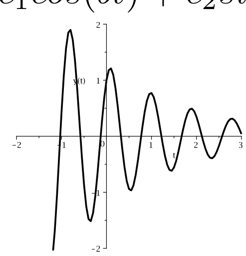
IV = A.)



III = B.)



I = C.)



II = D.)

3.) Solve the differential equation $t^3y' + 3t^2y = \frac{ln(e)}{t^2-4}$. Simplify your answer.

Note this is a first order LINEAR differential equation. Hence you can use an integrating factor in order to write the LHS as the derivative of a product.

Shortcut for this problem: Note the LHS is already the derivative of a product

$t^3y' + 3t^2y = \frac{ln(e)}{t^2-4}$.

$(t^3y)' = \frac{1}{t^2-4}$

$\int (t^3y)'dt = \int \frac{1}{t^2-4}dt$

$t^3y = \int \frac{1}{t^2-4}dt$

$\int \frac{1}{t^2-4}dt = \int [\frac{1}{4(t-2)} - \frac{1}{4(t+2)}]dt = \frac{1}{4}ln|t-2| - \frac{1}{4}ln|t+2| + C$

$= \frac{1}{4}(ln|t-2| - ln|t+2|) + C$

$= \frac{1}{4}ln|\frac{t-2}{t+2}| + C$

Hence $t^3y = \frac{1}{4}ln|\frac{t-2}{t+2}| + C$

Hence $y = \frac{1}{4t^3}ln|\frac{t-2}{t+2}| + Ct^{-3}$

Note to integrate the RHS, we needed to use partial fractions:

$\frac{A}{t-2} + \frac{B}{t+2} = \frac{1}{t^2-4}$

$A(t+2) + B(t-2) = 1$

$At + 2A + Bt - 2B = 1$

$t(A+B) + 2A - 2B = 1$

Thus $A + B = 0$, $2A - 2B = 1$. Thus $A = \frac{1}{4}$, $B = -\frac{1}{4}$

Answer: $y = \frac{ln|\frac{t-2}{t+2}|+C}{4t^3}$

4.) Solve $\frac{y''}{y'} - \frac{1}{y^2} = 0, \;\; y(2) = 1, \;\; y'(2) = -1$

Non-linear 2nd order differential equation. Hence you have only one option. Since you have no idea how to solve a non-linear 2nd order differential equation, you must transform it into something you can solve: a first order differential equation.

Let $v = y'$, $v' = y''$

Hence $\frac{y''}{y'} - \frac{1}{y^2} = 0$ becomes $\frac{v'}{v} - \frac{1}{y^2} = 0$

Now we have a first order differential equation, but it involves 3 variables. Since $v' = \frac{dv}{dt}$, our equation involves the variables v, t, and y. Fortunately we know how to eliminate one of these variables:

Recall $v = y' = \frac{dy}{dt}$. Hence $v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Thus the equation $\frac{v'}{v} - \frac{1}{y^2} = 0$ becomes $\frac{\frac{dv}{dy}v}{v} - \frac{1}{y^2} = 0$

We can simplify and separate variables.

$\frac{dv}{dy} = \frac{1}{y^2}$

$\int dv = \int y^{-2} dy$

$y' = v = -y^{-1} + c_1$

$y(2) = 1, \;\; y'(2) = -1$: when $t = 2$, $-1 = -1 + c_1$. Thus $c_1 = 0$

$\frac{dy}{dt} = -y^{-1}$

$\int y dy = \int -dt$

$\frac{1}{2}y^2 = -t + c_2$

$y^2 = -2t + c_2$

$y = \pm\sqrt{-2t + c_2}$

$y(2) = 1$: $1 = \pm\sqrt{-4 + c_2}$. Thus $c_2 = 5$ and $\quad y = \sqrt{-2t + 5}$

Answer: $\underline{\;y = \sqrt{-2t+5}\;}$

5.) A mass of 10 kg stretches a spring 9.8m. The mass is pushed upward, contracting the spring a distance of one meter and set in motion with an upward velocity of 4 m/sec. If the mass moves in a medium that imparts a viscous force of 100 N when the speed of the mass is 5 m/sec, find the equation of motion of the mass.

$$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$$
$$mg - kL = 0, \qquad F_{damping}(t) = -\gamma u'(t)$$

m= mass,
k = spring force proportionality constant,
γ = damping force proportionality constant

$m = 10$,

$mg = kL$: $98 = k(9.8)$ implies $k = 10$

$F_{damping}(t) = -\gamma u'(t)$: $100 = 5\gamma$. Thus $\gamma = 20$

$10u''(t) + 20u'(t) + 10u(t) = 0$

$u''(t) + 2u'(t) + u(t) = 0$, $u(0) = -1$, $u'(0) = -4$

If $u = e^{rt}$, then $u' = re^{rt}$ and $u'' = r^2e^{rt}$.

Hence $r^2e^{rt} + 2re^{rt} + e^{rt} = 0$

$r^2 + 2r + 1 = 0$

$(r + 1) = 0$. Hence $r = -1$

Hence general solution is $u(t) = c_1e^{-t} + c_2te^{-t}$

$u(0) = -1$, $u'(0) = -4$

$u(t) = c_1e^{-t} + c_2te^{-t}$

$u'(t) = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t}$

$-1 = c_1$

$-4 = -c_1 + c_2$, $-4 = 1 + c_2$. Thus $c_2 = -5$

$u(t) = -e^{-t} - 5te^{-t}$

Answer: $u(t) = -e^{-t} - 5te^{-t}$

6.) Show that L : set of all twice differentiable functions $\rightarrow$ set of all functions, $L(f) = af'' + bf' + cf$ is a linear function.

Hint: Calculate $L(rf + tg)$ where r, t are real numbers and f, g are twice differentiable functions.

$$\begin{aligned} L(rf + tg) &= a[rf + tg]'' + b[rf + tg]' + c[rf + tg] \\ &= a[rf'' + tg''] + b[rf' + tg'] + c[rf + tg] \\ &= arf'' + atg'' + brf' + btg' + crf + ctg \\ &= arf'' + brf' + crf + atg'' + btg' + ctg \\ &= r[af'' + bf' + cf] + t[ag'' + bg' + cg] \\ &= rL(f) + tL(g) \end{aligned}$$

Hence L is a linear function.

If $y = \phi(t)$ is a solution to $af'' + bf' + cf = 0$, then $L(\phi) = \underline{\quad 0 \quad}$.

If $y = \psi(t)$ is a solution to $af'' + bf' + cf = 0$, then $L(\psi) = \underline{\quad 0 \quad}$.

$L(c_1\phi + c_2\psi) = \underline{\quad 0 \quad}$.

Is $c_1\phi + c_2\psi$ a solution to $af'' + bf' + cf = 0$? $\underline{\quad yes \quad}$